

# FUNCTORIALITY FOR LAGRANGIAN CORRESPONDENCES IN FLOER THEORY

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ABSTRACT. We associate to every monotone Lagrangian correspondence a functor between Donaldson-Fukaya categories. The composition of such functors agrees with the functor associated to the geometric composition of the correspondences, if the latter is embedded. That is “categorification commutes with composition” for Lagrangian correspondences. This construction fits into a symplectic 2-category with a categorification 2-functor, in which all correspondences are composable, and embedded geometric composition is isomorphic to the actual composition. As a consequence, any functor from a bordism category to the symplectic category gives rise to a category valued topological field theory.

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## 1. INTRODUCTION

Correspondences arise naturally as generalizations of maps in a number of different settings: A correspondence between two sets is a subset of the Cartesian product of the sets – just like the graph of a map. In symplectic geometry, the natural class is that of Lagrangian correspondences, that is, Lagrangian submanifolds in the product of two symplectic manifolds (with the symplectic form on the first factor reversed). Lagrangian correspondences appear in Hörmander’s generalizations of pseudodifferential operators [7], and were investigated from the categorical point of view by Weinstein [24]. In gauge theory Lagrangian correspondences arise as moduli spaces of bundles associated to cobordisms [25].

One would hope that various constructions associated to symplectic manifolds, which are compatible with symplectomorphisms, can also be made functorial for Lagrangian correspondences. The constructions considered by Hörmander and Weinstein correspond to various notions of quantization, by which a symplectic manifold is replaced by a linear space; one then tries to attach to a Lagrangian correspondence a linear map. More recently, categorical invariants associated to a symplectic manifold have been introduced by Donaldson and Fukaya, see for example [5] and [15]. To the symplectic manifold is associated a category whose objects are certain Lagrangian submanifolds, and whose morphisms are certain chain complexes or Floer cohomology groups. The composition in this category

gives a way to understand various product structures in Floer theory, and plays a role in the homological mirror symmetry conjecture of Kontsevich [8].

In this paper we associate to every (compact monotone or geometrically bounded exact) symplectic manifold  $(M, \omega)$  a category  $\text{Don}^\#(M)$ , which is a slight enlargement of the usual Donaldson-Fukaya category. Its objects are certain sequences of (compact, oriented, relatively spin, monotone or exact) Lagrangian correspondences and its morphisms are quilted Floer cohomology classes, as introduced in [20]. Given two symplectic manifolds  $M_0$  and  $M_1$  of the same monotonicity type and an admissible Lagrangian correspondence  $L_{01} \subset M_0^- \times M_1$  we construct a functor

$$\Phi(L_{01}) : \text{Don}^\#(M_0) \rightarrow \text{Don}^\#(M_1).$$

On objects it is given by concatenation, e.g.  $\Phi(L_{01})(L_0) = (L_0, L_{01})$  for a Lagrangian submanifold  $L_0 \subset M_0$ . On morphisms the functor is given by a relative Floer theoretic invariant constructed from moduli spaces of pseudoholomorphic quilts introduced in [19].

Given a triple  $M_0, M_1, M_2$  of symplectic manifolds and admissible Lagrangian correspondences  $L_{01} \subset M_0^- \times M_1$  and  $L_{12} \subset M_1^- \times M_2$ , the algebraic composition  $\Phi(L_{01}) \circ \Phi(L_{12}) : \text{Don}^\#(M_0) \rightarrow \text{Don}^\#(M_2)$  is always defined. On the other hand, one may consider the *geometric composition* introduced by Weinstein [24]

$$L_{01} \circ L_{12} := \pi_{02}(L_{01} \times_{M_1} L_{12}) \subset M_0^- \times M_2,$$

given by the image under the projection  $\pi_{02} : M_0^- \times M_1 \times M_1^- \times M_2 \rightarrow M_0^- \times M_2$  of

$$(1) \quad L_{12} \times_{M_1} L_{01} := (L_{01} \times L_{12}) \cap (M_0^- \times \Delta_1 \times M_2).$$

If we assume transversality of the intersection then the restriction of  $\pi_{02}$  to  $L_{01} \times_{M_1} L_{12}$  is automatically an immersion, see [6, 20]. Using the strip-shrinking analysis from [18] we prove that if  $L_{01} \times_{M_1} L_{12}$  is a transverse intersection and embeds by  $\pi_{02}$  into  $M_0^- \times M_2$  then

$$(2) \quad \Phi(L_{01}) \circ \Phi(L_{12}) \cong \Phi(L_{01} \circ L_{12}).$$

This is the "categorification commutes with composition" result alluded to in the abstract. If  $M_1$  is not spin, there is also a shift of relative spin structures on the right-hand side.

There is a stronger version of this result, expressed in the language of 2-categories as follows. (See e.g. Section 8 for an introduction to this language.) We construct a *Weinstein-Floer* 2-category  $\text{Floer}^\#$  whose objects are symplectic manifolds, 1-morphisms are sequences of Lagrangian correspondences, and 2-morphisms are Floer cohomology classes. (Again, we impose monotonicity and certain further admissibility assumptions on all objects and 1-morphisms.) The composition of 1-morphisms in this category is concatenation, which we denote by  $\#$ . The construction of the functor  $\Phi(L_{01})$  above extends to a categorification 2-functor to the 2-category of categories

$$(3) \quad \text{Floer}^\# \longrightarrow \text{Cat}.$$

On objects and elementary 1-morphisms (i.e. sequences consisting of a single correspondence) it is given by associating to every symplectic manifold  $M$  its Donaldson-Fukaya category  $\text{Don}^\#(M)$ , and to every Lagrangian correspondence  $L_{01}$  the associated functor  $\Phi(L_{01})$ . The further 1-morphisms are concatenations of elementary Lagrangian correspondences, mapped to the composition of functors. The 2-morphisms are quilted Floer homology classes, to which we associate natural transformations. A refinement of (2) says that the concatenation  $L_{01} \# L_{12}$  is 2-isomorphic to the geometric composition  $L_{01} \circ L_{12}$  as

1-morphisms in  $\text{Floer}^\#$ . The formula (2) then follows by combining this result with the 2-functor axiom for 1-morphisms in (3).

Alternatively, one could identify the 1-morphisms  $L_{01} \# L_{12}$  and  $L_{01} \circ L_{12}$  if the latter is a transverse, embedded composition. This provides an elementary construction of a symplectic category  $\text{Symp}^\#$  explained in Section 2. It consists of symplectic manifolds and equivalence classes of sequences of Lagrangian correspondences, whose composition is always defined and coincides with geometric composition in transverse, embedded cases.

The categorical point of view fits in well with one of the applications of our results, which is the construction of topological field theories associated to various gauge theories. A corollary of our categorification functor (3) is that any functor from a bordism category to the (monotone subcategory of the) symplectic category  $\text{Symp}^\#$  gives rise to a category valued TFT. For example, in [22] we investigate the topological quantum field theory with corners (roughly speaking; not all the axioms are satisfied) in  $2 + 1 + 1$  dimensions arising from moduli spaces of flat bundles with compact structure group on punctured surfaces and three-dimensional cobordisms containing tangles. In particular, this gives rise to  $SU(N)$  Floer theoretic invariants for 3-manifolds that should be thought of as Lagrangian Floer versions of gauge-theoretic invariants investigated by Donaldson and Floer, in the case without knots, and Kronheimer-Mrowka [9] and Collin-Steer [3], in the case with knots. The construction of such theories was suggested by Fukaya in [4] and was one of the motivations for the development of Fukaya categories.

Many of our results have chain-level versions, that is, extensions to Fukaya categories. These will be published in [12], which is joint work with S. Mau. To each monotone Lagrangian correspondence with minimal Maslov number at least three we define an  $A_\infty$  functor  $\Psi(L_{01}) : \text{Fuk}^\#(M_0) \rightarrow \text{Fuk}^\#(M_1)$  between extended versions of the Fukaya categories. Moreover, we are working on extending this construction to an  $A_\infty$  functor

$$\text{Fuk}^\#(M_0, M_1) \longrightarrow \text{Fun}(\text{Fuk}^\#(M_0), \text{Fuk}^\#(M_1)),$$

where the Fukaya category on the left hand side should be a chain-level version of the morphism space of  $\text{Floer}^\#$  between  $M_0$  and  $M_1$ , i.e. its objects are Lagrangian correspondences and sequences thereof, starting at  $M_0$  and ending at  $M_1$ . On homology level, for the Donaldson-Fukaya categories, this functor is given as part of the 2-categorification functor (3). On chain level, it would finalize the proof of homological mirror symmetry for the four-torus by Abouzaid and Smith [1]. It should be seen as the symplectic analogue of the quasi-equivalence of dg-categories [17] in algebraic geometry  $D_\infty^b(X \times X) \simeq \text{Fun}(D_\infty^b(X), D_\infty^b(X))$  for (somewhat enhanced) derived categories of coherent sheaves on a projective variety  $X$ . Abouzaid and Smith utilize the conjectural symplectic functor to prove that a given subcategory  $\mathcal{A}$  (for which a fully faithful functor to a derived category of coherent sheaves is known) generates the Fukaya category  $\text{Fuk}^\#(T^4)$ , by resolving the diagonal  $\Delta \subset (T^4)^- \times T^4$  in terms of products of Lagrangians in  $\mathcal{A}$ .

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## 2. SYMPLECTIC CATEGORY WITH LAGRANGIAN CORRESPONDENCES

We begin by summarizing some results and elementary notions from [20]. Restricted to linear Lagrangian correspondences between symplectic vector spaces, the *geometric composition* of Lagrangian correspondences defined in (1) is a well defined composition and defines a linear symplectic category [6]. In general, however, even when the intersection (1) is transverse, the geometric composition only yields an immersed Lagrangian. While it

may be natural to allow immersed Lagrangian correspondences (and attempt a definition of Floer cohomology for these), a construction of a symplectic category based on geometric composition would require the inclusion of perturbation data. A simple resolution of the composition problem is given by passing to sequences of Lagrangian correspondences and defining a purely algebraic composition. Here and throughout we will write  $M$  for a symplectic manifold  $(M, \omega)$  consisting of a manifold with symplectic 2-form; and we denote by  $M^- := (M, -\omega)$  the same manifold equipped with the symplectic form  $-\omega$ .

**Definition 2.1.** Let  $M, M'$  be symplectic manifolds. A *generalized Lagrangian correspondence*  $\underline{L}$  from  $M$  to  $M'$  consists of

- (a) a sequence  $N_0, \dots, N_r$  of any length  $r + 1 \geq 2$  of symplectic manifolds with  $N_0 = M$  and  $N_r = M'$ ,
- (b) a sequence  $L_{01}, \dots, L_{(r-1)r}$  of Lagrangian correspondences with  $L_{(j-1)j} \subset N_{j-1}^- \times N_j$  for  $j = 1, \dots, r$ .

**Definition 2.2.** Let  $\underline{L}$  from  $M$  to  $M'$  and  $\underline{L}'$  from  $M'$  to  $M''$  be two generalized Lagrangian correspondences. Then we define composition

$$(\underline{L}, \underline{L}') := (L_{01}, \dots, L_{(r-1)r}, L'_{01}, \dots, L'_{(r'-1)r'})$$

as a generalized Lagrangian correspondence from  $M$  to  $M''$ .

We will however want to include geometric composition into our category – if it is well defined. For the purpose of obtaining well defined Floer cohomology we will restrict ourselves to the following class of compositions, for which the resulting Lagrangian correspondence is in fact a smooth submanifold.

**Definition 2.3.** We say that the composition  $L_{01} \circ L_{12}$  is *embedded* if the intersection  $(L_{01} \times L_{12}) \pitchfork (M_0^- \times \Delta_1 \times M_2)$  is transverse and the projection  $\pi_{02} : L_{12} \times_{M_1} L_{01} \rightarrow L_{01} \circ L_{12} \subset M_0^- \times M_2$  is injective (and hence automatically an embedding).

Using these notions we can now define a *symplectic category*  $\text{Symp}^\#$  which includes all Lagrangian correspondences. An extension of this approach, using Floer cohomology spaces to define a 2-category, is given in Section 8.

**Definition 2.4.** The symplectic category  $\text{Symp}^\#$  is defined as follows:

- (a) The objects of  $\text{Symp}^\#$  are smooth symplectic manifolds  $M = (M, \omega)$ .
- (b) The morphisms  $\text{Hom}(M, M')$  of  $\text{Symp}^\#$  are generalized Lagrangian correspondences from  $M$  to  $M'$  modulo the equivalence relation  $\sim$  generated by

$$(\dots, L_{(j-1)j}, L_{j(j+1)}, \dots) \sim (\dots, L_{(j-1)j} \circ L_{j(j+1)}, \dots)$$

for all sequences and  $j$  such that  $L_{(j-1)j} \circ L_{j(j+1)}$  is embedded.

- (c) The composition of morphisms  $[\underline{L}] \in \text{Hom}(M, M')$  and  $[\underline{L}'] \in \text{Hom}(M', M'')$  is defined by

$$[\underline{L}] \circ [\underline{L}'] := [(\underline{L}, \underline{L}')] \in \text{Hom}(M, M'').$$

Note that a sequence of Lagrangian correspondences in  $\text{Hom}(M, M')$  can run through any sequence  $(N_i)_{i=1, \dots, r-1}$  of intermediate symplectic manifolds of any length  $r - 1 \in \mathbb{N}_0$ . Nevertheless, the composition of two such sequences is always well defined. In (c) the new sequence of intermediate symplectic manifolds for  $\underline{L} \circ \underline{L}'$  is  $(N_1, \dots, N_{r-1}, N_r = M' = N'_0, N'_1, \dots, N'_{r'-1})$ . This definition descends to the quotient by the equivalence relation  $\sim$  since any equivalences within  $\underline{L}$  and  $\underline{L}'$  combine to an equivalence within  $\underline{L} \circ \underline{L}'$ .

*Remark 2.5.*

- (a) The composition in  $\text{Symp}^\#$  is evidently associative:  $[\underline{L}] \circ [\underline{L}'] \circ [\underline{L}'' ] = [(\underline{L}, \underline{L}', \underline{L}'')]$ .
- (b) The identity in  $\text{Hom}(M, M)$  is the equivalence class  $[\Delta_M]$  of the diagonal  $\Delta_M \subset M^- \times M$ . It composes as identity since e.g.  $L_{(r-1)r} \circ \Delta_{M_r} = L_{(r-1)r}$  is always smooth and embedded.
- (c) In order to make  $\text{Symp}^\#$  a small category, one should fix a set of smooth manifolds, for example those embedded in Euclidean space. Any smooth manifold can be so embedded by Whitney's theorem. For a fixed manifold, the possible symplectic forms again form a set, hence we have a set of objects. Given two symplectic manifolds, the finite sequences of objects  $(N_i)_{i=1, \dots, r-1}$  again form a set, and for each fixed sequence the generalized Lagrangian correspondences between them can be exhibited as subsets satisfying submanifold, isotropy, and coisotropy conditions. Finally, we take the quotient by a relation to obtain a set of morphisms.

- Lemma 2.6.**
- (a) *If  $L_a, L_b \subset M^- \times M'$  are distinct Lagrangian submanifolds, then the corresponding morphisms  $[L_a], [L_b] \in \text{Hom}(M, M')$  are distinct.*
  - (b) *The composition of smooth Lagrangian correspondences  $L \subset M^- \times M'$  and  $L' \subset M'^- \times M''$  coincides with the geometric composition,  $[L] \circ [L'] = [L \circ L']$  if  $L \circ L'$  is embedded.*

*Proof.* To see that  $L_a \neq L_b \subset M^- \times M'$  define distinct morphisms note that the projection to the (possibly singular) Lagrangian  $\pi([\underline{L}]) := L_{01} \circ \dots \circ L_{(r-1)r} \subset M^- \times M'$  is well defined for all  $[\underline{L}] \in \text{Hom}(M, M')$ . The rest follows directly from the definitions.  $\square$

*Remark 2.7.* Lagrangian correspondences appeared in the study of Fourier integral operators by Hörmander and others. Any immersed homogeneous<sup>1</sup> Lagrangian correspondence  $L_{01} \rightarrow T^*Q_0^- \times T^*Q_1$  gives rise to a class of operators  $FIO_\rho(L_{01})$  depending on a real parameter  $\rho > 1/2$ , mapping smooth functions on  $Q_0$  to distributions on  $Q_1$ . These operators satisfy the compositability property similar to (2). Namely, [7, Theorem 4.2.2] shows that if a pair  $L_{01} \rightarrow T^*Q_0^- \times T^*Q_1, L_{12} \rightarrow T^*Q_1^- \times T^*Q_2$  satisfies

$$(4) \quad \begin{aligned} &L_{01} \times L_{12} \text{ intersects } T^*Q_0^- \times \Delta_{T^*Q_1} \times T^*Q_2 \text{ transversally and} \\ &\text{the projection from the intersection to } T^*Q_0^- \times T^*Q_2 \text{ is proper,} \end{aligned}$$

then the corresponding operators are composable and

$$(5) \quad FIO_\rho(L_{01}) \circ FIO_\rho(L_{12}) \subset FIO_\rho(L_{01} \circ L_{12}).$$

Hence, similar to our construction of  $\text{Symp}^\#$ , one could define a category  $\text{Hörm}^\#$ , whose

- objects are compact smooth manifolds,
- morphisms are sequences of pairs  $(L_{01}, P_{01})$  of immersed homogeneous Lagrangian correspondences (between cotangent bundles) together with operators  $P_{01} \in FIO_\rho(L_{01})$ , modulo the equivalence relation that is generated by  $(\dots, (L_{01}, P_{01}), (L_{12}, P_{12}), \dots) \sim (\dots, (L_{01} \circ L_{12}, P_{01} \circ P_{12}), \dots)$  for  $L_{01}, L_{12}$  satisfying (4).

A morphism in this category might be called a *generalized Fourier integral operator*.

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<sup>1</sup>An immersed Lagrangian correspondence  $L_{01}$  is called homogeneous if its image lies in the complement of the zero sections,  $L_{01} \rightarrow (T^*Q_0^- \setminus 0_{Q_0}) \times (T^*Q_1 \setminus 0_{Q_1})$ , and if it is conic, i.e. invariant under positive scalar multiplication in the fibres.

## 3. DONALDSON-FUKAYA CATEGORY OF LAGRANGIANS

Throughout this paper we will use the notation and constructions for (quilted) Floer homology and relative invariants introduced in [20, 19]. In particular, we will be using the following standing assumptions on symplectic manifolds, Lagrangian submanifolds, and gradings; see [20] for details.

- (M1):  $(M, \omega)$  is *monotone*, that is  $[\omega] = \tau c_1(TM)$  for some  $\tau \geq 0$ .
- (M2): If  $\tau > 0$  then  $M$  is compact. If  $\tau = 0$  then  $M$  is (necessarily) noncompact but satisfies “bounded geometry” assumptions as in [15].
- (L1):  $L \subset M$  is *monotone*, that is the symplectic area and Maslov index are related by  $2A(u) = \tau I(u)$  for all  $u \in \pi_2(M, L)$ , where the  $\tau \geq 0$  is (necessarily) that from (M1).
- (L2):  $L$  is compact and oriented.
- (L3):  $L$  has minimal Maslov number  $N_L \geq 3$ .
- (G1):  $M$  is equipped with a Maslov covering  $\text{Lag}^N(M)$  for  $N$  even, and the induced 2-fold Maslov covering  $\text{Lag}^2(M)$  is the one given by oriented Lagrangian subspaces.
- (G2):  $L \subset M$  is equipped with a grading  $\sigma_L^N : L \rightarrow \text{Lag}^N(M)$ , and the induced 2-grading  $L \rightarrow \text{Lag}^2(M)$  is the one given by the orientation of  $L$ .

In the following we review the construction of the Donaldson-Fukaya category  $\text{Don}(M)$  for a symplectic manifold  $(M, \omega)$  satisfying (M1-2). The “closed” analog of this category, whose morphisms are symplectomorphisms, was introduced by Donaldson in a seminar talk [13, 12.6]. Subsequently Fukaya introduced an  $A_\infty$  category involving Lagrangian submanifolds. Here we describe the category arising from the Fukaya category by taking homology.

We fix a Maslov cover  $\text{Lag}^N(M) \rightarrow M$  as in (G1), which will be used to grade Floer cohomology groups, and a *background class*  $b \in H^2(M, \mathbb{Z}_2)$ , which will be used to fix orientations of moduli spaces and thus define Floer cohomology groups with  $\mathbb{Z}$  coefficients. In our examples,  $b$  will be either 0 or the second Stiefel-Whitney class  $w_2(M)$  of  $M$ .

**Definition 3.1.** We say that a Lagrangian submanifold  $L \subset M$  is *admissible* if it satisfies (L1-3), (G2), and the image of  $\pi_1(L)$  in  $\pi_1(M)$  is torsion.

The assumption on  $\pi_1(L)$  guarantees that any collection of admissible Lagrangian submanifolds is monotone with respect to any surface in the sense of [20]. Alternatively, one could work with Bohr-Sommerfeld monotone Lagrangians as described in [20]. The assumption (L3) implies that the Floer cohomology of any sequence is well-defined, and can be relaxed to  $N_L \geq 2$  by working with matrix factorizations as explained in [23].

**Definition 3.2.** A *brane structure* on an admissible  $L$  consists of an orientation, a grading, and a relative spin structure with background class  $b$ , see [20, 21] for details. An admissible Lagrangian equipped with a brane structure will be called a *Lagrangian brane*.

*Remark 3.3.* (a) We have not included in the definition of Lagrangian branes the data of a flat vector bundle, in order to save space. The extension of the constructions below to this case should be straight forward and is left to the reader.

- (b) If one wants only  $\mathbb{Z}_2$ -gradings on the morphism spaces of the Donaldson-Fukaya category, then the assumptions (G1-2) may be ignored.
- (c) If one wants only  $\mathbb{Z}_2$  coefficients, then the background class and relative spin structures may be ignored.

**Definition 3.4.** The *Donaldson-Fukaya category*  $\text{Don}(M) := \text{Don}(M, \text{Lag}^N(M), \omega, b)$  is defined as follows:

- (a) The objects of  $\text{Don}(M)$  are Lagrangian branes in  $M$ .
- (b) The morphism spaces of  $\text{Don}(M)$  are the  $\mathbb{Z}_N$ -graded Floer cohomology groups with  $\mathbb{Z}$  coefficients  $\text{Hom}(L, L') := HF(L, L')$  constructed using a choice of perturbation datum consisting of a pair  $(J, H)$  of a time-dependent almost complex structure  $J$  and a Hamiltonian  $H$ , as in e.g. [20].
- (c) The composition law in the category  $\text{Don}(M)$  is defined by

$$\begin{aligned} \text{Hom}(L, L') \times \text{Hom}(L', L'') &\longrightarrow \text{Hom}(L, L'') \\ (f, g) &\longmapsto f \circ g := \Phi_P(f \otimes g), \end{aligned}$$

where  $\Phi_P$  is the relative Floer theoretic invariant associated to the “half-pair of pants” surface  $P$ , that is, the disk with three markings on the boundary (two incoming ends, one outgoing end) as in Figure 1.

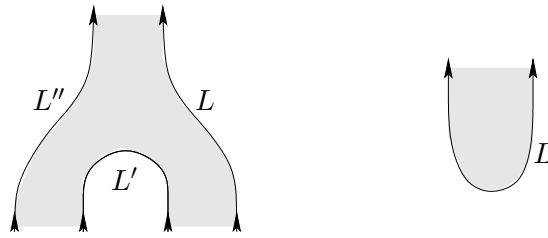


FIGURE 1. Composition and identity in the Donaldson-Fukaya category

*Remark 3.5.* (a) Associativity of the composition follows from the standard gluing theorem (see e.g. [19, Theorem 2.7]) applied to the surfaces in Figure 2: The two ways of composing correspond to two ways of gluing the pair of pants. The resulting surfaces are the same (up to a deformation of the complex structure), hence the resulting compositions are the same.

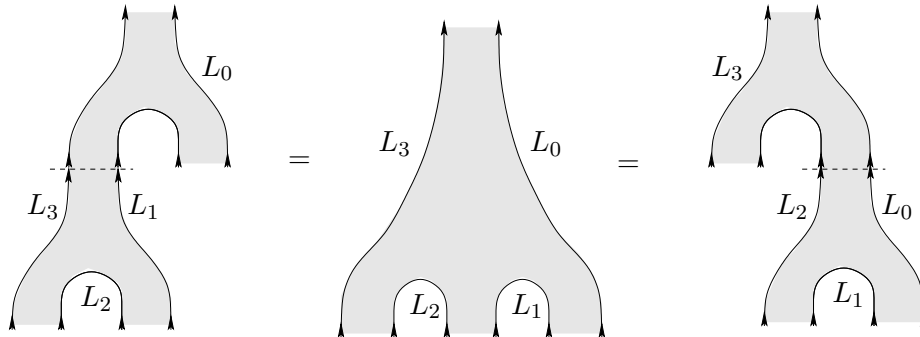


FIGURE 2. Associativity of composition

- (b) The identity  $1_L \in \text{Hom}(L, L)$  is the relative Floer theoretic invariant  $1_L := \Phi_S \in HF(L, L)$  associated to a disk  $S$  with a single marking (an outgoing end), see Figure 1. The identity axiom  $1_{L_0} \circ f = f = f \circ 1_{L_1}$  follows from the same gluing

argument applied to the surfaces on the left and right in Figure 3. Here – in contrast to the strips counted towards the Floer differential, where the equations are  $\mathbb{R}$ -invariant – the equation on the strip need not be  $\mathbb{R}$ -invariant and solutions are counted without quotienting by  $\mathbb{R}$ . However, as in the strip example [19, Example 2.5] one can choose  $\mathbb{R}$ -invariant perturbation data to make the equation  $\mathbb{R}$ -invariant. Then the only isolated solutions contributing to the count are constant, and hence the relative invariant is the identity.

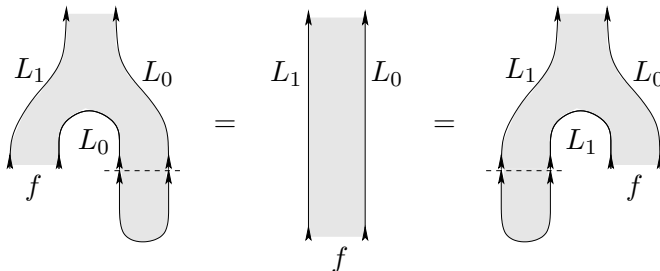


FIGURE 3. Identity axiom

*Remark 3.6.* The category  $\text{Don}(M)$  is independent of the choices of perturbation data involved in the definition of Floer homology and the relative invariants, up to isomorphism of categories: The relative invariants for the infinite strip with perturbation data interpolating between two different choices gives an isomorphism of the morphism spaces, see e.g. [20]. The gluing theorem implies compatibility of these morphisms with compositions and identities.

**3.1. Functor associated to symplectomorphisms.** Next, we recall that any graded symplectomorphism (see [15] or [20] for the grading)  $\psi : M_0 \rightarrow M_1$  induces a functor between Donaldson-Fukaya categories.

**Definition 3.7.** Let  $\Phi(\psi) : \text{Don}(M_0) \rightarrow \text{Don}(M_1)$  be the functor defined

- (a) on the level of objects by  $L \mapsto \psi(L)$ ,
- (b) on the level of morphisms by the map  $HF(L_0, L_1) \rightarrow HF(\psi(L_0), \psi(L_1))$  induced by the obvious map of chain complexes

$$CF(L_0, L_1) \rightarrow CF(\psi(L_0), \psi(L_1)), \quad \langle x \rangle \mapsto \langle \psi(x) \rangle$$

for all  $x \in \mathcal{I}(L_0, L_1)$ . (Here we use the Hamiltonians  $H \in \text{Ham}(L_0, L_1)$  and  $H \circ \psi^{-1} \in \text{Ham}(\psi(L_0), \psi(L_1))$ .)

Note that  $\Phi(\psi)$  satisfies the functor axioms

$$\Phi(\psi)(f \circ g) = \Phi(\psi)(f) \circ \Phi(\psi)(g), \quad \Phi(\psi)(1_L) = 1_{\psi(L)}.$$

Furthermore if  $\psi_{01} : M_0 \rightarrow M_1$  and  $\psi_{12} : M_1 \rightarrow M_2$  are symplectomorphisms then

$$\Phi(\psi_{12} \circ \psi_{01}) = \Phi(\psi_{01}) \circ \Phi(\psi_{12}).$$

In terms of Lagrangian correspondences this functor is  $L \mapsto L \circ \text{graph } \psi$  on objects. This suggests that one should extend the functor to more general Lagrangian correspondences  $L_{01} \subset M_0^- \times M_1$  by  $L \mapsto L \circ L_{01}$  on objects. However, these compositions are generically only immersed, so one would have to allow for singular Lagrangians as objects in  $\text{Don}(M_1)$ .



Moreover, it is not clear how to extend the functor on the level of morphisms, that is Floer cohomology groups. In the following sections we propose some alternative definitions of functors associated to general Lagrangian correspondences.

**3.2. First functor associated to Lagrangian correspondences.** We now define a first functor associated to a Lagrangian correspondence. Fix an integer  $N > 0$  and let  $\text{Ab}_N$  be the category of  $\mathbb{Z}_N$ -graded abelian groups. Let  $\text{Don}(M)^\vee$  be the category whose objects are functors from  $\text{Don}(M)$  to  $\text{Ab}_N$ , and whose morphisms are natural transformations.

Let  $(M_0, \omega_0)$  and  $(M_1, \omega_1)$  be symplectic manifolds satisfying (M1-2), equipped with  $N$ -fold Maslov coverings  $\text{Lag}^N(M_j)$  as in (G1) and background classes  $b_j \in H^2(M_j, \mathbb{Z}_2)$ , and let  $L_{01} \subset M_0^- \times M_1$  be an admissible Lagrangian correspondence in the sense of Definition 3.1, equipped with a grading as in (G2) and a relative spin structure with background class  $-\pi_0^*b_0 + \pi_1^*b_1$ .

**Definition 3.8.** The contravariant functor  $\Phi_{L_{01}} : \text{Don}(M_0) \rightarrow \text{Don}(M_1)^\vee$  associated to  $L_{01}$  is defined as follows:

- (a) On the level of objects, for every Lagrangian  $L_0 \subset M_0$  we define a functor  $\Phi_{L_{01}}(L_0) : \text{Don}(M_1) \rightarrow \text{Ab}_N$  by

$$L_1 \mapsto HF(L_0, L_{01}, L_1) = HF(L_0 \times L_1, L_{01})$$

on objects  $L_1 \subset M_1$ , and on morphisms

$$\begin{aligned} HF(L_1, L'_1) &\rightarrow \text{Hom}(HF(L_0, L_{01}, L_1), HF(L_0, L_{01}, L'_1)) \\ f &\mapsto \{g \mapsto \Phi_{\underline{S}_1}(g \otimes f)\} \end{aligned}$$

is defined by the relative invariant for the quilted surface  $\underline{S}_1$  shown in Figure 4,

$$\Phi_{\underline{S}_1} : HF(L_0, L_{01}, L_1) \otimes HF(L_1, L'_1) \rightarrow HF(L_0, L_{01}, L'_1).$$

- (b) The functor on the level of morphisms associates to every  $f \in HF(L_0, L'_0)$  a natural transformation

$$\Phi_{L_{01}}(f) : \Phi_{L_{01}}(L'_0) \rightarrow \Phi_{L_{01}}(L_0),$$

which maps objects  $L_1 \subset M_1$  to the  $\text{Ab}_N$ -morphism

$$\begin{aligned} \Phi_{L_{01}}(f)(L_1) : HF(L'_0, L_{01}, L_1) &\rightarrow HF(L_0, L_{01}, L_1) \\ g &\mapsto \Phi_{\underline{S}_0}(f \otimes g) \end{aligned}$$

defined by the relative invariant for the quilted surface  $\underline{S}_0$  shown in Figure 4,

$$\Phi_{\underline{S}_0} : HF(L_0, L'_0) \otimes HF(L'_0, L_{01}, L_1) \rightarrow HF(L_0, L_{01}, L_1).$$

The composition axiom for the functors  $\Phi_{L_{01}}(L_0)$  and the commutation axiom for the natural transformations follow from the quilted gluing theorem [19, Theorem 3.13]<sup>2</sup> applied to Figures 5 and 6.

Clearly the functor  $\Phi_{L_{01}}$  is unsatisfactory, since given two Lagrangian correspondences  $L_{01} \subset M_0^- \times M_1$ ,  $L_{12} \subset M_1^- \times M_2$  it is not clear how to define the composition of the associated functors  $\Phi_{L_{01}} : \text{Don}(M_0) \rightarrow \text{Don}(M_1)^\vee$  and  $\Phi_{L_{12}} : \text{Don}(M_1) \rightarrow \text{Don}(M_2)^\vee$ . As a solution (perhaps not the only one) we will define in Section 4 a category sitting in between  $\text{Don}(M)$  and  $\text{Don}(M)^\vee$ . This will allow for the definition of composable functors for general Lagrangian correspondences in Section 5.

<sup>2</sup>A complete account of the gluing analysis for quilted surfaces can be found in [11].

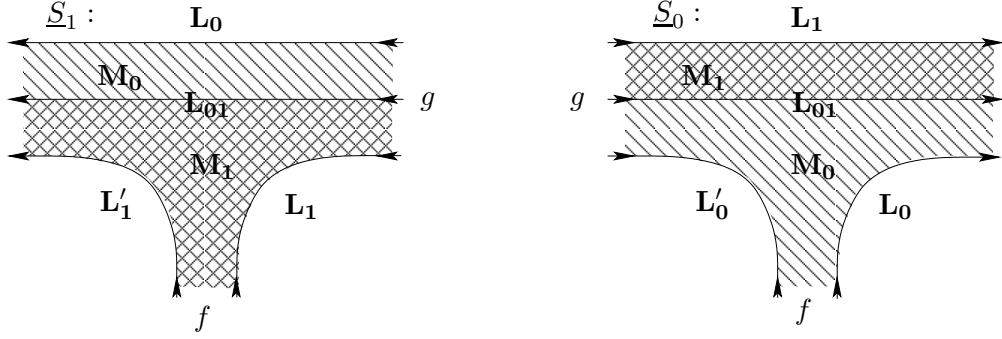


FIGURE 4. Lagrangian functor for morphisms

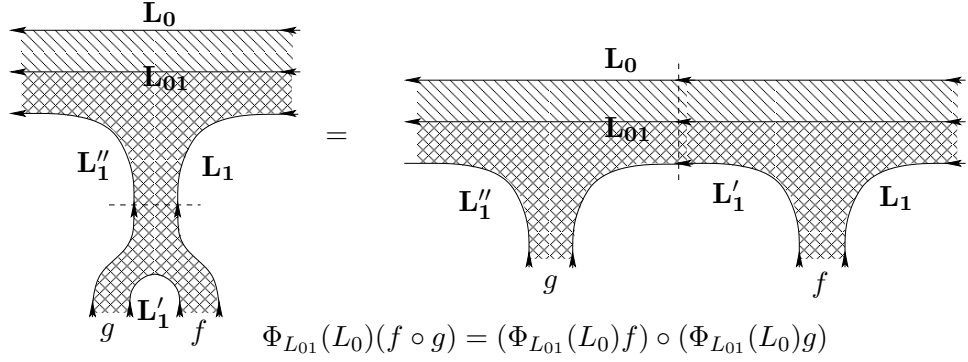


FIGURE 5. Composition axiom for Lagrangian functors

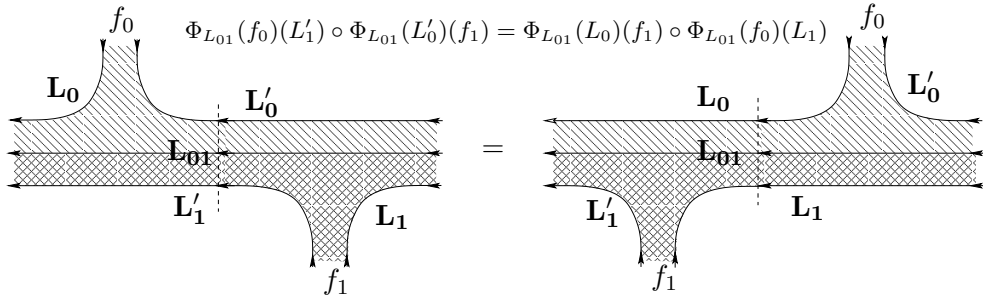


FIGURE 6. Commutation axiom for Lagrangian functors

#### 4. DONALDSON-FUKAYA CATEGORY OF GENERALIZED LAGRANGIANS

In this section we extend the Donaldson-Fukaya category  $\text{Don}(M)$  to a category  $\text{Don}^\#(M)$  which has generalized Lagrangian submanifolds as objects. Hence  $\text{Don}^\#(M)$  sits in between  $\text{Don}(M)$  and  $\text{Don}(M)^\vee$ . One might draw an analogy here with the way square-integrable functions sit between smooth functions and distributions.  $\text{Don}^\#(M)$  admits a functor to  $\text{Don}(M)^\vee$ , whose image is roughly speaking the subcategory of  $\text{Don}(M)^\vee$  generated by objects of geometric origin. This extension of the Donaldson-Fukaya category is particularly natural in our application to 2+1-dimensional topological field theory: One expects to associate a Lagrangian submanifold to any three-manifold with boundary, but our constructions

in fact yield generalized Lagrangian submanifolds that arise naturally from a decomposition into elementary cobordisms (or compression bodies).

Let  $(M, \omega)$  be a symplectic manifold satisfying (M1-2) with monotonicity constant  $\tau \geq 0$ . We fix a Maslov cover  $\text{Lag}^N(M) \rightarrow M$  as in (G1) and a background class  $b \in H^2(M, \mathbb{Z}_2)$ .

**Definition 4.1.** (a) A generalized Lagrangian submanifold of  $M$  is a generalized Lagrangian correspondence  $\underline{L}$  from  $\{\text{pt}\}$  to  $M$ , in the sense of Definition 2.1. That is,  $\underline{L} = (L_{(-r)(-r+1)}, \dots, L_{(-1)0})$  is a sequence of Lagrangian correspondences  $L_{(i-1)i} \subset N_{i-1}^- \times N_i$  for a sequence  $N_{-r}, \dots, N_0$  of any length  $r \geq 0$  of symplectic manifolds with  $N_{-r} = \{\text{pt}\}$  a point and  $N_0 = M$ .

(b) We call a generalized Lagrangian  $\underline{L}$  *admissible* if each  $N_i$  satisfies (M1-2) with the monotonicity constant  $\tau \geq 0$ , each  $L_{(i-1)i}$  satisfies (L1-3), and the image of each  $\pi_1(L_{(i-1)i})$  in  $\pi_1(N_{i-1}^- \times N_i)$  is torsion.

Again, one could replace the torsion assumption on fundamental groups by Bohr-Sommerfeld monotonicity as described in [20]. Note that an (admissible) Lagrangian submanifold  $L \subset M$  is an (admissible) generalized Lagrangian with  $r = 0$ . We picture a generalized Lagrangian  $\underline{L}$  as a sequence

$$\{\text{pt}\} \xrightarrow{L_{(-r)(-r+1)}} N_{-r} \xrightarrow{L_{(-r+1)(-r+2)}} \dots \xrightarrow{L_{(-2)(-1)}} N_{-1} \xrightarrow{L_{(-1)0}} N_0 = M .$$

Given two generalized Lagrangians  $\underline{L}, \underline{L}'$  of  $M$  we can transpose one and concatenate them to a sequence of Lagrangian correspondences from  $\{\text{pt}\}$  to  $\{\text{pt}\}$ ,

$$\{\text{pt}\} \xrightarrow{L_{(-r)(-r+1)}} \dots \xrightarrow{L_{(-1)0}} N_0 = M = N'_0 \xrightarrow{(L'_{(-1)0})^t} \dots \xrightarrow{(L'_{(-r')(-r'+1)})^t} \{\text{pt}\} .$$

The Floer cohomology of this sequence (as defined in [20]) is the natural generalization of the Floer cohomology for pairs of Lagrangian submanifolds. Hence we define

$$(6) \quad HF(\underline{L}, \underline{L}') := HF(L_{(-r)(-r+1)}, \dots, L_{(-1)0}, (L'_{(-1)0})^t, \dots, (L'_{(-r')(-r'+1)})^t).$$

Note here that every such sequence arising from a pair of admissible generalized Lagrangians is automatically monotone by a Lemma of [20].

**Definition 4.2.** The *generalized Donaldson-Fukaya category*

$$\text{Don}^\#(M) := \text{Don}^\#(M, \text{Lag}^N(M), \omega, b)$$

is defined as follows:

- (a) Objects of  $\text{Don}^\#(M)$  are admissible generalized Lagrangians of  $M$ , equipped with orientations, a grading, and a relative spin structure (see [20]).
- (b) Morphism spaces of  $\text{Don}^\#(M)$  are the  $\mathbb{Z}_N$ -graded Floer cohomology groups (see (6))

$$\text{Hom}(\underline{L}, \underline{L}') := HF(\underline{L}, \underline{L}')[d], \quad d = \frac{1}{2} \left( \sum_k \dim(N_k) + \sum_{k'} \dim(N_{k'}) \right),$$

given by choices of a perturbation datum and widths as described in [20] and degree shift  $d$ . For  $\mathbb{Z}$ -coefficients the Floer cohomology groups are modified by the inclusion of additional determinant lines as below in (7).

- (c) Composition of morphisms in  $\text{Don}^\#(M)$ ,

$$\begin{aligned} \text{Hom}(\underline{L}, \underline{L}') \times \text{Hom}(\underline{L}', \underline{L}'') &\longrightarrow \text{Hom}(\underline{L}, \underline{L}'') \\ (f, g) &\longmapsto f \circ g := \Phi_{\underline{L}}(f \otimes g) \end{aligned}$$

is defined by the relative invariant  $\Phi_{\underline{P}}$  associated to the quilted half-pair of pants surface  $\underline{P}$  in Figure 7, with the following orderings: The relative invariant is independent of the ordering of the patches with one outgoing end by a Remark in [19]. The remaining patches with two incoming ends are ordered from the top down, that is, starting with those furthest from the boundary.

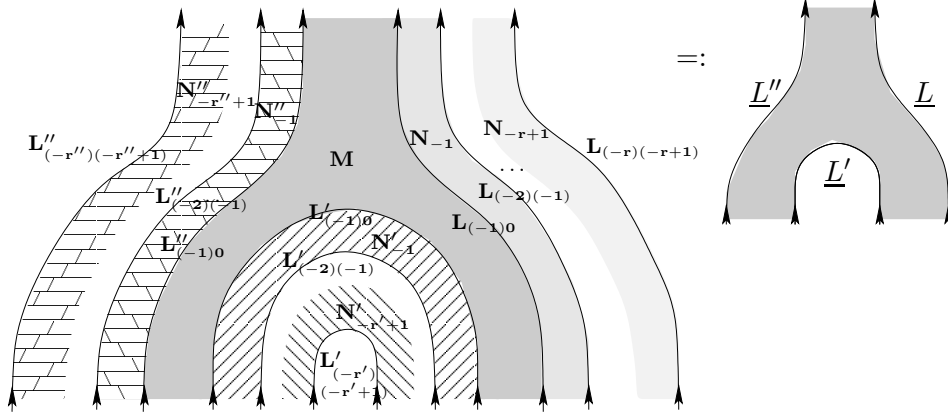


FIGURE 7. Quilted pair of pants

- Remark 4.3.*
- (a) Identities  $1_{\underline{L}} \in \text{Hom}(\underline{L}, \underline{L})$  are furnished by relative invariants  $1_{\underline{L}} := \Phi_{\underline{S}} \in \text{Hom}(\underline{L}, \underline{L})$  associated to the quilted disk  $\underline{S}$  in Figure 8, with patches ordered from the bottom up, that is, starting with those closest to the boundary.
  - (b) The identity and associativity axioms are satisfied with  $\mathbb{Z}_2$  coefficients by the quilted gluing theorem [19, Theorem 3.13] applied to the quilted versions of Figures 2, 3.
  - (c) Both the identity and composition are degree 0 by a Remark of [19].
  - (d)  $\text{Don}^\#(M)$  is a small category. The objects form a set by the same arguments as in Remark 2.5 (c); the morphisms are evidently constructed as set.

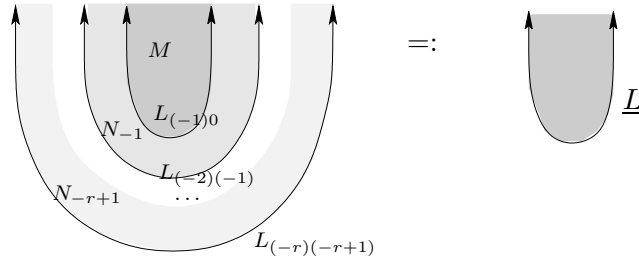


FIGURE 8. Quilted identity

*Remark 4.4.* To obtain the axioms with  $\mathbb{Z}$  coefficients requires a modification of the Floer cohomology groups, incorporating the determinant lines in a more canonical way. This will be treated in detail in [21], so we only give a sketch here: For each intersection point  $\underline{x} \in \mathcal{I}(\underline{L}, \underline{L}')$  we say that an *orientation* for  $\underline{x}$  consists of the following data: A partially

quilted surface<sup>3</sup>  $\underline{S}$  with a single end, complex vector bundles  $\underline{E}$  over  $\underline{S}$ , and totally real subbundles  $\mathcal{F}$  over the boundaries and seams, such that near infinity on the strip-like ends  $\underline{E}$  and  $\mathcal{F}$  are given by  $(T_{x_i}M_i)$  and  $T_{\underline{x}}\underline{L}, T_{\underline{x}}\underline{L}'$ ; a real Cauchy-Riemann operator  $D_{\underline{E}, \mathcal{F}}$ ; an orientation on the determinant line  $\det(D_{\underline{E}, \mathcal{F}})$ . We say that two orientations for  $\underline{x}$  are *isomorphic* if the two problems have isomorphic bundles  $\underline{E}$ , and the surfaces, boundary and seam conditions are deformation equivalent after a possible re-ordering of boundary components etc., and the orientations are related by the isomorphism of determinant lines arising from re-ordering. Let  $O(\underline{x})$  denote the space of isomorphism classes of orientations for  $\underline{x}$ . Define

$$(7) \quad \widetilde{CF}(\underline{L}, \underline{L}') = \bigoplus_{\underline{x} \in \mathcal{I}(\underline{L}, \underline{L}')} O(\underline{x}) \otimes_{\mathbb{Z}_2} \mathbb{Z}.$$

The Floer coboundary operator extends canonically to an operator of degree 1 on  $\widetilde{CF}(\underline{L}, \underline{L}')$ , and let  $\widetilde{HF}(\underline{L}, \underline{L}')$  denote its cohomology. This is similar to the definition given in e.g. Seidel [15, (12f)], except that we allow more general surfaces. The group  $\widetilde{HF}(\underline{L}, \underline{L}')$  is of infinite rank over  $\mathbb{Z}$ , but it has finite rank over a suitable graded-commutative Novikov ring generated by determinant lines.

The relative invariants extend to operators  $\widetilde{\Phi}_{\underline{S}}$  operating on the tensor product of (extended) Floer cohomologies. In particular, the quilted pair of pants defines an operator

$$\widetilde{\Phi}_P : \widetilde{HF}(\underline{L}, \underline{L}') \otimes \widetilde{HF}(\underline{L}', \underline{L}'') \rightarrow \widetilde{HF}(\underline{L}, \underline{L}'').$$

If we fix orientations for each generator  $\langle \underline{x} \rangle$ , as in the definition of  $HF$ , then the gluing sign for the first gluing (to the second incoming end) in the proof of associativity, Figure 2, is  $+1$ . For the second gluing (to the first incoming end) when applied to  $\langle \underline{x}_1 \rangle \otimes \langle \underline{x}_2 \rangle \otimes \langle \underline{x}_3 \rangle$  the sign is  $(-1)^{|\underline{x}_3| \frac{1}{2} \sum_i \dim(N_i^{(1)})}$ . Here  $N_i^{(j)}$  denotes the sequence of symplectic manifolds underlying the generalized Lagrangian correspondence  $\underline{L}_j$ . In addition, the two gluings induce different orderings of patches in the glued quilted surface, which are related by the additional sign  $(-1)^{\left(\frac{1}{2} \sum_i \dim(N_i^{(1)})\right) \left(\frac{1}{2} \sum_i \dim(N_i^{(2)})\right)}$ . Combined together, these factors cancel the sign arising from the re-ordering of determinants in the definitions of  $\widetilde{\Phi}_P(\widetilde{\Phi}_P(\langle \underline{x}_1 \rangle \otimes \langle \underline{x}_2 \rangle) \otimes \langle \underline{x}_3 \rangle)$  and  $\widetilde{\Phi}_P(\langle \underline{x}_1 \rangle \otimes \widetilde{\Phi}_P(\langle \underline{x}_2 \rangle \otimes \langle \underline{x}_3 \rangle))$ .

The identity axiom involves gluing a quilted cup with a quilted pair of pants; the orderings of the patches for the quilted cup and quilted pants above are chosen so that the gluing sign for gluing the quilted cup with quilted pants to obtain a quilted strip is  $+1$  for gluing into the second argument, and  $(-1)^{|\underline{x}| \frac{1}{2} \sum_i \dim(N_i)}$  for gluing into the first argument. Again, the additional sign is absorbed into the isomorphism of determinant lines induced by gluing.

*Convention 4.5.* To simplify pictures of quilts we will use the following conventions indicated in Figure 9 : A generalized Lagrangian submanifold  $\underline{L}$  of  $M$  can be used as “boundary condition” for a surface mapping to  $M$  in the sense that the boundary arc that is labeled by the sequence  $\underline{L} = (L_{(-r)(-r+1)}, \dots, L_{(-1)0})$  of Lagrangian correspondences from  $\{\text{pt}\}$  to  $M$  is replaced by a sequence of strips mapping to  $N_{-1}, \dots, N_{-r+1}$ , with seam conditions in  $L_{(-1)0}, \dots, L_{(-r+2)(-r+1)}$  and a final boundary condition in  $L_{(-r)(-r+1)}$ . Similarly, a generalized Lagrangian correspondence  $\underline{L}$  between  $M_-$  and  $M_+$  can be used as “seam condition” between surfaces mapping to  $M_{\pm}$  in the sense that the seam that is labeled by the sequence

<sup>3</sup>See [21] for the definition of partial quilts. For example, the standard cup orientation for  $\underline{x} = (x_1, \dots, x_N)$  will use unquilted cups  $S_i$  associated to each  $T_{x_i}M_i$ , and identified via seams on the strip-like ends.

$\underline{L} = (L_{01}, \dots, L_{(r-1)r})$  of Lagrangian correspondences from  $M_-$  to  $M_+$  is replaced by a sequence of strips mapping to  $M_1, \dots, M_{r-1}$  with seam conditions in  $L_{01}, \dots, L_{(r-1)r}$ .

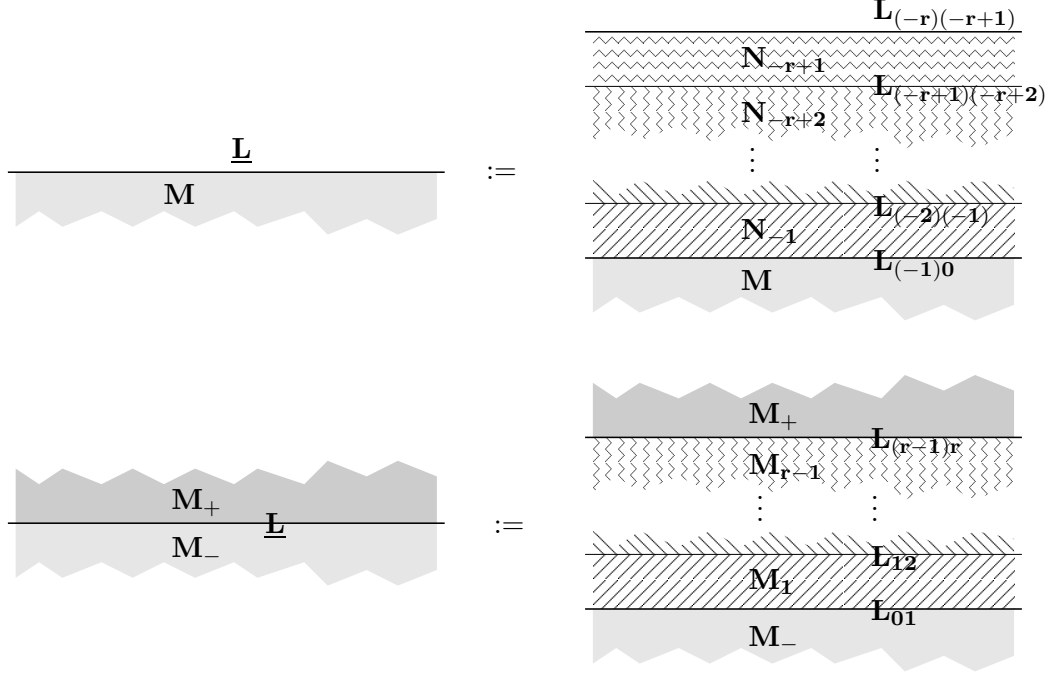


FIGURE 9. Conventions on using generalized Lagrangians and Lagrangian correspondences as boundary and seam conditions

*Remark 4.6.* As for  $\text{Don}(M)$ , the category  $\text{Don}^\#(M)$  is independent of the choices of perturbation data and widths up to isomorphism of categories, see Remark 3.6 and the proofs of independence of quilted Floer cohomology and relative quilt invariants in [20, 19].

**Proposition 4.7.** *The map  $\underline{L} \mapsto \underline{L}^\vee$ , for a generalized Lagrangian  $\underline{L}$  of  $M$  given by*

$$\underline{L}^\vee(L_0) := \text{Hom}(\underline{L}, L_0) = HF(L_{-r(-r+1)}, \dots, L_{(-1)0}, L_0)[d]$$

for all Lagrangian submanifolds  $L_0 \subset M$  and with degree shift  $d = \frac{1}{2} \sum_k \dim(N_k)$ , extends to a contravariant functor  $\text{Don}^\#(M) \rightarrow \text{Don}(M)^\vee$ .

*Proof.* The functor  $\underline{L}^\vee : \text{Don}(M) \rightarrow \text{Ab}_N$  can be defined on morphisms by

$$\underline{L}^\vee : \begin{array}{ccc} \text{Hom}(L_1, L'_1) & \rightarrow & \text{Hom}(\text{Hom}(\underline{L}, L_1), \text{Hom}(\underline{L}, L'_1)) \\ f & \mapsto & \{g \mapsto g \circ f = \Phi_{\underline{L}}(g \otimes f)\} \end{array}$$

using the composition on  $\text{Don}^\#(M)$ . To morphisms  $f \in \text{Hom}(\underline{L}, \underline{L}')$  of  $\text{Don}^\#(M)$  we can then associate the natural transformation  $f^\vee : \underline{L}'^\vee \rightarrow \underline{L}^\vee$ , which maps every object  $L_1 \subset M$  of  $\text{Don}(M)$  to the following  $\text{Ab}_N$ -morphism  $f^\vee(L_1)$ :

$$\text{Hom}(\underline{L}', L_1) \rightarrow \text{Hom}(\underline{L}, L_1), \quad g \mapsto f \circ g = \Phi_{\underline{L}}(f \otimes g),$$

again given by composition on  $\text{Don}^\#(M)$ . The axioms follow from the quilted gluing theorem [19, Theorem 3.13] applied to jazzed-up versions of Figures 5 and 6 (which show the

example  $\underline{L} = (L_0, L_{01})$ ,  $\underline{L}' = (L'_0, L_{01})$ ). In this case the orientations are independent of the ordering of patches since all have one boundary component and one outgoing end.  $\square$

### 5. COMPOSABLE FUNCTORS ASSOCIATED TO LAGRANGIAN CORRESPONDENCES

Let  $M_0$  and  $M_1$  be two symplectic manifolds satisfying (M1-2) with the same monotonicity constant  $\tau \geq 0$ . We fix Maslov covers  $\text{Lag}^N(M_i) \rightarrow M_i$  as in (G1) and background classes  $b_i \in H^2(M_i, \mathbb{Z}_2)$ . Given an admissible Lagrangian correspondence  $L_{01} \subset M_0^- \times M_1$  in the sense of Section 6, we can now define a functor  $\Phi(L_{01}) : \text{Don}^\#(M_0) \rightarrow \text{Don}^\#(M_1)$ . More precisely, we assume that  $L_{01}$  satisfies (L1-3), and the image of  $\pi_1(L_{01})$  in  $\pi_1(M_0^- \times M_1)$  is torsion.

**Definition 5.1.** The functor  $\Phi(L_{01}) : \text{Don}^\#(M_0) \rightarrow \text{Don}^\#(M_1)$  is defined as follows:

- (a) On the level of objects,  $\Phi(L_{01})$  is concatenation of the Lagrangian correspondence to the sequence of Lagrangian correspondences: For a generalized Lagrangian  $\underline{L} = (L_{-r(-r+1)}, \dots, L_{(-1)0})$  of  $M_0$  with corresponding sequence of symplectic manifolds  $(\{\text{pt}\}, N_{-r+1}, \dots, N_{-1}, M_0)$  we put

$$\Phi(L_{01})(\underline{L}) := (\underline{L}, L_{01}) := (L_{(-r)(-r+1)}, \dots, L_{(-1)0}, L_{01})$$

with the corresponding symplectic manifolds  $(\{\text{pt}\}, N_{-r+1}, \dots, N_{-1}, M_0, M_1)$ ;

- (b) On the level of morphisms, for any pair  $\underline{L}, \underline{L}'$  of generalized Lagrangians in  $M_0$ ,

$$\Phi(L_{01}) := \Phi_{\underline{S}} : \text{Hom}(\underline{L}, \underline{L}') \rightarrow \text{Hom}(\Phi(L_{01})(\underline{L}), \Phi(L_{01})(\underline{L}'))$$

is the relative invariant associated to the quilted surface  $\underline{S}$  with two punctures and one interior circle, as in Figure 10.

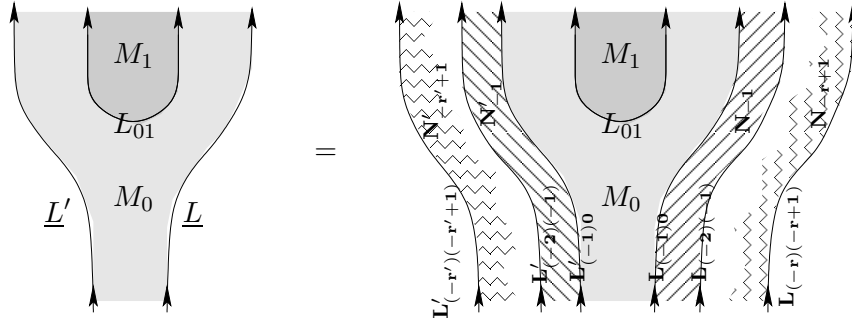
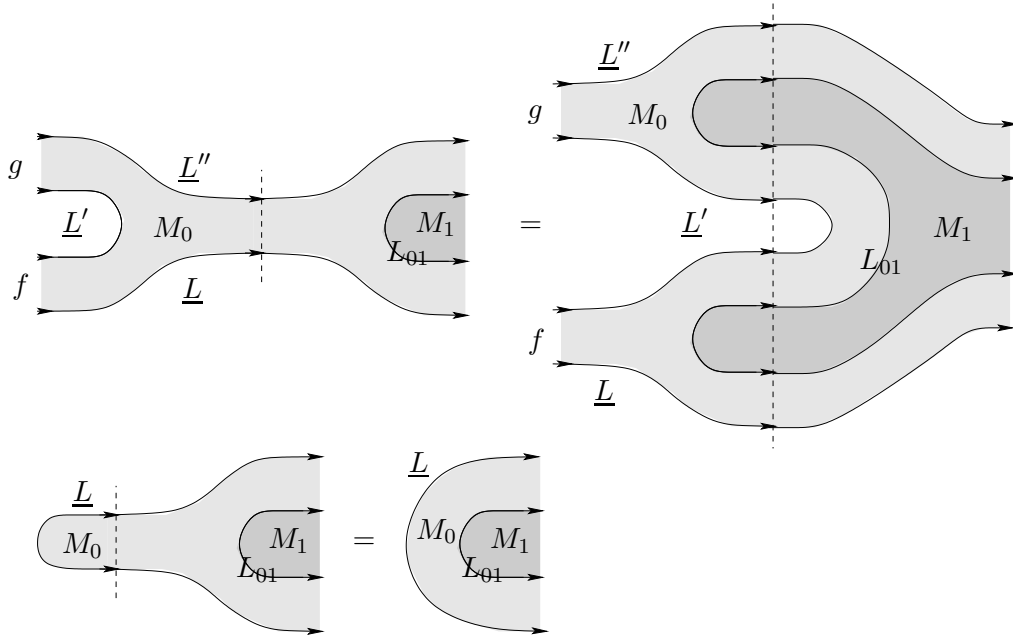


FIGURE 10. The Lagrangian correspondence functor  $\Phi(L_{01})$  on morphisms

*Remark 5.2.* In the case that  $M_1$  is a point, the map for morphisms is the dual of the pair of pants product.

For composable morphisms  $f \in \text{Hom}(\underline{L}, \underline{L}')$ ,  $g \in \text{Hom}(\underline{L}', \underline{L}')$  one shows  $\Phi_{L_{01}}(f \circ g) = \Phi_{L_{01}}(f) \circ \Phi_{L_{01}}(g)$  by applying the quilted gluing theorem [19, Theorem 3.13] to the gluings shown in Figure 11 (simplifying the picture by Convention 4.5), which yield homotopic quilted surfaces. The gluing signs for both gluings are positive. Similarly, the second gluing shows that  $\Phi(L_{01})(1_{\underline{L}}) = 1_{\Phi(L_{01})(\underline{L})}$ , since we have ordered the patches of the quilted cup from the outside in.

FIGURE 11. The functor axioms for  $\Phi_{L_{01}}$ 

*Remark 5.3.* The surfaces of the first gluing in Figure 11 can equivalently be represented as degenerations of one quilted disk. The corresponding one-parameter family in Figure 12 is the one-dimensional *multiplihedron* of Stasheff, see [16], [10, p. 113], to which we will return in [12].

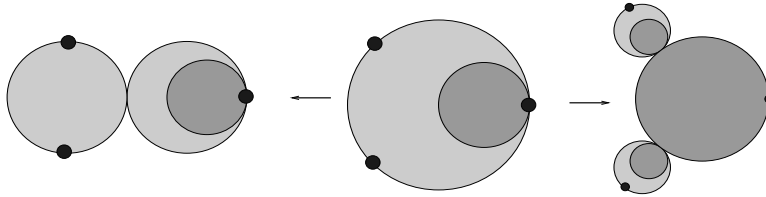


FIGURE 12. Degeneration view of the first functor axiom

With this new definition, any two functors associated to smooth, compact, admissible Lagrangian correspondences,  $\Phi(L_{01}) : \text{Don}^\#(M_0) \rightarrow \text{Don}^\#(M_1)$  and  $\Phi(L_{12}) : \text{Don}^\#(M_1) \rightarrow \text{Don}^\#(M_2)$ , are clearly composable. More generally, consider a sequence

$$\underline{L}_{0r} = (L_{01}, \dots, L_{(r-1)r})$$

of Lagrangian correspondences  $L_{(j-1)j} \subset M_{j-1}^- \times M_j$ . (That is,  $\underline{L}_{0r}$  is a generalized Lagrangian correspondence from  $M_0$  to  $M_r$  in the sense of Definition 2.1.) Assume that  $\underline{L}_{0r}$  is admissible in the sense of Section 6 below. We can then define a functor by composition

$$(8) \quad \Phi(\underline{L}_{0r}) := \Phi(L_{01}) \circ \dots \circ \Phi(L_{(r-1)r}) : \text{Don}^\#(M_0) \rightarrow \text{Don}^\#(M_r).$$



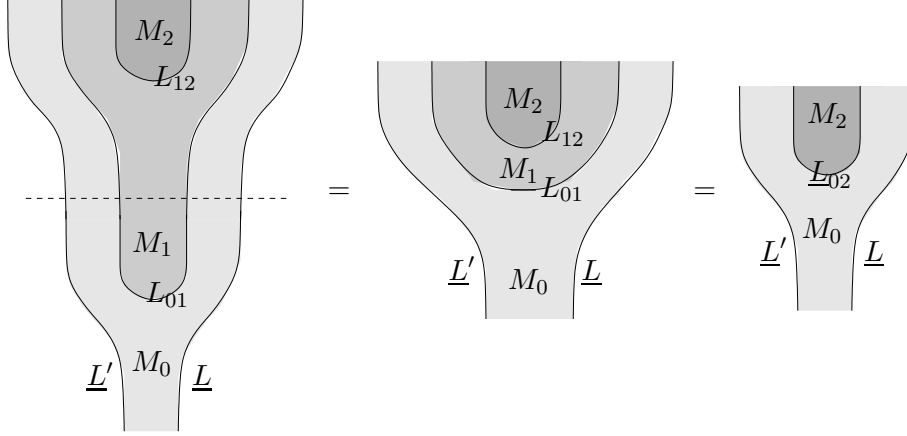


FIGURE 13. The composition  $\Phi(L_{01}) \circ \dots \circ \Phi(L_{(r-1)r})$  is given by a relative invariant for the sequence  $\underline{L}_{0r} = (L_{01}, \dots, L_{(r-1)r})$ . (Here  $r = 2$ .)

*Remark 5.4.* On the level of morphisms, the functor  $\Phi(\underline{L}_{0r})$  is given by the relative invariant associated to the quilted surface  $\underline{S}$  in Figure 13,

$$\Phi(\underline{L}_{0r}) = \Phi_{\underline{S}} : \text{Hom}(\underline{L}, \underline{L}') \rightarrow \text{Hom}(\Phi(\underline{L}_{0r})(\underline{L}), \Phi(\underline{L}_{0r})(\underline{L}'))$$

for all generalized Lagrangian submanifolds  $\underline{L}, \underline{L}' \in \text{Obj}(\text{Don}^{\#}(M_0))$ , with patches with two outgoing ends ordered from bottom up. This follows from the quilted gluing theorem applied to the gluing shown in Figure 13.

### 5.1. Functors associated to composed Lagrangian correspondences and graphs.

The next two strip-shrinking results are summarized from [18, 20, 19]. The first theorem describes the isomorphism of Floer cohomology under geometric composition, while the second describes the behavior of the relative invariants.

**Theorem 5.5.** *Let  $\underline{L} = (L_{01}, \dots, L_{r(r+1)})$  be a cyclic sequence of Lagrangian correspondences between symplectic manifolds  $M_0, \dots, M_{r+1} = M_0$ . Suppose that*

- (a) *the symplectic manifolds all satisfy (M1-2) with the same monotonicity constant  $\tau$ ,*
- (b) *the Lagrangian correspondences all satisfy (L1-3),*
- (c) *the sequence  $\underline{L}$  is monotone, relatively spin, and graded;*
- (d) *for some  $1 \leq j \leq r$  the composition  $L_{(j-1)j} \circ L_{j(j+1)}$  is embedded in the sense of Definition 2.3,*

*Then with respect to the induced relative spin structure, orientation, and grading on the modified sequence  $\underline{L}' = (L_{01}, \dots, L_{(j-1)j} \circ L_{j(j+1)}, \dots, L_{r(r+1)})$  there exists a canonical isomorphism of graded groups*

$$HF(\underline{L}) = HF(\dots L_{(j-1)j}, L_{j(j+1)} \dots) \xrightarrow{\sim} HF(\dots L_{(j-1)j} \circ L_{j(j+1)} \dots) = HF(\underline{L}'),$$

*induced by the canonical identification of intersection points.*

**Theorem 5.6.** *Consider a quilted surface  $\underline{S}$  containing a patch  $S_{\ell_1}$  that is diffeomorphic to  $\mathbb{R} \times [0, 1]$  and attached via seams  $\sigma_{01} = \{(\ell_0, I_0), (\ell_1, \mathbb{R} \times \{0\})\}$  and  $\sigma_{12} = \{(\ell_1, \mathbb{R} \times \{1\}), (\ell_2, I_2)\}$  to boundary components  $I_0, I_2$  of other surfaces  $S_{\ell_0}, S_{\ell_2}$ . Let  $\underline{M}$  be symplectic manifolds (satisfying (M1-2), (G1) with the same  $\tau \geq 0$  and  $N \in \mathbb{N}$ ) labeling the patches of  $\underline{S}$ , and  $\underline{\mathcal{L}}$  be Lagrangian boundary and seam conditions for  $\underline{S}$  such that all Lagrangians in*

$\mathcal{L}$  satisfy (L1-3), (G2), and  $\mathcal{L}$  is monotone and relative spin in the sense of [19]. Suppose that the Lagrangian correspondences  $L_{\sigma_{01}} \subset M_{\ell_0}^- \times M_{\ell_1}$ ,  $L_{\sigma_{12}} \subset M_{\ell_1}^- \times M_{\ell_2}$  associated to the boundary components of  $S_{\ell_1}$  are such that  $L_{\sigma_{01}} \circ L_{\sigma_{12}}$  is embedded.

Let  $\underline{S}'$  denote the quilted surface obtained by removing the patch  $S_{\ell_1}$  and corresponding seams and replacing it by a new seam  $\sigma_{02} := \{(\ell_0, I_0), (\ell_2, I_2)\}$ . We define Lagrangian boundary conditions  $\mathcal{L}'$  for  $\underline{S}'$  by  $L_{\sigma_{02}} := L_{\sigma_{01}} \circ L_{\sigma_{12}}$ . Then the isomorphisms in Floer cohomology  $\Psi_{\underline{e}} : HF(\underline{L}_{\underline{e}}) \rightarrow HF(\underline{L}'_{\underline{e}})$  for each end  $\underline{e} \in \mathcal{E}(\underline{S}) \cong \mathcal{E}(\underline{S}')$  intertwine with the relative invariants:

$$\Phi_{\underline{S}'} \circ \left( \bigotimes_{\underline{e} \in \mathcal{E}_-} \Psi_{\underline{e}} \right) = \left( \bigotimes_{\underline{e} \in \mathcal{E}_+} \Psi_{\underline{e}} \right) \circ \Phi_{\underline{S}}[n_{\ell_1}d].$$

Here  $2n_{\ell_1}$  is the dimension of  $M_{\ell_1}$ , and  $d = 1, 0$ , or  $-1$  according to whether the removed strip  $S_{\ell_1}$  has two outgoing ends, one in- and one outgoing, or two incoming ends.

As first application of these results we will show that the composed functor  $\Phi(L_{01}) \circ \Phi(L_{12}) : \text{Don}^\#(M_0) \rightarrow \text{Don}^\#(M_2)$  is isomorphic to the functor  $\Phi(L_{01} \circ L_{12})$  of the geometric composition  $L_{01} \circ L_{12} \subset M_0^- \times M_2$ , if the latter is embedded. More precisely and more generally, we have the following result.

**Theorem 5.7.** *Let  $\underline{L}_{0r} = (L_{01}, \dots, L_{(r-1)r})$  and  $\underline{L}'_{0r'} = (L'_{01}, \dots, L'_{(r'-1)r'})$  be two admissible generalized Lagrangian correspondence from  $M_0$  to  $M_r = M_{r'}$ . Suppose that they are equivalent in the sense of Section 2 through a series of embedded compositions of consecutive Lagrangian correspondences and such that each intermediate generalized Lagrangian correspondence is admissible. Then for any two admissible generalized Lagrangian submanifolds  $\underline{L}, \underline{L}' \in \text{Obj}(\text{Don}^\#(M_0))$  there is an isomorphism*

$$\Psi : \text{Hom}(\Phi(\underline{L}_{0r})(\underline{L}), \Phi(\underline{L}_{0r})(\underline{L}')) \rightarrow \text{Hom}(\Phi(\underline{L}'_{0r'})(\underline{L}), \Phi(\underline{L}'_{0r'})(\underline{L}'))$$

which intertwines the functors on the morphism level,

$$\Psi \circ \Phi(\underline{L}_{0r}) = \Phi(\underline{L}'_{0r'}) : \text{Hom}(\underline{L}, \underline{L}') \rightarrow \text{Hom}(\Phi(\underline{L}'_{0r'})(\underline{L}), \Phi(\underline{L}'_{0r'})(\underline{L}')).$$

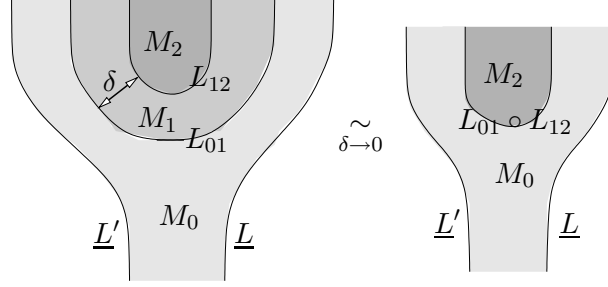
*Proof.* By assumption there exists a sequence of admissible generalized Lagrangian correspondences  $\underline{L}^j$  connecting  $\underline{L}^0 = \underline{L}_{0r}$  to  $\underline{L}^N = \underline{L}'_{0r'}$ . In each step two consecutive Lagrangian correspondences  $L_-, L_+$  in the sequence  $\underline{L}^j = (\dots, L_-, L_+, \dots)$  are replaced by their embedded composition  $L_- \circ L_+$  in  $\underline{L}^{j\pm 1} = (\dots, L_- \circ L_+, \dots)$ . To each  $\underline{L}^j$  we associate seam conditions for the quilted surface  $\underline{S}^j$  on the right of Figure 13. Replacing the consecutive correspondences by their composition corresponds to shrinking a strip in this surface. So Theorem 5.6 provides an isomorphism  $\Psi_{\underline{e}_+^j}$  associated to the outgoing end  $\underline{e}_+^j$  of each surface  $\underline{S}^j$  such that  $\Psi_{\underline{e}_+^j} \circ \Phi_{\underline{S}^j} = \Phi_{\underline{S}^{j\pm 1}}$ . Figure 14 shows an example of this degeneration. The isomorphism  $\Psi$  is given by concatenation of the isomorphisms  $\Psi_{\underline{e}_+^j}$  (and their inverses in case the composition is between  $\underline{L}^j$  and  $\underline{L}^{j-1}$ ). It intertwines  $\Phi_{\underline{S}^0} = \Phi(\underline{L}_{0r})$  and  $\Phi_{\underline{S}^N} = \Phi(\underline{L}'_{0r'})$  as claimed.  $\square$

Next, let  $\psi : M_0 \rightarrow M_1$  be a symplectomorphism and graph  $\psi \subset M_0^- \times M_1$  its graph. The functor  $\Phi(\psi)$  defined in Section 3.1 extends to a functor

$$\Phi(\psi) : \text{Don}^\#(M_0) \rightarrow \text{Don}^\#(M_1)$$

defined on the level of objects by

$$\underline{L} = (L_{-r(-r+1)}, \dots, L_{-10}) \mapsto (L_{-r(-r+1)}, \dots, (1_{N-1} \times \psi)(L_{-10})) =: \psi(\underline{L}).$$


 FIGURE 14. Isomorphism between the functors  $\Phi(L_{01}) \circ \Phi(L_{12})$  and  $\Phi(L_{01} \circ L_{12})$ 

On the level of morphisms, the functor  $\Phi(\psi) : \text{Hom}(\underline{L}, \underline{L}') \rightarrow \text{Hom}(\Phi(\psi)(\underline{L}), \Phi(\psi)(\underline{L}'))$  is defined by  $\langle (x_{-r}, \dots, x_{-1}, x_0, x'_{-1}, \dots, x'_{-r'}) \rangle \mapsto \langle (x_{-r}, \dots, x_{-1}, \psi(x_0), x'_{-1}, \dots, x'_{-r'}) \rangle$  on the generators  $\mathcal{I}(\underline{L}, \underline{L}')$  of the chain complex. As another application of Theorem 5.6 we will show that this functor is in fact isomorphic to the functor  $\Phi(\text{graph } \psi) : \text{Don}^\#(M_0) \rightarrow \text{Don}^\#(M_1)$  that we defined for the Lagrangian correspondence  $\text{graph } \psi$ .

**Proposition 5.8.**  *$\Phi(\psi)$  and  $\Phi(\text{graph } \psi)$  are canonically isomorphic as functors from  $\text{Don}^\#(M_0)$  to  $\text{Don}^\#(M_1)$ . More precisely, there exists a canonical natural transformation  $\alpha : \Phi(\psi) \rightarrow \Phi(\text{graph } \psi)$ , that is  $\alpha(\underline{L}) \in \text{Hom}(\Phi(\psi)(\underline{L}), \Phi(\text{graph } \psi)(\underline{L}))$  for every  $\underline{L} \in \text{Obj}(\text{Don}^\#(M_0))$  such that  $\alpha(\underline{L}) \circ \Phi(\text{graph } \psi)(f) = \Phi(\psi)(f) \circ \alpha(\underline{L}')$  for all  $f \in \text{Hom}(\underline{L}, \underline{L}')$ , and all  $\alpha(\underline{L})$  are isomorphisms in  $\text{Don}^\#(M_1)$ .*

*Proof.* Consider a generalized Lagrangian submanifold  $\underline{L} = (L_{(-r)(-r+1)}, \dots, L_{(-1)0}) \in \text{Obj}(\text{Don}^\#(M_0))$ . By Theorem 5.5 we have canonical isomorphisms from

$$\begin{aligned} \text{Hom}(\Phi(\psi)\underline{L}, \Phi(\text{graph } \psi)\underline{L}) &= \text{Hom}(\psi(\underline{L}), (\underline{L}, \text{graph } \psi)) \\ &= \text{Hom}(\dots (1 \times \psi)(L_{(-1)0}), (\text{graph } \psi)^t, (L_{(-1)0})^t \dots) \end{aligned}$$

to all three of

$$\begin{aligned} \text{Hom}(\dots (1 \times \psi)(L_{(-1)0}), (L_{(-1)0} \circ (\text{graph } \psi))^t \dots) &= \text{Hom}(\psi(\underline{L}), \psi(\underline{L})), \\ \text{Hom}(\dots L_{(-1)0}, \text{graph } \psi, (\text{graph } \psi)^t, (L_{(-1)0})^t \dots) &= \text{Hom}((\underline{L}, \text{graph } \psi)(\underline{L}, \text{graph } \psi)), \\ \text{Hom}(\dots (1 \times \psi)(L_{(-1)0}) \circ \text{graph }(\psi^{-1}), (L_{(-1)0})^t \dots) &= \text{Hom}(\underline{L}, \underline{L}), \end{aligned}$$

see Figure 15.<sup>4</sup> The isomorphisms are by  $(\psi(\underline{x}), \underline{x}) \mapsto \psi(\underline{x})$ ,  $(\underline{x}, \psi(x_0), \underline{x})$ , or  $\underline{x}$ , respectively, on the level of perturbed intersection points  $\underline{x} = (x_{-r}, \dots, x_0) \in \mathcal{I}(\underline{L}, \underline{L})$ . The first two isomorphisms also intertwine the identity morphisms  $1_{\psi(\underline{L})} \cong 1_{(\underline{L}, \text{graph } \psi)}$  by Theorem 5.5 and the degeneration of the quilted identity indicated in Figure 15; this is the identity axiom for the functor  $\Phi(\text{graph } \psi)$ . The identity axiom for  $\Phi(\psi)$  implies that the above isomorphisms (their composition which coincides with  $\Phi(\psi) : \text{Hom}(\underline{L}, \underline{L}) \rightarrow \text{Hom}(\psi(\underline{L}), \psi(\underline{L}))$ ) also intertwine  $1_{\underline{L}}$  with  $1_{\psi(\underline{L})}$ . We define  $\alpha(\underline{L}) \in \text{Hom}(\Phi(\psi)\underline{L}, \Phi(\text{graph } \psi)\underline{L})$  to be the element corresponding to the identities  $1_{\Phi(\psi)(\underline{L})} \cong 1_{\Phi(\text{graph } \psi)(\underline{L})} \cong 1_{\underline{L}}$  under these isomorphisms.

<sup>4</sup>Strictly speaking, one has to apply the shift functor  $\Psi_{M_0}$  of Definition 5.10 to adjust the relative spin structure on  $\underline{L}$ . However,  $HF(\Psi_{M_0}(\underline{L}), \Psi_{M_0}(\underline{L}))$  is canonically isomorphic to  $HF(\underline{L}, \underline{L})$ .

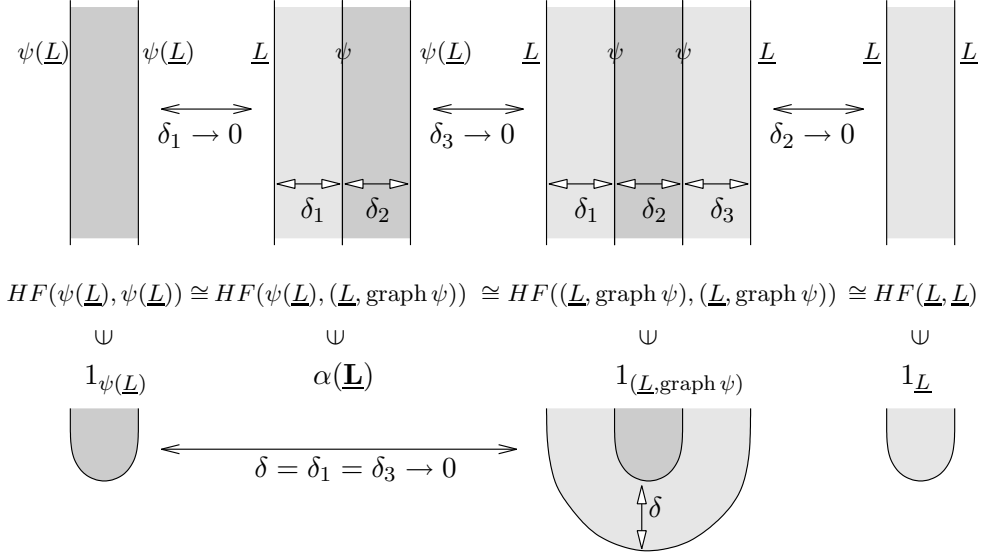


FIGURE 15. Natural isomorphisms of Floer cohomology groups and definition of the natural transformation  $\alpha$ : The light and dark shaded surfaces are mapped to  $M_0$  and  $M_1$  respectively and we abbreviate  $\text{graph } \psi$  by  $\psi$  and  $\Phi(\psi)(\underline{L})$  by  $\psi(\underline{L})$ .

Each  $\alpha(\underline{L})$  is an isomorphism since  $\alpha(\underline{L}) \circ f = I_1(f)$  for all  $f \in HF(\Phi(\text{graph } \psi)\underline{L}, \underline{L}'')$  and  $f \circ \alpha(\underline{L}) = I_2(f)$  for all  $f \in HF(\underline{L}'', \Phi(\psi)\underline{L})$ , with the isomorphisms from Theorem 5.5

$$I_1 : HF((\underline{L}, \text{graph } \psi), \underline{L}'') \rightarrow HF(\psi(\underline{L}), \underline{L}''),$$

$$I_2 : HF(\underline{L}'', \psi(\underline{L})) \rightarrow HF(\underline{L}'', (\underline{L}, \text{graph } \psi)).$$

These identities can be seen from the gluing theorem in [19] and Theorem 5.5, applied to the gluings and degenerations indicated in Figure 16. The quilted surfaces can be deformed to a strip resp. a quilted strip (which corresponds to a strip in  $M_0^- \times M_1$ ). These relative invariants both are the identity since the solutions are counted without quotienting by  $\mathbb{R}$ , see the strip example [19, Example 2.5]. For  $f \in \text{Hom}(\underline{L}, \underline{L}')$  this already shows the

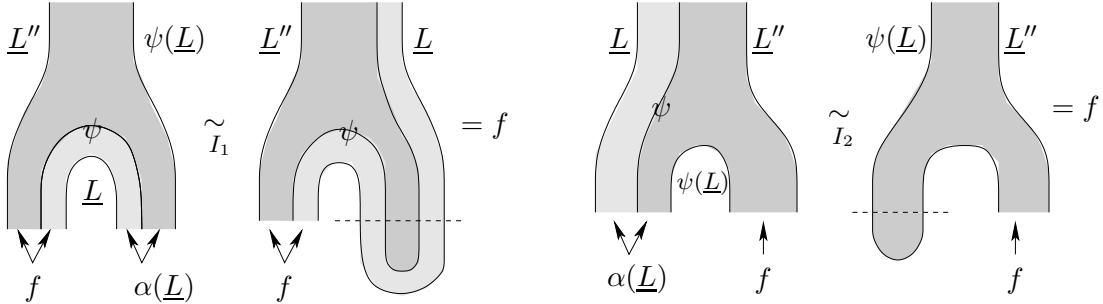


FIGURE 16.  $\alpha(\underline{L})$  is an isomorphism in  $\text{Don}^\#(M_1)$

first equality in  $\Phi(\psi)(f) \circ \alpha(\underline{L}') = I(f) = \alpha(\underline{L}) \circ \Phi(\text{graph } \psi)(f)$  with the isomorphism  $I : HF(\underline{L}, \underline{L}') \rightarrow HF(\psi(\underline{L}), (\underline{L}', \text{graph } \psi))$ . More precisely, on the chain level for  $x \in \mathcal{I}(\underline{L}, \underline{L}')$

$$\Phi(\psi)(x) \circ \alpha(\underline{L}') = (\psi(x), x) = \alpha(\underline{L}) \circ \Phi(\text{graph } \psi)(x).$$

The second identity is proven by repeatedly using Theorem 5.6 and the quilted gluing theorem [19, Theorem 3.13], see Figure 17.  $\square$

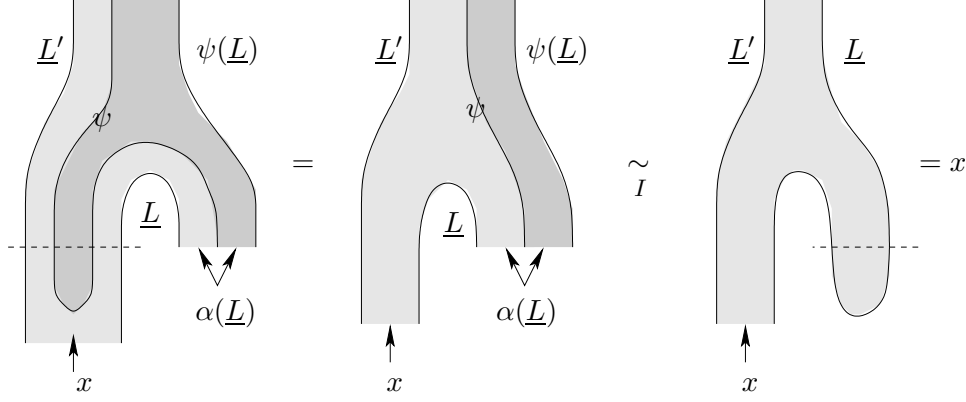


FIGURE 17. Isomorphism of functors for a symplectomorphism and its graph, using shrinking strips

*Remark 5.9.* There is an analytically easier proof of the previous Proposition 5.8 since it deals only with the special case when one of the Lagrangian correspondences is the graph of a symplectomorphism: Instead of shrinking a strip as in Theorems 5.5 and Theorem 5.6 one can apply the symplectomorphism to the whole strip; for a suitable choice of perturbation data it then attaches smoothly to the other surface in the quilt, and the seam can be removed.

The functor  $\Phi(\text{Id}_{M_0})$  associated to the identity map on  $M_0$  clearly is the identity functor on  $\text{Don}^\#(M_0)$ . So Proposition 5.8 gives a (rather indirect) isomorphism between the functor for the diagonal and the identity functor. To be more precise, taking into account the relative spin structure of the diagonal, we need to introduce the following shift functor.

**Definition 5.10.** We define a shift functor

$$\Psi_{M_0} : \text{Don}^\#(M_0, \text{Lag}^N(M_0), \omega_0, b_0) \rightarrow \text{Don}^\#(M_0, \text{Lag}^N(M_0), \omega_0, b_0 - w_2(M_0)).$$

- (a) On the level of objects,  $\Psi_{M_0}$  maps every generalized Lagrangian  $\underline{L} \in \text{Don}^\#(M_0)$  to itself but shifts the relative spin structure to one with background class  $b_0 - w_2(M_0)$ , as explained in [21].
- (b) On the level of morphisms,  $\Psi_{M_0} : \text{Hom}(\underline{L}, \underline{L}') \rightarrow \text{Hom}(\Psi_{M_0}(\underline{L}), \Psi_{M_0}(\underline{L}'))$  is the canonical isomorphism for shifted spin structures from [21].

*Remark 5.11.* Let  $\Delta \subset M_0^- \times M_0$  denote the diagonal. Throughout, we will equip  $\Delta$  with the orientation and relative spin structure that are induced by the projection to the second factor (see [21]). Then  $\Delta$  is an admissible Lagrangian correspondence from  $M_0$  to  $M_1$ , where  $M_1 = M_0$  with the same symplectic structure  $\omega_1 = \omega_0$  and Maslov cover  $\text{Lag}^N(M_1) = \text{Lag}^N(M_0)$ , but with a shifted background class  $b_1 = b_0 - w_2(M_0)$ . In other words,  $\Delta$  is an object in the category  $\text{Don}^\#(M_0, M_1)$  that is introduced in Section 6 below.

In the following, we will drop the Maslov cover and symplectic form from the notation.

**Corollary 5.12.** *The functor  $\Phi(\Delta) : \text{Don}^\#(M_0, b_0) \rightarrow \text{Don}^\#(M_0, b_0 - w_2(M_0))$  associated to the diagonal is canonically isomorphic to the shift functor  $\Psi_{M_0}$ .*

## 6. COMPOSITION FUNCTOR FOR CATEGORIES OF CORRESPONDENCES

The set of generalized Lagrangian correspondences forms a category in its own right, which we define in close analogy to the generalized Donaldson category in Section 4. We will then be able to define a composition functor for these categories.

Let  $M_a$  and  $M_b$  be symplectic manifolds satisfying (M1-2) with the same monotonicity constant  $\tau \geq 0$ . We fix an integer  $N > 0$ ,  $N$ -fold Maslov covers  $\text{Lag}^N(M_{(\cdot)}) \rightarrow M_{(\cdot)}$  as in (G1), and background classes  $b_{(\cdot)} \in H^2(M_{(\cdot)}, \mathbb{Z}_2)$ . Recall from Definition 2.1 that a generalized Lagrangian correspondence from  $M_a$  to  $M_b$  is a sequence  $\underline{L} = (L_{01}, L_{12}, \dots, L_{(r-1)r})$  of Lagrangian correspondences  $L_{(i-1)i} \subset N_{i-1}^- \times N_i$  for a sequence  $N_0, \dots, N_r$  of any length  $r \geq 0$  of symplectic manifolds with  $N_0 = M_a$  and  $N_r = M_b$ . We picture  $\underline{L}$  as sequence

$$M_a = N_0 \xrightarrow{L_{01}} N_1 \xrightarrow{L_{12}} \dots \xrightarrow{L_{(r-1)r}} N_r = M_b .$$

As in Definition 4.1 we call a generalized Lagrangian correspondence  $\underline{L}$  from  $M_a$  to  $M_b$  *admissible* if each  $N_i$  satisfies (M1-2) with the monotonicity constant  $\tau \geq 0$ , each  $L_{(i-1)i}$  satisfies (L1-3), and the image of each  $\pi_1(L_{(i-1)i})$  in  $\pi_1(N_{i-1}^- \times N_i)$  is torsion.

**Definition 6.1.** The *Donaldson-Fukaya category of correspondences*

$$\text{Don}^\#(M_a, M_b) := \text{Don}^\#(M_a, M_b, \text{Lag}^N(M_a), \text{Lag}^N(M_b), \omega_a, \omega_b, b_a, b_b)$$

is defined as follows:

- (a) The objects of  $\text{Don}^\#(M_a, M_b)$  are admissible generalized Lagrangian correspondences from  $M_a$  to  $M_b$ , equipped with orientations, gradings, and relative spin structures.<sup>5</sup>
- (b) The morphism spaces of  $\text{Don}^\#(M_a, M_b)$  are the  $\mathbb{Z}_N$ -graded Floer cohomology groups (defined in [20])

$$\text{Hom}(\underline{L}, \underline{L}') := HF(\underline{L}, \underline{L}')[d],$$

where the second group is shifted by  $d = \frac{1}{2}(\sum_k \dim(N_k) + \sum_{k'} \dim(N_{k'}))$ . For  $\mathbb{Z}$ -coefficients one has to introduce determinant lines as in Remark 4.4. See Figure 18 for views of the quilted holomorphic cylinders which are counted (modulo  $\mathbb{R}$ -shift) as Floer trajectories.

- (c) The composition of morphisms in  $\text{Don}^\#(M_a, M_b)$ ,

$$\begin{aligned} \text{Hom}(\underline{L}, \underline{L}') \times \text{Hom}(\underline{L}', \underline{L}'') &\longrightarrow \text{Hom}(\underline{L}, \underline{L}'') \\ (f, g) &\longmapsto f \circ g := \Phi_{\underline{L}}(f \otimes g) \end{aligned}$$

is defined by the relative invariant  $\Phi_{\underline{L}}$  associated to the quilted pair of pants surface  $\underline{P}$  (this time the pair of pants is an honest one, not just the front) in Figure 19, where the patches without outgoing ends are ordered from  $M_a$  to  $M_b$ .

*Convention 6.2.* In Figure 18 and the following pictures, the outer circles will always be outgoing ends. The inner circles are usually incoming ends, indicated by a  $\otimes$  or marked with

<sup>5</sup>In the previous notation, a grading on  $\underline{L}$  is a collection of  $N$ -fold Maslov covers  $\text{Lag}^N(N_j) \rightarrow N_j$  for  $j = 0, \dots, r$  and gradings of the Lagrangian correspondences  $L_{(j-1)j}$ . Here the gradings on  $N_0 = M_a$  and  $N_r = M_b$  are the fixed ones. A relative spin structure on  $\underline{L}$  is a collection of background classes  $b_j \in H^2(N_j, \mathbb{Z}_2)$  for  $j = 0, \dots, r$  and relative spin structures on  $L_{(j-1)j}$  with background classes  $-\pi_{j-1}^* b_{j-1} + \pi_j^* b_j$ . Here  $b_0 = b_a$  and  $b_r = b_b$  are the fixed background classes in  $M_a$  and  $M_b$ . See [20] for more details.

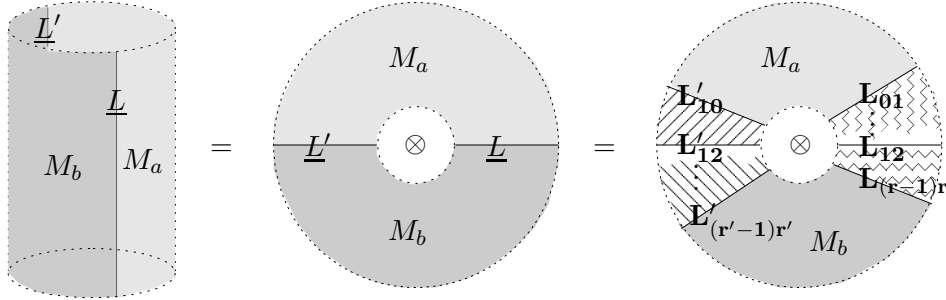


FIGURE 18. Floer trajectories for pairs of generalized Lagrangian correspondences

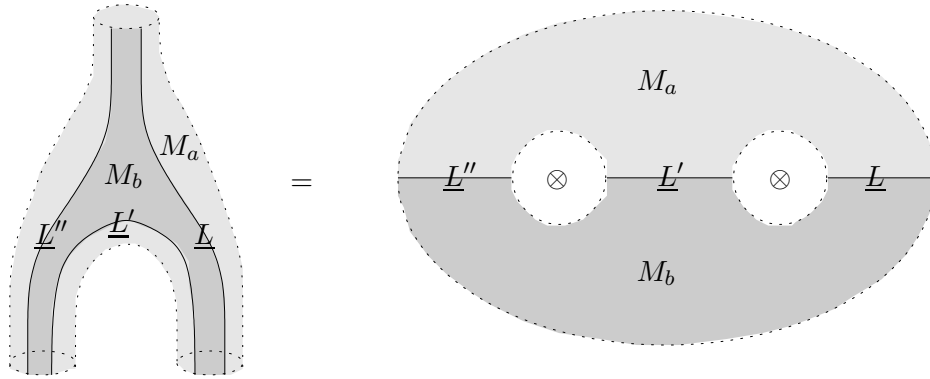


FIGURE 19. Quilted pair of pants: Composition of morphisms for Lagrangian correspondences

the incoming morphism. Ends at the top resp. bottom of pictures will always be outgoing resp. incoming, unless otherwise indicated by arrows.

- Remark 6.3.*
- (a) The identity  $1_{\underline{L}} \in \text{Hom}(\underline{L}, \underline{L})$  for a generalized Lagrangian correspondence  $\underline{L}$  is given by the relative invariant  $1_{\underline{L}} := \Phi_{\underline{L}}$  associated to the quilted cap in Figure 20, where the patches without outgoing ends are ordered from  $M_b$  to  $M_a$ .
  - (b) The associativity and identity axiom for  $\text{Don}^{\#}(M_a, M_b)$  follow from the quilted gluing theorem [19, Theorem 3.13] applied to the gluings (indicated by dashed lines) in Figure 21. Note that – in contrast to Figure 18 – the solutions on the quilted annulus (i.e. cylinder) are counted without quotienting by  $\mathbb{R}$ , hence as in the strip example [19, Example 2.5] this relative invariant is the identity.
  - (c)  $\text{Don}^{\#}(M_a, M_b)$  is a small category by the same arguments as in Remark 2.5 (c).

*Remark 6.4.* Consider the case where the symplectic manifolds  $M_a = M_b = M$  agree (including Maslov cover and background class). Then for any admissible generalized Lagrangian correspondence  $\underline{L} \in \text{Obj}(\text{Don}^{\#}(M, M))$  the composition of morphisms in (c) defines a ring structure on  $\text{Hom}(\underline{L}, \underline{L})$ , and (d) provides an identity element. Another application of the strip shrinking theorems shows that this ring structure is isomorphic under

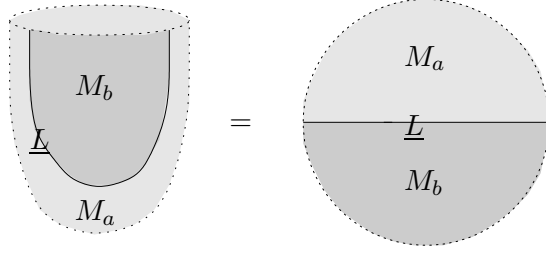


FIGURE 20. Quilted cap: Identity for Lagrangian correspondences

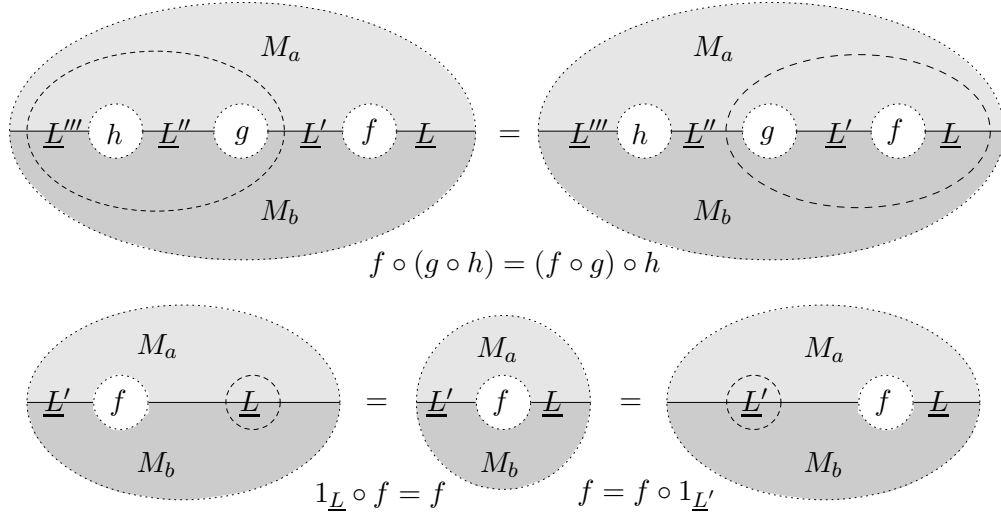


FIGURE 21. Axioms for Donaldson-Fukaya category of correspondences

embedded compositions of correspondences: Let  $\underline{L}$  and  $\underline{L}'$  be two admissible generalized Lagrangian correspondences from  $M$  to itself. Suppose that they are equivalent in the sense of Section 2 through a series of embedded compositions of consecutive Lagrangian correspondences, and such that each intermediate generalized Lagrangian correspondence is admissible. Then there is a canonical ring isomorphism  $(\text{Hom}(\underline{L}, \underline{L}), \circ) \simeq (\text{Hom}(\underline{L}', \underline{L}'), \circ)$  which intertwines the identity elements  $1_{\underline{L}}$  and  $1_{\underline{L}'}$ .

Indeed, by assumption there exists a sequence of admissible generalized Lagrangian correspondences  $\underline{L}^j$  connecting  $\underline{L}^0 = \underline{L}$  to  $\underline{L}^N = \underline{L}'$  as in the proof of Theorem 5.7. In each step two consecutive Lagrangian correspondences in the sequence  $\underline{L}^j = (\dots, L_-, L_+, \dots)$  are replaced by their embedded, monotone composition in  $\underline{L}^{j+1} = (\dots, L_- \circ L_+, \dots)$ . Theorem 5.5 provides isomorphisms  $\Psi^j : HF(\underline{L}^j, \underline{L}^j) \rightarrow HF(\underline{L}^{j+1}, \underline{L}^{j+1})$  by shrinking the strip between  $L_-$  and  $L_+$ . Theorem 5.6 applies to the corresponding strips in the pair of pants surface and the quilted cap surface of Definition 6.1 (c) and (d) and shows that the isomorphisms  $\Psi^j$  intertwine the ring structures and identity morphisms. The full ring isomorphism is given by a composition of these isomorphisms or their inverses.

Next, consider a triple of symplectic manifolds  $M_a, M_b, M_c$  satisfying (M1-2) with the same monotonicity constant  $\tau$ , equipped with Maslov covers  $\text{Lag}^N(M_{(\cdot)}) \rightarrow M_{(\cdot)}$  (with the same  $N$ ) and background classes  $b_{(\cdot)} \in H^2(M_{(\cdot)}, \mathbb{Z}_2)$ . We denote by  $\text{Don}^\#(M_a, M_b) \times$



$\text{Don}^\#(M_b, M_c)$  the product category. That is, objects are pairs  $(\underline{L}_{ab}, \underline{L}_{bc})$  of objects of  $\text{Don}^\#(M_a, M_b)$  and  $\text{Don}^\#(M_b, M_c)$ . Morphisms are pairs  $(f, g)$  with  $f \in \text{Hom}(\underline{L}_{ab}, \underline{L}'_{ab}), g \in \text{Hom}(\underline{L}_{bc}, \underline{L}'_{bc})$ . Composition is given by

$$(f, g) \circ (f', g') := (-1)^{|f'| |g|} (f \circ f', g \circ g')$$

for  $f \in \text{Hom}(\underline{L}_{ab}, \underline{L}'_{ab}), f' \in \text{Hom}(\underline{L}'_{ab}, \underline{L}''_{ab}), g \in \text{Hom}(\underline{L}_{bc}, \underline{L}'_{bc}), g' \in \text{Hom}(\underline{L}'_{bc}, \underline{L}''_{bc})$ .

**Definition 6.5.** The *composition functor*

$$(9) \quad \# : \text{Don}^\#(M_a, M_b) \times \text{Don}^\#(M_b, M_c) \rightarrow \text{Don}^\#(M_a, M_c)$$

is defined as follows.

(a) On the level of objects  $\#$  is defined by concatenation:

$$\begin{aligned} \text{Obj}(\text{Don}^\#(M_a, M_b)) \times \text{Obj}(\text{Don}^\#(M_b, M_c)) &\rightarrow \text{Obj}(\text{Don}^\#(M_a, M_c)) \\ (\underline{L}_{ab}, \underline{L}_{bc}) &\mapsto \underline{L}_{ab} \# \underline{L}_{bc}, \end{aligned}$$

where

$$(L_{01}^{ab}, \dots, L_{(r-1)r}^{ab}) \# (L_{01}^{bc}, \dots, L_{(r'-1)r'}^{bc}) := (L_{01}^{ab}, \dots, L_{(r-1)r}^{ab}, L_{01}^{bc}, \dots, L_{(r'-1)r'}^{bc}).$$

(b) On the level of morphisms,  $\#$  is defined for  $\underline{L}_{ab}, \underline{L}'_{ab} \in \text{Obj}(\text{Don}^\#(M_a, M_b))$  and  $\underline{L}_{bc}, \underline{L}'_{bc} \in \text{Obj}(\text{Don}^\#(M_b, M_c))$  by

$$\begin{aligned} \text{Hom}(\underline{L}_{ab}, \underline{L}'_{ab}) \times \text{Hom}(\underline{L}_{bc}, \underline{L}'_{bc}) &\rightarrow \text{Hom}(\underline{L}_{ab} \# \underline{L}_{bc}, \underline{L}'_{ab} \# \underline{L}'_{bc}) \\ (f, g) &\mapsto f \# g := \Phi_{\underline{P}}(f \otimes g), \end{aligned}$$

where  $\Phi_{\underline{P}}$  is the relative invariant associated to the quilted pair of pants  $\underline{P}$ , where now every seam connects one of the incoming cylindrical ends to the outgoing cylindrical end, as in Figure 22.

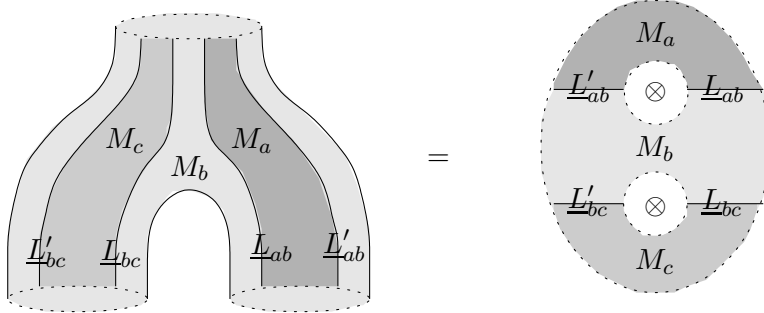


FIGURE 22. Composition functor on Donaldson categories of correspondences

The composition axiom for the functor  $\#$  follows from the quilted gluing theorem [19, Theorem 3.13] applied to the two degenerations of the five-holed sphere shown in Figure 23: For all  $f \in \text{Hom}(\underline{L}_{ab}, \underline{L}'_{ab}), f' \in \text{Hom}(\underline{L}'_{ab}, \underline{L}''_{ab}), g \in \text{Hom}(\underline{L}_{bc}, \underline{L}'_{bc}), g' \in \text{Hom}(\underline{L}'_{bc}, \underline{L}''_{bc})$  we obtain

$$\#((f, g) \circ (f', g')) = (-1)^{|f'| |g|} (f \circ f') \# (g \circ g') = (f \# g) \circ (f' \# g').$$

The identity axiom for the concatenation functor,  $1_{\underline{L}_{ab}} \# 1_{\underline{L}_{bc}} = 1_{\underline{L}_{ab} \# \underline{L}_{bc}}$ , follows similarly from the quilted gluing theorem applied to the degenerations shown in Figure 24.

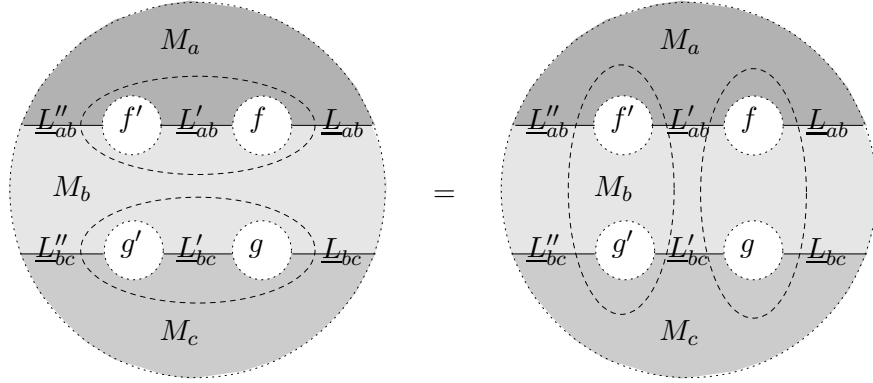


FIGURE 23. Composition axiom for the concatenation functor

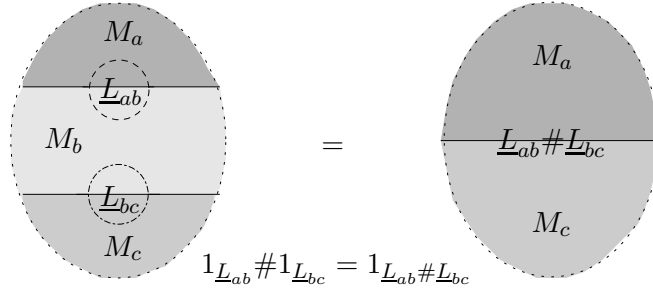


FIGURE 24. Identity axiom for the concatenation functor

*Remark 6.6.* The construction of functors associated to Lagrangian correspondences in Section 5 has an obvious extension (8) for generalized Lagrangian correspondences. For  $\underline{L}_{ab} \in \text{Don}^\#(M_a, M_b)$  the functor  $\Phi(\underline{L}_{ab}) : \text{Don}^\#(M_a) \rightarrow \text{Don}^\#(M_b)$  acts on objects  $\underline{L} \in \text{Obj}(\text{Don}^\#(M_a))$  by concatenation  $\Phi(\underline{L}_{ab}) = \underline{L} \# \underline{L}_{ab}$ , and on morphisms  $\Phi(\underline{L}_{ab}) : HF(\underline{L}, \underline{L}') \rightarrow HF(\underline{L} \# \underline{L}_{ab}, \underline{L}' \# \underline{L}_{ab})$  is defined by composition  $\Phi(L_{01}) \circ \dots \circ \Phi(L_{(r-1)r})$  of the functors associated to the elementary Lagrangian correspondences  $(L_{01}, \dots, L_{(r-1)r}) = \underline{L}_{ab}$ . Alternatively, the map  $\Phi(\underline{L}_{ab})$  on morphisms can be defined directly by the relative invariant in Figure 13, see Remark 5.4. Using the first definition, we have a tautological equality of functors

$$(10) \quad \Phi(\underline{L}_{ab}) \circ \Phi(\underline{L}_{bc}) = \Phi(\underline{L}_{ab} \# \underline{L}_{bc})$$

for any two objects  $\underline{L}_{ab} \in \text{Don}^\#(M_a, M_b)$  and  $\underline{L}_{bc} \in \text{Don}^\#(M_b, M_c)$ .

## 7. NATURAL TRANSFORMATIONS ASSOCIATED TO FLOER COHOMOLOGY CLASSES

Let  $M_a$  and  $M_b$  be as in the previous section and let  $\underline{L}_{ab}, \underline{L}'_{ab}$  be objects in  $\text{Don}^\#(M_a, M_b)$ .

**Definition 7.1.** Given a morphism  $T \in \text{Hom}(\underline{L}_{ab}, \underline{L}'_{ab})$  we define a natural transformation

$$\Phi_T : \Phi(\underline{L}_{ab}) \rightarrow \Phi(\underline{L}'_{ab})$$

as follows: To any object  $\underline{L}$  in  $\text{Don}^\#(M_a)$  we assign the morphism

$$\Phi_T(\underline{L}) \in \text{Hom}(\Phi(\underline{L}_{ab})(\underline{L}), \Phi(\underline{L}'_{ab})(\underline{L}))$$

given by the relative invariant associated to the surface in Figure 25, which is independent of the ordering of the patches. (Note that the end where  $T$  is inserted is cylindrical in the sense that the strip-like ends glue together to a cylindrical end.)

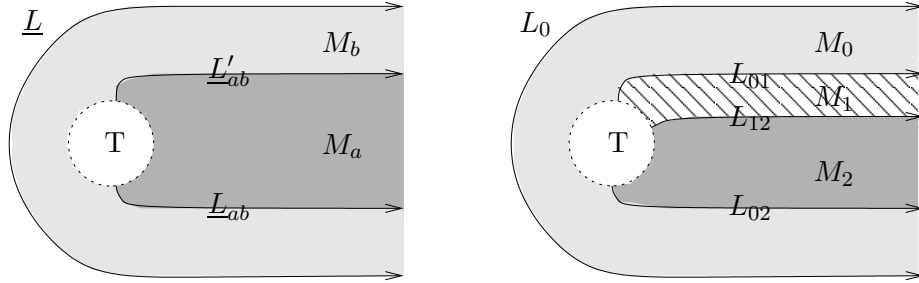


FIGURE 25. Natural transformation associated to a Floer cohomology class: General case and an example, where  $\underline{L}$  consists of a single Lagrangian  $L_0$ ,  $\underline{L}_{ab}$  consists of a single Lagrangian  $L_{02}$ , and  $\underline{L}'_{ab}$  consists of a pair  $(L_{01}, L_{12})$ .

To see that  $\Phi_T$  is a natural transformation of functors  $\Phi(\underline{L}_{ab}) \rightarrow \Phi(\underline{L}'_{ab})$  we must show that for any two objects  $\underline{L}, \underline{L}'$  in  $\text{Don}^\#(M_a)$  and any morphism  $f \in \text{Hom}(\underline{L}, \underline{L}')$  we have

$$(11) \quad \Phi(\underline{L}_{ab})(f) \circ \Phi_T(\underline{L}') = (-1)^{|T||f|} \Phi_T(\underline{L}) \circ \Phi(\underline{L}'_{ab})(f).$$

This identity follows from the quilted gluing theorem [19, Theorem 3.13] applied to the gluing shown in Figure 26.

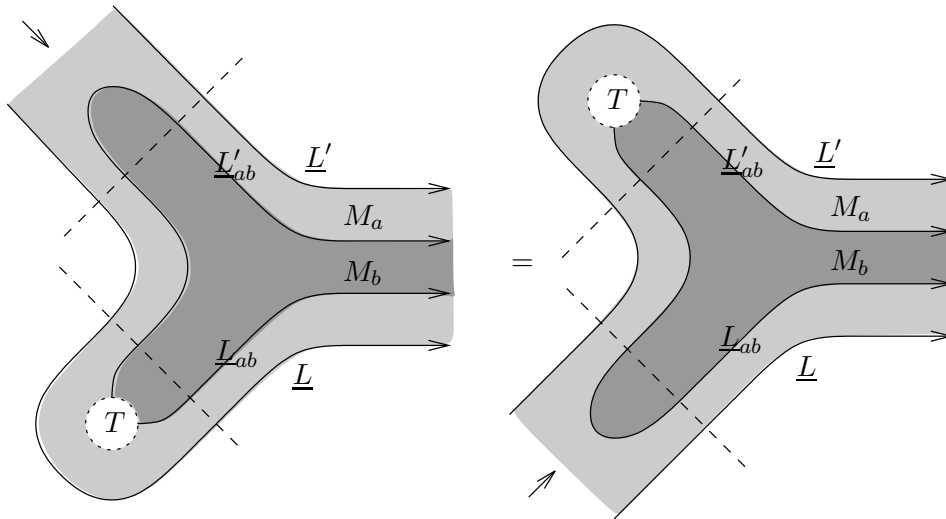


FIGURE 26. Natural transformation axiom

**Proposition 7.2.** *The maps  $\underline{L}_{ab} \mapsto \Phi(\underline{L}_{ab})$  and  $T \mapsto \Phi_T$  define a functor*

$$\text{Don}^\#(M_a, M_b) \rightarrow \text{Fun}(\text{Don}^\#(M_a), \text{Don}^\#(M_b)).$$

*Proof.* We apply the quilted gluing theorem [19, Theorem 3.13] to the quilted surfaces in Figure 27 to deduce the composition axiom  $\Phi_T(\underline{L}) \circ \Phi_{T'}(\underline{L}) = \Phi_{T \circ T'}(\underline{L})$  for all  $T \in \text{Hom}(\underline{L}_{ab}, \underline{L}'_{ab})$ ,  $T' \in \text{Hom}(\underline{L}'_{ab}, \underline{L}''_{ab})$ , and  $\underline{L} \in \text{Obj}(\text{Don}^\#(M_a))$ . The identity axiom  $\Phi_{1_{\underline{L}_{ab}}}(\underline{L}) = 1_{\Phi(\underline{L}_{ab})(\underline{L})}$  for  $T = 1_{\underline{L}_{ab}} \in \text{Hom}(\underline{L}_{ab}, \underline{L}_{ab})$  and  $\underline{L} \in \text{Obj}(\text{Don}^\#(M_a))$  follows from the quilted gluing theorem applied to the quilted surface in Figure 28.  $\square$

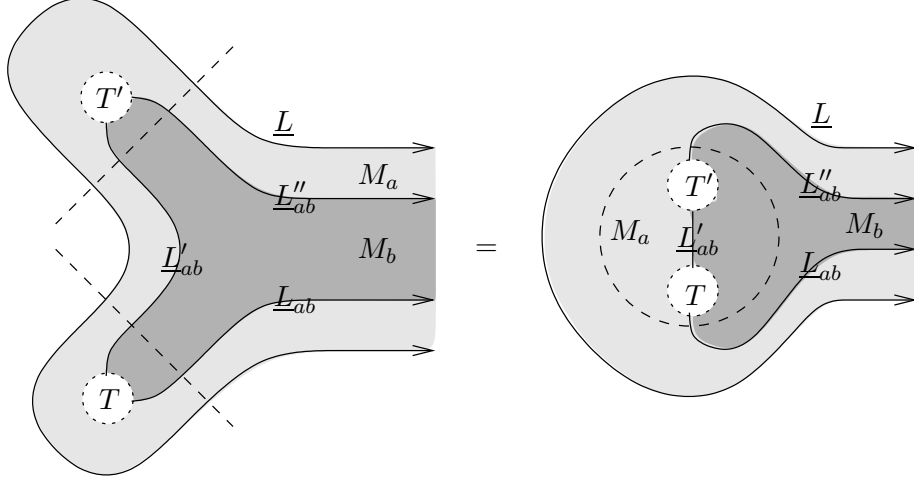


FIGURE 27. Composition axiom for natural transformations

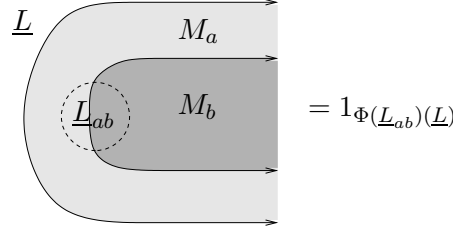


FIGURE 28. Identity axiom for natural transformations

*Remark 7.3.* In this remark we discuss the special case of the diagonal  $\Delta \subset M^- \times M$ , which gives rise to the so-called open-closed maps in 2D TQFT. By [14] there is a ring isomorphism between the Floer cohomology of the diagonal  $HF(\Delta, \Delta)$  and the quantum cohomology  $HF(\text{Id})$ . Our construction gives for any element  $\alpha \in HF(\Delta, \Delta) \simeq HF(\text{Id})$  an automorphism of the identity functor  $\Phi(\Delta)$  (more precisely, of the shift functor  $\Phi(\Delta) \simeq \Psi_M$  in case  $w_2(M) \neq 0$ ). In particular, we obtain elements  $\Phi_\alpha(L) \in HF((L, \Delta), (L, \Delta)) \simeq HF(L, L)$  for each admissible Lagrangian submanifold  $L \subset M$ . (Here  $HF((L, \Delta), (L, \Delta)) \simeq HF(L, L)$  is a ring isomorphism by Remark 6.4.) Proposition 7.2 gives  $\Phi_{\alpha \circ \beta}(L) = \Phi_\alpha(L) \circ \Phi_\beta(L)$ . That is, the closed-open map  $HF(\text{Id}) \rightarrow HF(L, L)$  is a ring homomorphism. The closed-open maps in Floer theory are discussed in more detail in Albers [2, Theorem 3.1].

For any pair of Lagrangians  $L^0, L^1 \subset M$ , combining the ring homomorphism  $HF(\text{Id}) \rightarrow HF(L^k, L^k)$  with the composition  $HF(L^0, L^0) \times HF(L^0, L^1) \rightarrow HF(L^0, L^1)$  resp.  $HF(L^0, L^1) \times HF(L^1, L^1) \rightarrow HF(L^0, L^1)$  gives a module structure on  $HF(L^0, L^1)$  over  $HF(\text{Id})$ . The

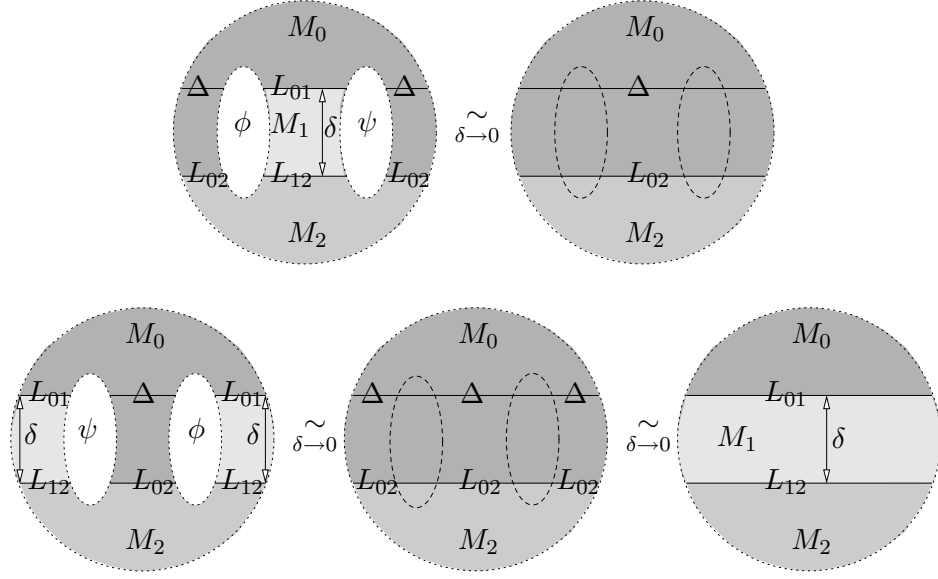


FIGURE 29. Isomorphism of composition and concatenation

module structure is independent of  $k = 0, 1$ , by the natural transformation axiom (11) with  $\underline{L}_{ab} = \underline{L}'_{ab} = \Delta$ . It is equal to the module structure induced by the isomorphism  $HF(L^0 \times L^1, \Delta) \rightarrow HF(L^0, L^1)$  of [21].

Note that if  $HF(\text{Id}) \rightarrow HF(L, L)$  is a surjection and  $HF(\text{Id})$  is semisimple then  $HF(L, L)$  is again semisimple, and in particular nilpotent free.

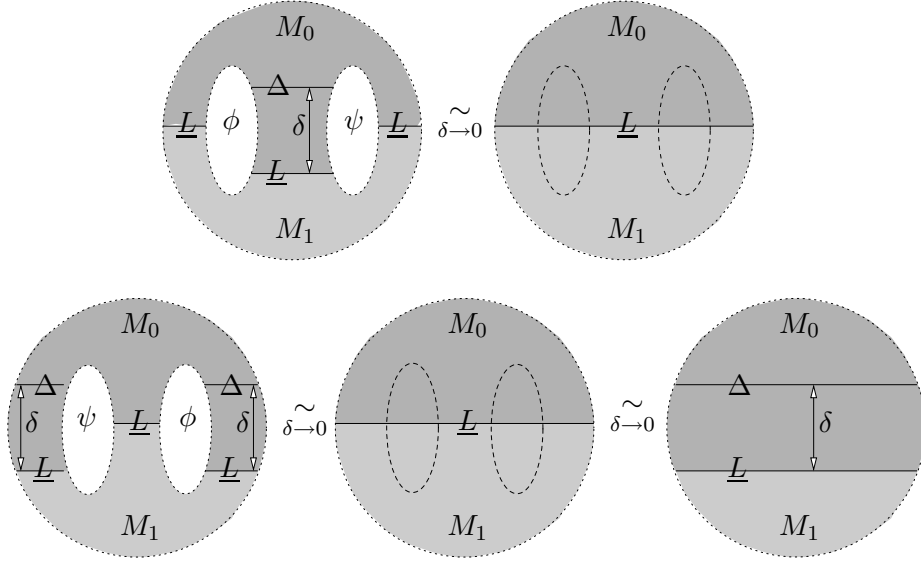
Next, we show that embedded composition of Lagrangian correspondences gives rise to isomorphic objects in the Donaldson-Fukaya category. For simplicity we restrict to the case of elementary Lagrangian correspondences, i.e. sequences of length 1. The statement and argument for the general case is analogous.

**Theorem 7.4.** *Let  $L_{01} \in \text{Obj}(\text{Don}^\#(M_0, M_1))$  and  $L_{12} \in \text{Obj}(\text{Don}^\#(M_1, M_2))$  be admissible Lagrangian correspondences. Suppose that  $L_{01} \times_{M_1} L_{12} \rightarrow M_0^- \times M_2$  is cut out transversally and embeds to a smooth, admissible Lagrangian correspondence  $L_{02} := L_{01} \circ L_{12} \in \text{Obj}(\text{Don}^\#(M_0, M_2))$ . Then  $\Delta_{M_0} \# L_{02}$ ,  $L_{02} \# \Delta_{M_2}$ , and  $L_{01} \# L_{12}$  are all isomorphic in  $\text{Don}^\#(M_0, M_2)$ .*

*Remark 7.5.* If in Theorem 7.4 we moreover assume  $w_2(M_0) = 0$  or  $w_2(M_2) = 0$ , then we in fact have an isomorphism between  $L_{01} \# L_{12}$  and  $L_{01} \circ L_{12}$ , by Proposition 7.6 below.

*Proof.* By Theorem 5.5,  $\text{Hom}(L_{01} \# L_{12}, \Delta_{M_0} \# L_{02})$  resp.  $\text{Hom}(\Delta_{M_0} \# L_{02}, L_{01} \# L_{12})$  is isomorphic to  $\text{Hom}(\Delta_{M_0} \# L_{02}, \Delta_{M_0} \# L_{02})$ ; let  $\phi$  resp.  $\psi$  denote the inverse image of the identity  $1_{\Delta_{M_0} \# L_{02}}$ . To establish the isomorphism  $L_{01} \# L_{12} \simeq \Delta_{M_0} \# L_{02}$  we show that  $\psi \circ \phi = 1_{\Delta_{M_0} \# L_{02}}$  and  $\phi \circ \psi = 1_{L_{01} \# L_{12}}$  for the composition by the pair of pants products. These are special cases of Theorem 5.6 applied to the degenerations shown in Figure 29. The isomorphism  $L_{01} \# L_{12} \simeq L_{02} \# \Delta_{M_2}$  is proven in the same way.  $\square$

**Proposition 7.6.** *Suppose that  $M_0$  satisfies  $w_2(M_0) = 0$ . Then the diagonal  $\Delta_{M_0} \in \text{Don}^\#(M_0, M_0)$  is an identity of the composition  $\#$  up to isomorphism. That is, for every*

FIGURE 30. Isomorphism of  $\Delta_{M_0} \# \underline{L}$  and  $\underline{L}$ 

generalized Lagrangian  $\underline{L} \in \text{Obj}(\text{Don}^\#(M_0, M_1))$  the objects  $\Delta_{M_0} \# \underline{L}$  and  $\underline{L}$  are isomorphic in  $\text{Don}^\#(M_0, M_1)$ , and for every generalized Lagrangian  $\underline{L} \in \text{Obj}(\text{Don}^\#(M_1, M_0))$  the objects  $\underline{L} \# \Delta_{M_0}$  and  $\underline{L}$  are isomorphic in  $\text{Don}^\#(M_1, M_0)$ .

*Proof.* By Theorem 5.5, both  $\text{Hom}(\Delta_{M_0} \# \underline{L}, \underline{L})$  and  $\text{Hom}(\underline{L}, \Delta_{M_0} \# \underline{L})$  are isomorphic to  $\text{Hom}(\underline{L}, \underline{L})$ ; let  $\phi$  resp.  $\psi$  denote the inverse image of the identity  $1_{\underline{L}}$ . Then the identities  $\phi \circ \psi = 1_{\underline{L}}$  and  $\phi \circ \psi = 1_{\Delta_{M_0} \# \underline{L}}$  follow from Theorem 5.6 applied to the degenerations shown in Figure 30. (Alternatively, as mentioned in Section 5.8, one could glue the strips instead of shrinking them.) This proves  $\Delta_{M_0} \# \underline{L} \simeq \underline{L}$ . The isomorphism  $\underline{L} \# \Delta_{M_0} \simeq \underline{L}$  is proven in the same way.  $\square$

**Corollary 7.7.** *Under the assumptions of Theorem 7.4 the functors  $\Psi_{M_0} \circ \Phi(L_{01} \circ L_{12})$ ,  $\Phi(L_{01} \circ L_{12}) \circ \Psi_{M_2}$ , and  $\Phi(L_{01}) \circ \Phi(L_{12})$  are all isomorphic in the category of functors from  $\text{Don}^\#(M_0)$  to  $\text{Don}^\#(M_2)$ .*

*Proof.* From Theorem 7.4 and (10) we obtain isomorphisms between  $\Phi(\Delta_{M_0} \# L_{02}) = \Phi(\Delta_{M_0}) \circ \Phi(L_{02})$ ,  $\Phi(L_{02} \# \Delta_{M_2}) = \Phi(L_{02}) \circ \Phi(\Delta_{M_2})$ , and  $\Phi(L_{01} \# L_{12}) = \Phi(L_{01}) \circ \Phi(L_{12})$ . By Proposition 5.12 the functors  $\Phi(\Delta_{M_k})$  are isomorphic to the shift functors  $\Psi_{M_k}$ . Since isomorphisms commute with composition of functors, this proves the corollary.  $\square$

## 8. 2-CATEGORY OF MONOTONE SYMPLECTIC MANIFOLDS

We can rephrase and summarize the constructions of the previous sections, using the language of 2-categories.

**Definition 8.1.** A 2-category  $\mathcal{C}$  consists of the following data:

- (a) A class of objects  $\text{Obj}(\mathcal{C})$ .
- (b) For each pair of objects  $X, Y \in \text{Obj}(\mathcal{C})$ , a small category  $\text{Hom}(X, Y)$ .
- (c) For each triple of objects  $X, Y, Z \in \text{Obj}(\mathcal{C})$ , a composition functor

$$\circ : \text{Hom}(X, Y) \times \text{Hom}(Y, Z) \rightarrow \text{Hom}(X, Z).$$

(d) For every  $X \in \text{Obj}(\mathcal{C})$  an identity functor  $1_X \in \text{Hom}(X, X)$ .

These data should satisfy the following axioms:

(Identity): For all  $X, Y \in \text{Obj}(\mathcal{C})$  and  $f \in \text{Hom}(X, Y)$

$$1_X \circ f = f, \quad f \circ 1_Y = f.$$

(Associativity): For all composable morphisms  $f, g, h$

$$f \circ (g \circ h) = (f \circ g) \circ h.$$

Objects resp. morphisms in  $\text{Hom}(X, Y)$  are called 1-morphisms resp. 2-morphisms. We say that  $\mathcal{C}$  has *weak identities* if equality in the identity axiom is replaced by 2-isomorphism.

The basic example of a 2-category is  $\text{Cat}$ , whose objects are small categories, 1-morphisms are functors, and 2-morphisms are natural transformations.

**Definition 8.2.** A 2-functor  $\mathcal{F} : \mathcal{C}_1 \rightarrow \mathcal{C}_2$  between 2-categories  $\mathcal{C}_1$  and  $\mathcal{C}_2$  consists of

- (a) a map  $\mathcal{F} : \text{Obj}(\mathcal{C}_1) \rightarrow \text{Obj}(\mathcal{C}_2)$ ,
- (b) for each pair  $X, Y \in \text{Obj}(\mathcal{C}_1)$ , a functor

$$\mathcal{F}(X, Y) : \text{Hom}(X, Y) \rightarrow \text{Hom}(\mathcal{F}(X), \mathcal{F}(Y)),$$

respecting composition and identities.

In the following we restrict ourselves to symplectic manifolds that are spin, i.e.  $w_2(M) = 0$ . Their advantage is that the shift functor  $\Psi_M : \text{Don}^\#(M, b) \rightarrow \text{Don}^\#(M, b)$  of Definition 5.10 is trivial and the diagonal  $\Delta_M \subset M^- \times M$  is an object of the category of correspondences  $\text{Don}^\#(M, M)$  from  $(M, b)$  to itself. We moreover drop the Maslov cover from the data, thus working with ungraded Floer cohomology groups.

**Definition 8.3.** Fix a constant  $\tau \geq 0$ . Let the *Weinstein-Floer 2-category*  $\text{Floer}_\tau^\#$  be the category given as follows:

- (a) Objects are symplectic manifolds  $(M, \omega)$  that satisfy (M1-2) with monotonicity constant  $\tau$  and  $w_2(M) = 0$ , and that are equipped with a background class  $b \in H^2(M, \mathbb{Z}_2)$ .
- (b) The morphism categories of  $\text{Floer}_\tau^\#$  are the Donaldson categories of Lagrangian correspondences,  $\text{Hom}(M_0, M_1) := \text{Don}^\#(M_0, M_1)$ ; without grading.
- (c) Composition is defined by the functor (9),

$$\# : \text{Don}^\#(M_0, M_1) \times \text{Don}^\#(M_1, M_2) \rightarrow \text{Don}^\#(M_0, M_2).$$

- (d) The diagonal defines a weak identity  $\Delta_M \in \text{Don}^\#(M, M)$ .

*Remark 8.4.* One could define  $\text{Floer}_\tau^\#$  by restricting to nonempty symplectic manifolds. However, for future applications, we wish to include the empty set  $\emptyset$  as object. The only elementary Lagrangian correspondence from  $\emptyset$  to  $M$  is  $L = \emptyset$ , but in the sequence of a generalized Lagrangian correspondences, we must now allow any number of  $\emptyset$  as symplectic manifolds as well as Lagrangian correspondences. However, the Floer cohomology of any generalized Lagrangian correspondence containing  $\emptyset$  is the trivial group  $HF(\dots \xrightarrow{\emptyset} \dots) = \{0\}$ .

The associativity axiom on  $\text{Floer}_\tau^\#$  is immediate on the level of objects: For any triple  $\underline{L}_{01} \in \text{Obj}(\text{Don}^\#(M_0, M_1))$ ,  $\underline{L}_{12} \in \text{Obj}(\text{Don}^\#(M_1, M_2))$ ,  $\underline{L}_{23} \in \text{Obj}(\text{Don}^\#(M_2, M_3))$  we have  $(\underline{L}_{01} \# \underline{L}_{12}) \# \underline{L}_{23} = \underline{L}_{01} \# (\underline{L}_{12} \# \underline{L}_{23})$ . On the level of morphisms we apply the quilted

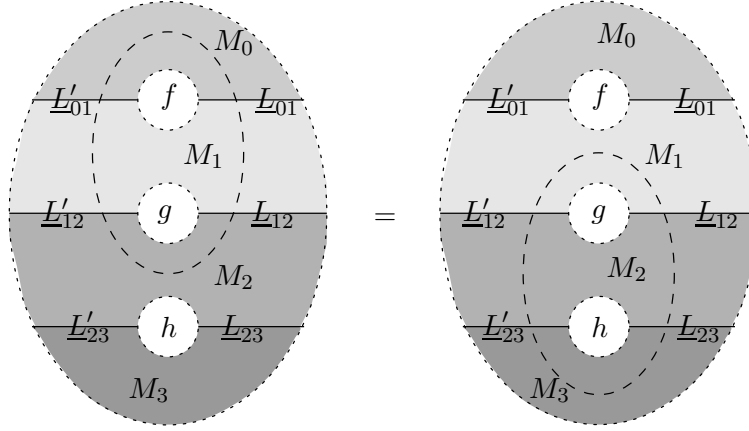


FIGURE 31. Associativity of the concatenation functor

gluing theorem [19, Theorem 3.13] to the gluings indicated by dashed lines in Figure 31 to prove that  $(f\#g)\#h = f\#(g\#h)$  for all  $f \in \text{Hom}(\underline{L}_{01}, \underline{L}'_{01})$ ,  $g \in \text{Hom}(\underline{L}_{12}, \underline{L}'_{12})$ ,  $h \in \text{Hom}(\underline{L}_{23}, \underline{L}'_{23})$ . The weak identity axiom follows from Proposition 7.6. Hence  $\text{Floer}^\#$  is a 2-category with weak identities.

*Remark 8.5.*  $\text{Floer}_\tau^\#$  is independent up to 2-isomorphism of 2-categories of the choices of perturbation data and strip widths, as in Remarks 3.6, 4.6, and the proofs of independence of quilted Floer cohomology and relative quilt invariants in [20, 19].

Theorem 7.4 implies that the definition of composition in the Weinstein-Floer 2-category  $\text{Floer}_\tau^\#$  agrees with the geometric definition, in the case that geometric composition is smooth, embedded, and monotone.

**Theorem 8.6.** *The map  $M_0 \mapsto \text{Don}^\#(M_0)$  and the functors*

$$\text{Don}^\#(M_0, M_1) \rightarrow \text{Fun}(\text{Don}^\#(M_0), \text{Don}^\#(M_1))$$

*as in Proposition 7.2 define a categorification 2-functor  $\text{Floer}_\tau^\# \rightarrow \text{Cat}$  for every  $\tau \geq 0$ .*

*Proof.* Compatibility with the composition follows from the identity (10). The weak identities  $\Delta_M \in \text{Hom}(M, M)$  are mapped to weak identities  $\Phi(\Delta) \simeq 1_{\text{Don}^\#(M)}$  by Corollary 5.12. Here the shift functor  $\Psi_M$  is the identity since  $w_2(M) = 0$ .  $\square$

*Remark 8.7.* (a) For any genuinely monotone symplectic manifold (i.e. with  $\tau > 0$ ) we can achieve  $\tau = 1$  by rescaling. It thus suffices to consider the *exact Weinstein-Floer 2-category*  $\text{Floer}_0^\#$  and the *monotone Weinstein-Floer 2-category*  $\text{Floer}_1^\#$ . Note however that we cannot incorporate Lagrangian correspondences between monotone symplectic manifolds with different monotonicity constants. This is due to bubbling effects which in our present setup are true obstructions to the equivalence of algebraic composition  $L_{01}\#L_{12}$  and embedded geometric composition  $L_{01} \circ L_{12}$ . We expect that the  $A_\infty$ -setup, incorporating all bubbling effects, has better behavior.

- (b) One can define an analogous *graded Weinstein-Floer 2-category*  $\text{Floer}_{N,\tau}^\#$  for any  $\tau \geq 0$  and integer  $N$ , whose objects are monotone symplectic manifolds with the additional structure of a Maslov cover  $\text{Lag}^N(M) \rightarrow M$ . Its 1-morphisms are graded generalized Lagrangian correspondences, and its 2-morphism spaces are the graded Floer cohomology groups.



*Remark 8.8.* (a) One can define a strong identity  $1_M \in \text{Hom}(M, M)$  by allowing the empty sequence  $1_M := \emptyset$  as a generalized Lagrangian correspondence. The various constructions in this Section extend to the case of empty sequences by allowing cylindrical ends.

(b) In the case  $w_2(M) \neq 0$ , the diagonal is not an automorphism but a morphism  $\Delta_M \in \text{Hom}((M, b), (M, b - w_2(M)))$ , see Remark 5.11. Hence

$$\underline{L} \# \Delta_M \in \text{Hom}((M_1, b_1), (M, b - w_2(M))), \quad \underline{L} \in \text{Hom}((M_1, b_1), (M, b))$$

lie in different morphism spaces that are not related by a simple shift in the background class. However, the categorification functor in Theorem 8.6 generalizes directly to this setup as follows. The functor maps the special Floer $^\#_\tau$  1-morphisms  $\Delta_M \in \text{Don}^\#((M, b), (M, b - w_2(M)))$  to Cat 1-morphisms that are isomorphic to the shift functors  $\Psi_M \in \text{Fun}(\text{Don}^\#(M, b), \text{Don}^\#(M, b - w_2(M)))$ .

(c) One can make the diagonal a strong identity by modding out by the equivalence relation discussed Section 2. Let  $\text{Brane}^\#_\tau$  denote the 2-category whose objects and 1-morphisms are those of Floer $^\#_\tau$ , modulo the equivalence relation  $L_{01} \# L_{12} \sim L_{01} \circ L_{12}$  for embedded compositions as in Section 2, and whose 2-morphisms are defined as follows. Given a pair  $[\underline{L}_{01}], [\underline{L}'_{01}]$  of 1-morphisms from  $M_0$  to  $M_1$ , define the space of 2-morphisms  $\text{Hom}([\underline{L}_{01}], [\underline{L}'_{01}])$  by  $\text{Hom}([\underline{L}_{01}], [\underline{L}'_{01}]) = HF(\underline{L}_{01}, \underline{L}'_{01})$  for some choice of representatives  $\underline{L}_{01}, \underline{L}'_{01}$ . Define composition by concatenation  $\#$ , as in (9). The equivalence classes of the diagonal  $[\Delta_M]$  define true identities in case  $w_2(M) = 0$ . Our main result, Theorem 5.5, implies that  $\text{Brane}^\#_\tau$  is independent of the choice of representatives up to 2-isomorphism of 2-categories. Theorem 7.4 implies that the categorification 2-functor of Theorem 8.6 induces a 2-functor  $\text{Brane}^\#_\tau \rightarrow \text{Cat}$  to the 2-category of categories Cat.

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