

Banach space valued Cauchy-Riemann equations with totally real boundary conditions – corrected

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Abstract

The main purpose of this paper is to give a general regularity result for Cauchy-Riemann equations in complex Banach spaces with totally real boundary conditions. The usual elliptic L^p -regularity results hold true under one crucial assumption: The Banach space is isomorphic to a closed subspace of an L^p -space. (Equivalently, the totally real submanifold is modelled on a closed subspace of an L^p -space.) **Some minor corrections are in order on the Sobolev arithmetic in the estimates.**

Secondly, we describe a class of examples of such totally real submanifolds, namely gauge invariant Lagrangian submanifolds in the space of connections over a Riemann surface. These pose natural boundary conditions for the anti-self-duality equation on 4-manifolds with a boundary space-time splitting, leading towards the definition of a Floer homology for 3-manifolds with boundary, which is the first step in a program by Salamon for the proof of the Atiyah-Floer conjecture. The principal part of such a boundary value problem is an example of a Banach space valued Cauchy-Riemann equation with totally real boundary condition.

1 Introduction

A complex Banach space is a Banach space X equipped with a complex structure, i.e. $J \in \text{End } X$ that satisfies $J^2 = -\mathbf{1}$. The Cauchy-Riemann equation for a map $u : \Omega \rightarrow X$ on a domain $\Omega \subset \mathbb{R}^2$ with coordinates (s, t) is $\partial_s u + J\partial_t u = 0$. We will also study the equation with an inhomogeneous term on the right hand side. As in the finite dimensional case, totally real boundary conditions are natural for this Cauchy-Riemann equation. A Banach submanifold $\mathcal{L} \subset X$ is called totally real with respect to the complex structure J if for all $x \in \mathcal{L}$ one has the direct sum decomposition

$$X = T_x \mathcal{L} \oplus J T_x \mathcal{L}. \quad (1)$$

Let $\Omega \subset \mathbb{H}$ be a compact 2-dimensional submanifold in the half space

$$\mathbb{H} := \{(s, t) \in \mathbb{R}^2 \mid t \geq 0\}.$$

i.e. Ω has smooth boundary that might intersect $\partial\mathbb{H} = \{t = 0\}$. We will consider Cauchy-Riemann equations for maps $u : \Omega \rightarrow X$ that satisfy totally real boundary conditions on the boundary part $\partial\Omega \cap \partial\mathbb{H}$. Fix an inhomogeneous term $G : \Omega \rightarrow X$, a family $J : \Omega \rightarrow \text{End } X$ of complex structures on X , and let $\mathcal{L} \subset X$ be a Banach submanifold that is totally real with respect to $J_{s,t}$ for all $(s, t) \in \Omega$. Then we study the following boundary value problem for $u : \Omega \rightarrow X$,

$$\begin{cases} \partial_s u + J_{s,t} \partial_t u = G, \\ u(s, 0) \in \mathcal{L} \quad \forall (s, 0) \in \partial\Omega \cap \partial\mathbb{H}. \end{cases} \quad (2)$$

The Cauchy-Riemann equation itself is linear, but for the linearization of this boundary value problem one has to linearize the boundary conditions. So fix a path $x : \mathbb{R} \rightarrow \mathcal{L}$, then we will also study the Cauchy-Riemann equation with linearized totally real boundary conditions for $u : \Omega \rightarrow X$,

$$\begin{cases} \partial_s u + J_{s,t} \partial_t u = G, \\ u(s, 0) \in T_{x(s)}\mathcal{L} \quad \forall (s, 0) \in \partial\Omega \cap \partial\mathbb{H}. \end{cases} \quad (3)$$

In this case, there also is a weak formulation of the boundary value problem. We denote by X^* the dual space of X and denote by $J^* \in \text{End } X^*$ the dual operator of the complex structure $J \in \text{End } X$. Then the weak formulation of (3) for $u : \Omega \rightarrow X$ is

$$\int_{\Omega} \langle u, \partial_s \psi + \partial_t (J^* \psi) \rangle = - \int_{\Omega} \langle G, \psi \rangle$$

for all $\psi \in \mathcal{C}^\infty(\Omega, X^*)$ with $\text{supp } \psi \subset \text{int } \Omega$ and $\psi(s, 0) \in (J(s, 0)T_{x(s)}\mathcal{L})^\perp$ for all $(s, 0) \in \partial\Omega \cap \partial\mathbb{H}$. In order to obtain regularity results for any of the above boundary value problems we make the following crucial assumption.

(H_p) Throughout we suppose that the Banach submanifold $\mathcal{L} \subset X$ is modelled on a closed subspace $Y \subset Z$ of an L^p -space $Z = L^p(M, \mathbb{R}^m)$ for some $p > 1$, $m \in \mathbb{N}$, and a closed manifold M .

Remark. Consider (1) to see that for a totally real submanifold $\mathcal{L} \subset X$ the assumption (H_p) is equivalent to X being isomorphic to a closed subspace of an L^p -space.

To show that this assumption still allows \mathcal{L} to be modelled on a wide variety of Banach spaces, we give the following examples.

Example 1.1

- (i) Every finite dimensional space \mathbb{R}^m is isometric to the subspace of constants in $L^p(M, \mathbb{R}^m)$ for $\text{Vol } M = 1$.
- (ii) The Sobolev space $W^{\ell, p}(M)$ (and thus every closed subspace thereof) is isomorphic to a closed subspace of $L^p(M, \mathbb{R}^m)$.

To see this, choose vector fields $X_1, \dots, X_k \in \Gamma(\mathrm{TM})$ that span $T_x M$ for all $x \in M$. Then the map $u \mapsto (u, \nabla_{X_1} u, \dots, \nabla_{X_k}^\ell u)$ running through all derivatives of u in the direction of the X_i up to order ℓ gives an isomorphism between $W^{\ell,p}(M)$ and a closed subspace of $L^p(M, \mathbb{R}^m) =: Z$.

(iii) Finite products of closed subspaces in $L^p(M_i, \mathbb{R}^{m_i})$ are isometric to a closed subspace of $L^p(\bigcup M_i, \mathbb{R}^{\max\{m_i\}})$.

Our first main theorem gives regularity results and estimates for solutions of (2) depending on the regularity of the inhomogeneous term in the Cauchy-Riemann equation. Here and throughout the interior of Ω is defined with respect to the topology of \mathbb{H} , so $\mathrm{int} \Omega$ still contains $\partial\Omega \cap \partial\mathbb{H}$. We use the notation $\mathbb{N} = \{1, 2, \dots\}$.

Theorem 1.2 Fix $1 < p < \infty$ and a compact subset $K \subset \mathrm{int} \Omega$. Let $\mathcal{L} \subset X$ be a Banach submanifold that satisfies (H_p) .

(i) Fix $k \in \mathbb{N}$ and let

$$q := \begin{cases} p & ; \text{if } k \geq 3 \text{ or } k = 2, p > 2, \\ \text{any } q > 2 & ; \text{if } k = 2, p = 2, \\ \frac{4p}{2+p} & ; \text{if } k = 2, p < 2, \\ 2p & ; \text{if } k = 1. \end{cases}$$

Suppose that $u \in W^{k,q}(\Omega, X)$ solves (2) for $G \in W^{k,q}(\Omega, X)$ and with a family $J \in W^{k+1,\infty}(\Omega, \mathrm{End} X)$ of complex structures on X , with respect to which \mathcal{L} is totally real. Then $u \in W^{k+1,p}(K, X)$.

(ii) Let $J_0 \in C^\infty(\Omega, \mathrm{End} X)$ be a smooth family of complex structures on X , with respect to which \mathcal{L} is totally real. Let $u_0 \in C^\infty(\Omega, X)$ be such that $u_0(s, 0) \in \mathcal{L}$ for all $(s, 0) \in \partial\Omega \cap \partial\mathbb{H}$. Then there exists a constant $\delta > 0$ with the following significance: For every constant c and for every $k \in \mathbb{N}$ there exists a constant¹ C such that the following holds: If u , G , and J satisfy the hypotheses of (i) and

$$\begin{aligned} \|u - u_0\|_{L^\infty(\Omega, X)} &\leq \delta, & \|J - J_0\|_{L^\infty(\Omega, \mathrm{End} X)} &\leq \delta, \\ \|u - u_0\|_{W^{k,q}(\Omega, X)} &\leq c, & \|J - J_0\|_{W^{k+1,\infty}(\Omega, \mathrm{End} X)} &\leq c, \end{aligned}$$

then

$$\|u - u_0\|_{W^{k+1,p}(K, X)} \leq C(\|G\|_{W^{k,q}(\Omega, X)} + \|u - u_0\|_{W^{k,q}(\Omega, X)}).$$

Firstly note the special form of this theorem for $k = 1$ and $k = 2$. In order to deduce $W^{2,p}$ -regularity of u one needs to assume that u and G are $W^{1,2p}$ -regular, due to nonlinearities introduced by the coordinates. If now $p > 2$ then

¹More precisely, the constant C can be bounded by a polynomial in c of order $2k$.

$q \leq p$, so $W^{2,p}$ -regularity directly implies $W^{3,p}$ -regularity. If however $p \leq 2$, then $W^{3,p}$ -regularity would only follow if we had $W^{2,q}$ -regularity for the given $q > p$. For maps with values in a finite dimensional space, such arguments can be iterated together with Sobolev embeddings to prove smoothness, starting from $W^{1,2p}$ -regularity for any $p > 1$. For functions with values in a Banach space, and the Lagrangian modelled on an L^p -space, this iteration only works if $p > 2$.

Secondly, note that the u_0 in (ii) satisfies the Lagrangian boundary condition but is not a solution of the Cauchy-Riemann equation. It will be required as reference for the construction of coordinates near \mathcal{L} that straighten out the boundary condition but do not depend on the solution u and hence allow to deduce an estimate for u . In order that the constant in the estimate becomes independent of the complex structure J , this construction moreover requires that J is \mathcal{C}^0 -close to a fixed family J_0 of complex structures. The $W^{k+1,\infty}$ -bound on the complex structure is only required in order to obtain uniform constants.

Moreover, for fixed $k \in \mathbb{N}$ in theorem 1.2 it would actually suffice to have $W^{k+1,\infty}$ -regularity of u_0 and J_0 . For the interior regularity and estimates it even suffices to have $W^{k,\infty}$ -regularity and bounds on J since one does not need to write u in coordinates that are adapted to the boundary condition and hence depend on J . This is the same situation as in the finite dimensional case, c.f. [MS]. Finally, the submanifold \mathcal{L} need only be totally real with respect to $J_{s,t}$ for $(s,t) \in \partial\Omega \cap \partial\mathbb{H}$. Since this is an open condition, it is then automatically totally real in a neighbourhood of $\partial\Omega \cap \partial\mathbb{H}$.

The second main result concerns the linearized boundary value problem (3). We use its weak formulation to state the following regularity result.

Theorem 1.3 *Fix $1 < p < \infty$, a compact subset $K \subset \text{int } \Omega$, and a Banach submanifold $\mathcal{L} \subset X$ that satisfies (H_p) . Fix a path $x \in W^{1,\infty}(\mathbb{R}, \mathcal{L})$ in \mathcal{L} and let $J \in W^{1,\infty}(\Omega, \text{End } X)$ be a family of complex structures on X , with respect to which \mathcal{L} is totally real. Then there is a constant C such that the following holds:*

Suppose that $u \in L^p(\Omega, X)$ and that there exists a constant c_u such that for all $\psi \in W^{1,\infty}(\Omega, X^)$ with $\text{supp } \psi \subset \text{int } \Omega$ and $\psi(s, 0) \in (J(s, 0)\mathbb{T}_{x(s)}\mathcal{L})^\perp$ for all $(s, 0) \in \partial\Omega \cap \partial\mathbb{H}$*

$$\left| \int_{\Omega} \langle u, \partial_s \psi + \partial_t (J^* \psi) \rangle \right| \leq c_u \|\psi\|_{L^{p^*}(\Omega, X^*)}.$$

Then $u \in W^{1,p}(K, X)$ and

$$\|u\|_{W^{1,p}(K, X)} \leq C(c_u + \|u\|_{L^p(\Omega, X)}).$$

For strong solutions of the linearized boundary value problem (3) the more suitable formulation of theorem 1.3 is the following estimate.

Corollary 1.4 *In the setting of theorem 1.3 there exists a constant C such that the following holds: Suppose that $u \in W^{1,p}(\Omega, X)$ satisfies $u(s, 0) \in T_{x(s)}\mathcal{L}$ for all $(s, 0) \in \partial\Omega \cap \partial\mathbb{H}$, then*

$$\|u\|_{W^{1,p}(K, X)} \leq C(\|\partial_s u + J\partial_t u\|_{L^p(\Omega, X)} + \|u\|_{L^p(\Omega, X)}).$$

A first application of the above results is the elliptic theory for anti-self-dual instantons with Lagrangian boundary conditions. It is developed in [W2], where theorem 1.2 is used to obtain nonlinear regularity and compactness results, whereas theorem 1.3 enters in the Fredholm theory. Since the Fredholm theory is performed on a special compact model domain, we make the following remark.

Remark. Theorem 1.3 and corollary 1.4 remain true when \mathbb{R} is replaced by S^1 , i.e. when one considers compact domains $K \subset \text{int } \Omega \subset S^1 \times [0, \infty)$ in the half cylinder and a loop $x \in \mathcal{C}^\infty(S^1, \mathcal{L})$.

To see this, identify $S^1 \cong \mathbb{R}/\mathbb{Z}$, identify K with a compact subset $K' \subset \mathbb{H}$ in $[0, 1] \times [0, \infty)$, and periodically extend x and u for $s \in [-1, 2]$. Then u is defined and satisfies the weak equation on some open domain $\Omega' \subset \mathbb{H}$ such that $K' \subset \text{int } \Omega'$, so theorem 1.3 and corollary 1.4 apply. These assert regularity and estimates on K' and hence also on K .

We now proceed to describe a class of examples, to which the above regularity theory for the Cauchy-Riemann equation can be applied.

A symplectic Banach space (Z, ω) consists of a Banach space Z and a symplectic structure ω , that is a nondegenerate,² skewsymmetric, bilinear form $\omega : Z \times Z \rightarrow \mathbb{R}$. In the finite dimensional case there always exists an ω -compatible complex structure $J \in \text{End } Z$, i.e. a complex structure such that $\omega(\cdot, J\cdot)$ defines a positive definite inner product on Z . In the case of an infinite dimensional Banach space this is not necessarily true. If an ω -compatible complex structure exists, then the norm on Z that is induced by the inner product will be bounded but not necessarily complete. The completion of Z with respect to that norm is then a complex Hilbert space. In the example below, this Hilbert space will always be the same – only the complex structure varies.

Our example of a symplectic Banach space will be the space of connections over a Riemann surface Σ . We restrict the discussion to the trivial G -bundle over Σ , where G is a compact Lie group.³ Then the space of L^p -regular connections is given by the L^p -regular 1-forms with values in the Lie algebra \mathfrak{g} of G . We denote this space

$$\mathcal{A}^{0,p}(\Sigma) = L^p(\Sigma; T^*\Sigma \otimes \mathfrak{g}).$$

²Nondegeneracy means that for all $z \in Z \setminus \{0\}$ there exists a $y \in Z$ such that $\omega(z, y) \neq 0$.

³The discussion directly generalizes to nontrivial bundles, where the connections can be described as 1-forms with values in an associated bundle.

(For more details on gauge theory and the notation see section 3 and [W1].) For $p \geq 2$ the Banach space $\mathcal{A}^{0,p}(\Sigma)$ is equipped with the symplectic structure

$$\omega(\alpha, \beta) = \int_{\Sigma} \langle \alpha \wedge \beta \rangle \quad \forall \alpha, \beta \in L^p(\Sigma; \mathbb{T}^*\Sigma \otimes \mathfrak{g}). \quad (4)$$

Moreover, for $p > 2$ the gauge group $\mathcal{G}^{1,p}(\Sigma) = W^{1,p}(\Sigma, \mathbb{G})$ acts on $\mathcal{A}^{0,p}(\Sigma)$ by

$$u^*A = u^{-1}Au + u^{-1}du \quad \forall A \in \mathcal{A}^{0,p}(\Sigma), u \in \mathcal{G}^{1,p}(\Sigma).$$

This gauge action leaves ω invariant. So throughout we assume $p > 2$. Now for any metric on Σ , the Hodge $*$ operator induces an ω -compatible complex structure on $\mathcal{A}^{0,p}(\Sigma)$. The associated inner product is the L^2 -inner product of \mathfrak{g} -valued 1-forms, and the completion of $\mathcal{A}^{0,p}(\Sigma)$ with respect to the induced norm is always $L^2(\Sigma, \mathbb{T}^*\Sigma \otimes \mathfrak{g})$.

We call a Banach submanifold $\mathcal{L} \subset (\mathcal{A}^{0,p}(\Sigma), \omega)$ Lagrangian if it is isotropic, i.e. $\omega|_{\mathcal{L}} \equiv 0$, and if $\mathbb{T}_A\mathcal{L}$ is maximal for all $A \in \mathcal{L}$. By the latter we mean that for all $\alpha \in \mathcal{A}^{0,p}(\Sigma)$ the following implication holds:

$$(\forall \beta \in \mathbb{T}_A\mathcal{L} \quad \omega(\alpha, \beta) = 0) \implies \alpha \in \mathbb{T}_A\mathcal{L}.$$

In section 3 we will introduce the space of weakly flat L^p -connections $\mathcal{A}_{\text{flat}}^{0,p}(\Sigma)$. In particular, we prove that every weakly flat connection is gauge equivalent to a smooth connection. Then we shall show in section 4 that a gauge invariant Lagrangian submanifold of $\mathcal{A}^{0,p}(\Sigma)$ that also satisfies $\mathcal{L} \subset \mathcal{A}_{\text{flat}}^{0,p}(\Sigma)$ is automatically totally real with respect to the Hodge $*$ operator for any metric on Σ , i.e. for all $A \in \mathcal{L}$

$$\mathcal{A}^{0,p}(\Sigma) = \mathbb{T}_A\mathcal{L} \oplus *\mathbb{T}_A\mathcal{L}.$$

Moreover, such Lagrangian submanifolds satisfy the assumption (H_p) for theorems 1.2 and 1.3. The assumptions of gauge invariance and flatness also ensure that the Lagrangian submanifold \mathcal{L} descends to a Lagrangian submanifold in the (singular) symplectic manifold $M_{\Sigma} = \mathcal{A}_{\text{flat}}^{0,p}(\Sigma)/\mathcal{G}^{1,p}(\Sigma)$, the moduli space of gauge equivalence classes of flat connections. The latter can be viewed as symplectic quotient, as was first observed by [AB]. Note that both M_{Σ} and the quotient $\mathcal{L}/\mathcal{G}^{1,p}(\Sigma)$ are allowed to have singularities. These do not enter the discussion since we will be working in the total space.

Now a pseudoholomorphic curve $u : \Omega \rightarrow M_{\Sigma}$ with Lagrangian boundary conditions on $\partial\Omega \cap \partial\mathbb{H}$ lifts to a solution $B : \Omega \times \Sigma \rightarrow \mathbb{T}^*\Sigma \otimes \mathfrak{g}$ of the boundary value problem

$$\begin{cases} F_B = 0, \\ \partial_s B + *\partial_t B = d_B\Phi + *d_B\Psi, \\ B|_{(s,0) \times \Sigma} \in \mathcal{L} \quad \forall (s,0) \in \partial\Omega \cap \partial\mathbb{H}. \end{cases} \quad (5)$$

Here $\Phi, \Psi : \Omega \times \Sigma \rightarrow \mathfrak{g}$ are determined by the solution B . For given Φ, Ψ , the above boundary value problem without the first equation is a Cauchy-Riemann equation with totally real boundary conditions as studied in this paper.

Changing the first equation in (5) to $*F_B = \partial_t \Phi - \partial_s \Psi + [\Phi, \Psi]$ leads to the anti-self-duality equation for the connection $A = \Phi ds + \Psi dt + B$ on $\Omega \times \Sigma$ with Lagrangian boundary conditions,

$$\begin{cases} *F_A + F_A = 0, \\ A|_{(s,0) \times \Sigma} \in \mathcal{L} \quad \forall (s,0) \in \partial\Omega \cap \partial\mathbb{H}. \end{cases} \quad (6)$$

This boundary value problem arises naturally from the Chern-Simons 1-form on a 3-manifold Y with boundary Σ : This 1-form becomes closed and it is in fact the differential of the (multivalued) Chern-Simons functional, when it is restricted to the space $\mathcal{A}(Y, \mathcal{L})$ of connections A on Y with Lagrangian boundary conditions $A|_{\Sigma} \in \mathcal{L}$. Now the gradient flow lines of the Chern-Simons functional are just the solutions of (6) in a special gauge.

It is a program by Salamon [S] to use the boundary value problem (6) to define a Floer homology $\mathrm{HF}_*^{\mathrm{inst}}(Y, L)$ for 3-manifolds Y with boundary $\partial Y = \Sigma$ and Lagrangian submanifolds $L = \mathcal{L}/\mathcal{G}^{1,p}(\Sigma) \subset M_{\Sigma}$, i.e. a generalized Morse homology for the Chern-Simons functional on $\mathcal{A}(Y, \mathcal{L})$. As a first indication for the wellposedness of (6) we prove in corollary 4.5 that every $W^{1,p}$ -regular connection satisfying the boundary condition in (6) can be approximated by smooth connections satisfying the same boundary condition. The elliptic theory for the definition of this Floer homology is set up in [W2], where the regularity theorems 1.2 and 1.3 play a crucial role.

Another approach to the definition of a Floer homology for 3-manifolds with boundary was introduced by Fukaya [F]. This also uses Lagrangian boundary conditions, but the construction is restricted to the case of nontrivial bundles, in which case the quotient $\mathcal{L}/\mathcal{G}^{1,p}(\Sigma)$ is smooth.

Finally, a concrete example of a totally real submanifold in a complex Banach space is given in lemma 4.6. Let $\Sigma = \partial Y$ be the boundary of a handle body Y and consider the L^p -closure of the set of smooth flat connections on Σ that can be extended to a flat connection on Y ,

$$\mathcal{L}_Y := \mathrm{cl} \{ A \in \mathcal{A}_{\mathrm{flat}}(\Sigma) \mid \exists \tilde{A} \in \mathcal{A}_{\mathrm{flat}}(Y) : \tilde{A}|_{\Sigma} = A \} \subset \mathcal{A}^{0,p}(\Sigma).$$

This is a Lagrangian submanifold and it is gauge invariant and contained in the space of flat connections, so as above it also is totally real with respect to the Hodge operator as complex structure.

These submanifolds occur in the Atiyah-Floer conjecture for homology 3-spheres as follows: A Heegaard splitting $Y = Y_0 \cup_{\Sigma} Y_1$ of a homology 3-sphere Y into two handlebodies Y_0 and Y_1 with common boundary Σ gives rise to two Lagrangian submanifolds $\mathcal{L}_{Y_i} \subset \mathcal{A}^{0,p}(\Sigma)$ in the space of $\mathrm{SU}(2)$ -connections. One then has a symplectic Floer homology $\mathrm{HF}_*^{\mathrm{symp}}(M_{\Sigma}, L_{Y_0}, L_{Y_1})$ for the quotients $L_{Y_i} := \mathcal{L}_{Y_i}/\mathcal{G}^{1,p}(\Sigma) \subset M_{\Sigma}$. (This is generated by the intersection points of the Lagrangian submanifolds and the boundary operator arises from counting pseudoholomorphic strips with Lagrangian boundary conditions, i.e. solutions of a boundary value problem like (5).) It was conjectured by Atiyah [At] and Floer that this should be isomorphic to the instanton Floer homology $\mathrm{HF}_*^{\mathrm{inst}}(Y)$,

the generalized Morse homology for the Chern-Simons functional on the space of $SU(2)$ -connections on Y . Now the program by Salamon [S] is to establish this isomorphism in two steps via the intermediate $\mathrm{HF}_*^{\mathrm{inst}}([0, 1] \times \Sigma, L_{Y_0} \times L_{Y_1})$ by adiabatic limit type arguments similar to [DS2]. These adiabatic limits will again require elliptic estimates for boundary value problems including a Cauchy-Riemann equation with totally real boundary conditions as studied in this paper.

This paper is organized as follows: In section 2 we prove theorems 1.2 and 1.3 and corollary 1.4. Section 3 is of preliminary nature: We introduce the notion of a weakly flat connection, prove the fundamental regularity result for weakly flat connections, and discuss the moduli space of flat connections over a Riemann surface. Section 4 deals with gauge invariant Lagrangian submanifolds in the space of connections. We establish their basic properties and prove the approximation result mentioned above. Moreover, we show that the \mathcal{L}_Y are indeed examples of Lagrangian and totally real submanifolds.

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2 Regularity

In this section we prove the regularity theorems 1.2, 1.3, and corollary 1.4. Let $\Omega \subset \mathbb{H}$ be a compact 2-dimensional submanifold of the half space. Consider a Banach space X with a family $J : \Omega \rightarrow \mathrm{End} X$ of complex structures. Let $\mathcal{L} \subset X$ be a Banach submanifold that satisfies (H_p) , i.e. it is modelled on a closed subspace $Y \subset Z$ of an L^p -space $Z = L^p(M, \mathbb{R}^m)$ for some $p > 1$, $m \in \mathbb{N}$, and a closed manifold M , and suppose that \mathcal{L} is totally real with respect to all $J_{s,t}$ for $(s, t) \in \Omega$. Then we consider maps $u : \Omega \rightarrow X$ that solve the boundary value problem (2), restated here:

$$\begin{cases} \partial_s u + J_{s,t} \partial_t u = G, \\ u(s, 0) \in \mathcal{L} \quad \forall (s, 0) \in \partial\Omega \cap \partial\mathbb{H}. \end{cases} \quad (7)$$

The idea for the proof of theorem 1.2 is to straighten out the boundary condition by going to local coordinates in $Y \times Y$ near $u(s, 0) \in X$ such that $Y \times \{0\}$ corresponds to the submanifold \mathcal{L} and the complex structure becomes standard along $Y \times \{0\}$. For theorem 1.3, concerning the linearization of (7), one chooses \mathbb{R} -dependent coordinates for X that identify $Y \times \{0\}$ with $\mathbb{T}_{x(s)}\mathcal{L}$ along the path $x : \mathbb{R} \rightarrow \mathcal{L}$. Then the boundary value problem (7) or its linearization yields Dirichlet and Neumann boundary conditions for the two components of u

and one can use regularity results for the Laplace equation with such boundary conditions.

However, there are two difficulties. Firstly, by straightening out the totally real submanifold, the complex structure J becomes explicitly dependent on u , so one has to deal carefully with nonlinearities in the equation. Secondly, this approach requires a Caldéron-Zygmund inequality for functions with values in a Banach space. In general, the Caldéron-Zygmund inequality is only true for values in Hilbert spaces. However, due to the assumption that \mathcal{L} is modelled on an L^p -space, we only need the L^p -inequality for functions with values in L^p -spaces. In that case, the Caldéron-Zygmund inequality holds, as can be seen by integrating over the real valued inequality. This will be made precise in the following lemma, in which (i),(iii) are regularity results for the homogeneous Dirichlet problem and (ii),(iv) concern the Neumann problem with possibly inhomogeneous boundary conditions. In (i),(ii) the minimum regularity of u is $W^{1,p}$ – in the case of lower regularity one has to use the weak formulation in (iii), (iv). We abbreviate $\Delta := d^*d$ and denote by ν the outer unit normal to $\partial\Omega$. We write Z^* for the dual space of any Banach space Z and write $\langle \cdot, \cdot \rangle$ for the pairing of Z and Z^* . The Sobolev spaces of Banach space valued functions considered below are all defined as completions of the smooth functions with respect to the respective Sobolev norm. Moreover, we use the notation

$$\begin{aligned} \mathcal{C}_\delta^\infty(\Omega, Z^*) &:= \{\psi \in \mathcal{C}^\infty(\Omega, Z^*) \mid \psi|_{\partial\Omega} = 0\}, \\ \mathcal{C}_\nu^\infty(\Omega, Z^*) &:= \{\psi \in \mathcal{C}^\infty(\Omega, Z^*) \mid \frac{\partial\psi}{\partial\nu}|_{\partial\Omega} = 0\}. \end{aligned}$$

Lemma 2.1 *Fix $1 < p < \infty$ and $k \in \mathbb{N}$ and let Ω be a compact Riemannian manifold with boundary. Let $Z = L^p(M)$ for some closed manifold M . Then there exists a constant C such that the following holds.*

(i) *Let $f \in W^{k-1,p}(\Omega, Z)$ and suppose that $u \in W^{k,p}(\Omega, Z)$ solves*

$$\int_\Omega \langle u, \Delta\psi \rangle = \int_\Omega \langle f, \psi \rangle \quad \forall \psi \in \mathcal{C}_\delta^\infty(\Omega, Z^*).$$

Then $u \in W^{k+1,p}(\Omega, Z)$ and $\|u\|_{W^{k+1,p}} \leq C\|f\|_{W^{k-1,p}}$.

(ii) *Let $f \in W^{k-1,p}(\Omega, Z)$, $g \in W^{k,p}(\Omega, Z)$, and suppose that $u \in W^{k,p}(\Omega, Z)$ solves*

$$\int_\Omega \langle u, \Delta\psi \rangle = \int_\Omega \langle f, \psi \rangle + \int_{\partial\Omega} \langle g, \psi \rangle \quad \forall \psi \in \mathcal{C}_\nu^\infty(\Omega, Z^*).$$

Then $u \in W^{k+1,p}(\Omega, Z)$ and

$$\|u\|_{W^{k+1,p}} \leq C(\|f\|_{W^{k-1,p}} + \|g\|_{W^{k,p}} + \|u\|_{L^p}).$$

(iii) *Suppose that $u \in L^p(\Omega, Z)$ and there exists a constant c_u such that*

$$\left| \int_{\Omega \times M} u \cdot \Delta_\Omega \psi \right| \leq c_u \|\psi\|_{W^{1,p^*}(\Omega, Z^*)} \quad \forall \psi \in \mathcal{C}_\delta^\infty(\Omega \times M).$$

Then $u \in W^{1,p}(\Omega, Z)$ and $\|u\|_{W^{1,p}(\Omega, Z)} \leq Cc_u$.

(iv) Suppose that $u \in L^p(\Omega, Z)$ and there exists a constant c_u such that

$$\left| \int_{\Omega \times M} u \cdot \Delta_{\Omega} \psi \right| \leq c_u \|\psi\|_{W^{1,p^*}(\Omega, Z^*)} \quad \forall \psi \in \mathcal{C}_v^{\infty}(\Omega \times M).$$

Then $u \in W^{1,p}(\Omega, Z)$ and $\|u\|_{W^{1,p}(\Omega, Z)} \leq C(c_u + \|u\|_{L^p(\Omega, Z)})$.
If moreover $\int_{\Omega} u = 0$ then in fact $\|u\|_{W^{1,p}(\Omega, Z)} \leq Cc_u$.

The key to the proof of (i) and (ii) is the fact that the functions f and g can be approximated not only by smooth functions with values in the Banach space $L^p(M)$, but by smooth functions on $\Omega \times M$.

Lemma 2.2 *Let Ω be a compact manifold (possibly with boundary), let M be a closed manifold, let $1 < p, q < \infty$, and $k, \ell \in \mathbb{N}_0$. Then the following holds.*

- (i) $\mathcal{C}^{\infty}(\Omega \times M)$ is dense in $W^{k,q}(\Omega, W^{\ell,p}(M))$.
- (ii) A function $u \in W^{k,q}(\Omega, W^{\ell,p}(M))$ with zero boundary values $u|_{\partial\Omega} = 0$ can be approximated by $u^{\nu} \in \mathcal{C}^{\infty}(\Omega \times M)$ with $u^{\nu}|_{\partial\Omega \times M} = 0$.
- (iii) If $\ell p > \dim M$ and $z \in M$, then a function $u \in W^{k,q}(\Omega, W^{\ell,p}(M))$ with $u(\cdot, z) = 0 \in W^{k,q}(\Omega)$ can be approximated by $u^{\nu} \in \mathcal{C}^{\infty}(\Omega \times M)$ with $u^{\nu}(\cdot, z) \equiv 0$.

Proof of lemma 2.2:

We first prove (i). By definition $\mathcal{C}^{\infty}(\Omega, W^{\ell,p}(M))$ is dense in $W^{k,q}(\Omega, W^{\ell,p}(M))$. So we fix $g \in \mathcal{C}^{\infty}(\Omega, W^{\ell,p}(M))$ and show that in every $W^{k,q}(\Omega, W^{\ell,p}(M))$ -neighbourhood of g there exists a $\tilde{g} \in \mathcal{C}^{\infty}(\Omega \times M)$. Firstly, we prove this in the case $k = 0$ for closed manifolds M as well as in the following case (that will be needed for the proof in the case $k \geq 1$): $M = \mathbb{R}^n$, g is supported in $\Omega \times V$ and \tilde{g} is required to have support in $\Omega \times U$ for some open bounded domains $V, U \subset \mathbb{R}^n$ such that $\bar{V} \subset U$.

Fix $\delta > 0$. Since Ω is compact one finds a finite covering $\Omega = \bigcup_{i=1}^N U_i$ by neighbourhoods U_i of $x_i \in \Omega$ such that

$$\|g(x) - g(x_i)\|_{W^{\ell,p}(M)} \leq \frac{\delta}{2} \quad \forall x \in U_i.$$

Next, choose $g_i \in \mathcal{C}^{\infty}(M)$ such that $\|g_i - g(x_i)\|_{W^{\ell,p}(M)} \leq \frac{\delta}{2}$. In the case $M = \mathbb{R}^n$ one has $\text{supp } g(x_i) \subset V$ and hence can choose g_i such that it is supported in U (e.g. using mollifiers with compact support). Then choose a partition of unity $\sum_{i=1}^N \phi_i = 1$ by $\phi_i \in \mathcal{C}^{\infty}(\Omega, [0, 1])$ with $\text{supp } \phi_i \subset U_i$. Now one can define $\tilde{g} \in \mathcal{C}^{\infty}(\Omega \times M)$ by

$$\tilde{g}(x, z) := \sum_{i=1}^N \phi_i(x) g_i(z) \quad \forall x \in \Omega, z \in M.$$

In the case $M = \mathbb{R}^n$ this satisfies $\text{supp } \tilde{g} \subset \Omega \times U$ as required. Moreover,

$$\begin{aligned} \|\tilde{g} - g\|_{L^q(\Omega, W^{\ell,p}(M))}^q &= \int_{\Omega} \left\| \sum_{i=1}^N \phi_i (g_i - g) \right\|_{W^{\ell,p}(M)}^q \\ &\leq \int_{\Omega} (\sum_{i=1}^N \phi_i \cdot \sup_{x \in U_i} \|g_i - g(x)\|_{W^{\ell,p}(M)})^q \\ &\leq \int_{\Omega} \delta^q = \delta^q \text{Vol } \Omega. \end{aligned}$$

Thus we have proven the lemma in the case $k = 0$. For $k \geq 1$ this method does not work since one picks up derivatives of the cutoff functions ϕ_i . Instead, one has to use mollifiers and the result for $k = 0$ on $M = \mathbb{R}^n$.

So we assume $k \geq 1$, fix $g \in C^\infty(\Omega, W^{\ell,p}(M))$ and pick some $\delta > 0$. Let $M = \bigcup_{i=1}^N \Phi_i(U_i)$ be an atlas with bounded open domains $U_i \subset \mathbb{R}^n$ and charts $\Phi_i : U_i \rightarrow M$. Let $V_i \subset \bar{V}_i \subset U_i$ be open sets such that still $M = \bigcup_{i=1}^N \Phi_i(V_i)$. Then there exists a partition of unity $\sum_{i=1}^N \psi_i \circ \Phi_i^{-1} = 1$ by $\psi_i \in C^\infty(\mathbb{R}^n, [0, 1])$ such that $\text{supp } \psi_i \subset V_i$. Now $g = \sum_{i=1}^N g_i \circ (\text{id}_\Omega \times \Phi_i^{-1})$ with

$$g_i(x, y) = \psi_i(y) \cdot g(x, \Phi_i(y)) \quad \forall x \in \Omega, y \in U_i.$$

Here $g_i \in C^\infty(\Omega, W^{\ell,p}(\mathbb{R}^n))$ is extended by 0 outside of $\text{supp } g_i \subset \Omega \times V_i$, and it suffices to prove that each of these functions can be approximated in $W^{k,q}(\Omega, W^{\ell,p}(\mathbb{R}^n))$ by $\tilde{g}_i \in C^\infty(\Omega \times \mathbb{R}^n)$ with $\text{supp } \tilde{g}_i \subset \Omega \times U_i$. So drop the subscript i and consider $g \in C^\infty(\Omega, W^{\ell,p}(\mathbb{R}^n))$ that is supported in $\Omega \times V$, where $V, U \subset \mathbb{R}^n$ are open bounded domains such that $\bar{V} \subset U$.

Let $\sigma_\varepsilon(y) = \varepsilon^{-n} \sigma(y/\varepsilon)$ be a family of compactly supported mollifiers for $\varepsilon > 0$, i.e. $\sigma \in C^\infty(\mathbb{R}^n, [0, \infty))$ such that $\text{supp } \sigma \subset B_1(0)$ and $\int \sigma = 1$. Then for all $\varepsilon > 0$ define $\tilde{g}_\varepsilon \in C^\infty(\Omega \times \mathbb{R}^n)$ by

$$\tilde{g}_\varepsilon(x, y) := [\sigma_\varepsilon * g(x, \cdot)](y) \quad \forall x \in \Omega, y \in \mathbb{R}^n.$$

Firstly, $\text{supp } \sigma_\varepsilon \subset B_\varepsilon(0)$, so for sufficiently small $\varepsilon > 0$ the support of \tilde{g}_ε lies within $\Omega \times U$. Secondly, we abbreviate for $j \leq k$, $m \leq \ell$

$$f_{j,m} := \nabla_\Omega^j \nabla_{\mathbb{R}^n}^m g \in C^\infty(\Omega, L^p(\mathbb{R}^n)),$$

which are supported in $\Omega \times V$. Then

$$\begin{aligned} \|\tilde{g}_\varepsilon - g\|_{W^{k,q}(\Omega, W^{\ell,p}(\mathbb{R}^n))}^q &= \sum_{j \leq k} \int_{\Omega} \left\| \nabla_\Omega^j (\sigma_\varepsilon * g(x, \cdot) - g(x, \cdot)) \right\|_{W^{\ell,p}(\mathbb{R}^n)}^q \\ &\leq (\ell + 1)^{\frac{q}{p}} \sum_{j \leq k} \sum_{m \leq \ell} \int_{\Omega} \left\| \sigma_\varepsilon * f_{j,m}(x, \cdot) - f_{j,m}(x, \cdot) \right\|_{L^p(\mathbb{R}^n)}^q. \end{aligned}$$

Now use the result for $k = 0$ on $M = \mathbb{R}^n$ (with values in a vector bundle) to find $\tilde{f}_{j,m} \in C^\infty(\Omega \times \mathbb{R}^n)$ supported in $\Omega \times U$ such that

$$\|\tilde{f}_{j,m} - f_{j,m}\|_{L^q(\Omega, L^p(\mathbb{R}^n))} \leq \delta.$$

Then for all $x \in \Omega$ and sufficiently small $\varepsilon > 0$ the functions $\sigma_\varepsilon * \tilde{f}_{j,m}(x, \cdot)$ are supported in some fixed bounded domain $U' \subset \mathbb{R}^n$ containing U . Moreover, the $\tilde{f}_{j,m}$ are Lipschitz continuous, hence one finds a constant C (depending on the $\tilde{f}_{j,m}$, i.e. on g and δ) such that for all $x \in \Omega$

$$\begin{aligned} & \|\sigma_\varepsilon * \tilde{f}_{j,m}(x, \cdot) - \tilde{f}_{j,m}(x, \cdot)\|_{L^p(\mathbb{R}^n)}^p \\ &= \int_{U'} \left| \int_{\mathbb{R}^n} \sigma_\varepsilon(y' - y) (\tilde{f}_{j,m}(x, y') - \tilde{f}_{j,m}(x, y)) d^n y' \right|^p d^n y \\ &\leq \int_{U'} \left(\int_{\mathbb{R}^n} \sigma_\varepsilon(y' - y) \sup_{|y-y'| \leq \varepsilon} |\tilde{f}_{j,m}(x, y') - \tilde{f}_{j,m}(x, y)| d^n y' \right)^p d^n y \\ &\leq \text{Vol } U' (C\varepsilon)^p. \end{aligned}$$

Now use the fact that the convolution with σ_ε is continuous with respect to the L^p -norm, $\|\sigma_\varepsilon * f\|_p \leq \|f\|_p$ (see e.g. [Ad, Lemma 2.18]) to estimate

$$\begin{aligned} & \int_{\Omega} \|\sigma_\varepsilon * f_{j,m}(x, \cdot) - f_{j,m}(x, \cdot)\|_{L^p(\mathbb{R}^n)}^q \\ &\leq \int_{\Omega} \left(\|\sigma_\varepsilon * (f_{j,m}(x, \cdot) - \tilde{f}_{j,m}(x, \cdot))\|_{L^p(\mathbb{R}^n)} + \|f_{j,m}(x, \cdot) - \tilde{f}_{j,m}(x, \cdot)\|_{L^p(\mathbb{R}^n)} \right. \\ &\quad \left. + \|\sigma_\varepsilon * \tilde{f}_{j,m}(x, \cdot) - \tilde{f}_{j,m}(x, \cdot)\|_{L^p(\mathbb{R}^n)} \right)^q \\ &\leq 2 \cdot 3^q \|f_{j,m} - \tilde{f}_{j,m}\|_{L^q(\Omega, L^p(\mathbb{R}^n))}^q + 3^q \text{Vol } \Omega (\text{Vol } U)^{\frac{q}{p}} (C\varepsilon)^q \leq 3 \cdot 3^q \delta^q. \end{aligned}$$

Here we have chosen $0 < \varepsilon \leq C^{-1} (\text{Vol } \Omega)^{-\frac{1}{q}} (\text{Vol } U)^{-\frac{1}{p}} \delta$. Thus we obtain

$$\|\tilde{g}_\varepsilon - g\|_{W^{k,q}(\Omega, W^{\ell,p}(\mathbb{R}^n))} \leq 3(\ell + 1)^{\frac{1}{p}} (3(k + 1)(\ell + 1))^{\frac{1}{q}} \delta.$$

This proves (i). To show (ii) one first approximates in $\mathcal{C}^\infty(\Omega, W^{\ell,p}(M))$ with zero boundary values and then mollifies on M as in (i) as follows.

In case $k = 0$ the boundary condition is meaningless, but the approximation with zero boundary values can be done elementary by cutting off in small neighbourhoods of the boundary. For $k \geq 1$ consider a local chart of Ω in $[0, 1] \times \mathbb{R}^n$ such that $\{t = 0\}$ corresponds to the boundary, where t denotes the $[0, 1]$ -coordinate. Let $f \in W^{k,q}([0, 1] \times \mathbb{R}^n, Z)$ for any vector space Z with $f|_{t=0} = 0$ and compact support. Let σ_ε be mollifiers on \mathbb{R}^n as above, then $f_\varepsilon(t, \cdot) := \sigma_\varepsilon * f(t, \cdot)$ defines $f_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^n, W^{k,q}([0, 1], Z))$ for all $\varepsilon > 0$. One checks that $\|f_\varepsilon - f\|_{W^{k,q}([0,1] \times \mathbb{R}^n)} \rightarrow 0$ as $\varepsilon \rightarrow 0$. We choose the σ_ε with compact support, then the f_ε are also compactly supported and hence have finite $W^{k,q}([0, 1], W^{\ell,q}(\mathbb{R}^n))$ -norm for any $\ell \in \mathbb{N}$. Moreover, note that still $f_\varepsilon|_{t=0} = 0$. In order to approximate f_ε with zero boundary values one chooses $\ell = k$, then (i) gives a smooth approximation $g^\nu \rightarrow \partial_t f_\varepsilon$ in the $W^{k-1,q}([0, 1], W^{k,q}(\mathbb{R}^n))$ -norm. Now $f_\varepsilon^\nu(t, x) := \int_0^t g^\nu(\tau, x) d\tau$ defines functions in $\mathcal{C}^\infty([0, 1] \times \mathbb{R}^n, Z)$ that vanish at $t = 0$ and approximate f_ε in the $W^{k,q}([0, 1], W^{k,q}(\mathbb{R}^n))$ -norm, which is even stronger than the $W^{k,q}$ -norm on $[0, 1] \times \mathbb{R}^n$.

Finally, to prove (iii), we choose an approximation by $u^\nu \in \mathcal{C}^\infty(\Omega \times M)$. Then $u^\nu(\cdot, z) \rightarrow 0$ in $W^{k,q}(\Omega)$ since the evaluation at z is a continuous map $W^{\ell,p}(M) \rightarrow \mathbb{R}$. Now $u^\nu - u^\nu(\cdot, z) \in \mathcal{C}^\infty(\Omega \times M)$ still converges to u in $W^{k,q}(\Omega, W^{\ell,p}(M))$ but it vanishes at z . \square

In the case $q = p$ lemma 2.2 provides the continuous inclusion

$$W^{k,p}(\Omega, W^{\ell,p}(M)) \subset W^{\ell,p}(M, W^{k,p}(\Omega))$$

since the norms on these spaces are identical.⁴ Moreover, for $p = q$ and $k = \ell = 0$ the lemma identifies $L^p(\Omega, L^p(M)) = L^p(\Omega \times \Sigma)$ as the completion of $\mathcal{C}^\infty(\Omega \times M)$ under the L^p -norm.

Proof of lemma 2.1 (i) and (ii) :

We first give the proof of the regularity for the inhomogeneous Neumann problem (ii) in full detail; (i) is proven in complete analogy – using the regularity theory for the Laplace equation on \mathbb{R} -valued functions with Dirichlet boundary condition instead of the Neumann condition.

Fix $f \in W^{k-1,p}(\Omega, Z)$, $g \in W^{k,p}(\Omega, Z)$, and let $f^i, g^i \in \mathcal{C}^\infty(\Omega \times M)$ be approximating sequences given by lemma 2.2. Testing the weak equation with $\psi \equiv \alpha$ for all $\alpha \in Z^*$ implies $\int_\Omega f + \int_{\partial\Omega} g = 0$ and thus $h^i := \int_\Omega f^i + \int_{\partial\Omega} g^i \rightarrow 0$ in Z as $i \rightarrow \infty$, so one can replace the f^i by $f^i - h^i/\text{Vol } \Omega \in \mathcal{C}^\infty(\Omega, Z)$ to achieve

$$\int_\Omega f^i(\cdot, y) + \int_{\partial\Omega} g^i(\cdot, y) = 0 \quad \forall y \in M, i \in \mathbb{N}.$$

Now for each $y \in M$ there exist unique solutions $u^i(\cdot, y) \in \mathcal{C}^\infty(\Omega)$ of

$$\begin{cases} \Delta u^i(\cdot, y) = f^i(\cdot, y), \\ \frac{\partial}{\partial \nu} u^i(\cdot, y)|_{\partial\Omega} = g^i(\cdot, y)|_{\partial\Omega}, \\ \int_\Omega u^i(\cdot, y) = 0. \end{cases}$$

For each of these Laplace equations with Neumann boundary conditions one obtains an L^p -estimate for the solution, see e.g. [W1, Theorems 3.1, 3.2]. The constant can be chosen independently of $y \in M$ since it varies continuously with y and M is compact. Then integration of those estimates yields (with different constants C)

$$\begin{aligned} \|u^i\|_{W^{k+1,p}(\Omega, Z)}^p &= \int_M \|u^i\|_{W^{k+1,p}(\Omega)}^p \\ &\leq \int_M C(\|f^i\|_{W^{k-1,p}(\Omega)} + \|g^i\|_{W^{k,p}(\Omega)})^p \\ &\leq C(\|f^i\|_{W^{k-1,p}(\Omega, Z)} + \|g^i\|_{W^{k,p}(\Omega, Z)})^p. \end{aligned}$$

⁴The spaces are actually equal. The proof requires an extension of the approximation argument to manifolds with boundary. We do not carry this out here because we will only need this one inclusion.

Here one uses the crucial fact that $L^p(\Omega, L^p(M)) \subset L^p(M, L^p(\Omega))$ with identical norms. (Note that this is not the case if the integrability indices over Ω and M are different.) Similarly, one obtains for all $i, j \in \mathbb{N}$

$$\|u^i - u^j\|_{W^{k+1,p}(\Omega, Z)} \leq C(\|f^i - f^j\|_{W^{k-1,p}(\Omega, Z)} + \|g^i - g^j\|_{W^{k,p}(\Omega, Z)}).$$

So u^i is a Cauchy sequence and hence converges to some $\tilde{u} \in W^{k+1,p}(\Omega, Z)$. Now suppose that $u \in W^{k,p}(\Omega, Z)$ solves the weak Neumann equation for f and g , then we claim that in fact $u = \tilde{u} + c \in W^{k+1,p}(\Omega, Z)$, where $c \in Z$ is given by

$$c(y) := \frac{1}{\text{Vol } \Omega} \int_{\Omega} (u(\cdot, y) - \tilde{u}(\cdot, y)) \quad \forall y \in M.$$

In order to see that indeed $c \in L^p(M) = Z$ and that for some constant C one has $\|c\|_{L^p(M)} \leq C(\|u\|_{L^p(\Omega, Z)} + \|\tilde{u}\|_{L^p(\Omega, Z)})$ note that lemma 2.2 yields the continuous inclusion $W^{k,p}(\Omega, L^p(M)) \subset L^p(M, W^{k,p}(\Omega)) \subset L^p(M, L^1(\Omega))$. To establish the identity $u = \tilde{u} + c$, we first note that for all $\phi \in \mathcal{C}^\infty(M) \subset Z^*$

$$\int_{\Omega} \langle \tilde{u} + c - u, \phi \rangle = \int_M \phi \cdot \left(\text{Vol } \Omega \cdot c - \int_{\Omega} (u - \tilde{u}) \right) = 0.$$

Next, for any $\phi \in \mathcal{C}^\infty(\Omega \times M)$ let

$$\phi_0 := \frac{1}{\text{Vol } \Omega} \int_{\Omega} \phi \quad \in \mathcal{C}^\infty(M).$$

Then one finds $\psi \in \mathcal{C}_\nu^\infty(\Omega \times M)$ such that $\phi = \Delta_\Omega \psi + \phi_0$. (There exist unique solutions $\psi(\cdot, y)$ of the Neumann problem for $\phi(\cdot, y) - \phi_0(y)$, and these depend smoothly on $y \in M$.) So we find that for all $\phi \in \mathcal{C}^\infty(\Omega \times M)$, abbreviating $\Delta_\Omega = \Delta$

$$\begin{aligned} \int_{\Omega} \langle u - \tilde{u} - c, \phi \rangle &= \int_{\Omega} \langle u, \Delta \psi \rangle - \int_{\Omega} \langle \tilde{u} + c, \Delta \psi \rangle + \int_{\Omega} \langle u - \tilde{u} - c, \phi_0 \rangle \\ &= \int_{\Omega} \langle f, \psi \rangle + \int_{\partial \Omega} \langle g, \psi \rangle - \lim_{i \rightarrow \infty} \int_{\Omega} \langle u^i, \Delta \psi \rangle \\ &= \lim_{i \rightarrow \infty} \left(\int_{\Omega} \langle f - \Delta u^i, \psi \rangle + \int_{\partial \Omega} \langle g - \frac{\partial u^i}{\partial \nu}, \psi \rangle \right) = 0. \end{aligned}$$

This proves $u = \tilde{u} + c \in W^{k+1,p}(\Omega, Z)$ and the estimate for u^i yields in the limit

$$\begin{aligned} \|u\|_{W^{k+1,p}(\Omega, Z)} &\leq \|\tilde{u}\|_{W^{k+1,p}(\Omega, Z)} + (\text{Vol } \Omega)^{\frac{1}{p}} \|c\|_{L^p(M)} \\ &\leq C(\|f\|_{W^{k-1,p}(\Omega, Z)} + \|g\|_{W^{k,p}(\Omega, Z)} + \|u\|_{L^p(\Omega, Z)}). \end{aligned}$$

This finishes the proof of (ii), and analogously of (i). \square

Proof of lemma 2.1 (iii) and (iv) :

Let $u \in L^p(\Omega, Z)$ be as supposed in (iii) or (iv), where $Z = L^p(M)$ and thus

$Z^* = L^{p^*}(M)$. Then we have $u \in L^p(\Omega \times M)$ and the task is to prove that $d_\Omega u$ also is of class L^p on $\Omega \times M$. So we have to consider $\int_{\Omega \times M} u \cdot d_\Omega^* \tau$ for $\tau \in \mathcal{C}_\delta^\infty(\Omega \times M, T^*\Omega)$ (which are dense in $L^{p^*}(\Omega \times M, T^*\Omega)$). In the case (iii) one finds for any such smooth family τ of 1-forms on Ω a smooth function $\psi \in \mathcal{C}_\delta^\infty(\Omega \times M)$ such that $d_\Omega^* \tau = \Delta_\Omega \psi$. Then there is a constant C such that for all $y \in M$ (see e.g. [W1, Theorem D.2']

$$\|\psi(\cdot, y)\|_{W^{1,p^*}} \leq C \|\Delta_\Omega \psi(\cdot, y)\|_{(W^{1,p})^*} \leq C \|\tau(\cdot, y)\|_{p^*}.$$

In the case (iv) one similarly finds $\psi \in \mathcal{C}_\nu^\infty(\Omega \times M)$ such that $d_\Omega^* \tau = \Delta_\Omega \psi$ and $\|\psi(\cdot, y)\|_{W^{1,p^*}} \leq C \|\tau(\cdot, y)\|_{p^*}$ for all $y \in M$ and some constant C . (Note that $\int_\Omega d_\Omega^* \tau \equiv 0$ since τ vanishes on $\partial\Omega \times M$ and we have used e.g. [W1, Theorems 2.2, 2.3].) In both cases we can thus estimate for all $\tau \in \mathcal{C}_\delta^\infty(\Omega \times M, T^*\Omega)$ using the assumption

$$\begin{aligned} \left| \int_{\Omega \times M} u \cdot d_\Omega^* \tau \right| &= \left| \int_{\Omega \times M} u \cdot \Delta_\Omega \psi \right| \leq c_u \left(\int_M \|\psi\|_{W^{1,p^*}(\Omega)}^{p^*} \right)^{\frac{1}{p^*}} \\ &\leq C c_u \left(\int_M \|\tau\|_{L^{p^*}(\Omega)}^{p^*} \right)^{\frac{1}{p^*}} \leq C c_u \|\tau\|_{L^{p^*}(\Omega \times M)}. \end{aligned}$$

Now in both cases the Riesz representation theorem (e.g. [Ad, Theorem 2.33]) asserts that $\int_{\Omega \times M} u \cdot d_\Omega^* \tau = \int_{\Omega \times M} f \cdot \tau$ for all τ with some $f \in L^p(\Omega \times M)$. This proves the L^p -regularity of $d_\Omega u$ and yields the estimate

$$\|d_\Omega u\|_{L^p(\Omega \times M)} \leq C c_u.$$

In the case (iii), one can moreover deduce $u|_{\partial\Omega} = 0$. Indeed, partial integration in the weak equation gives for all $\psi \in \mathcal{C}_\delta^\infty(\Omega \times M)$

$$\begin{aligned} \left| \int_{\partial\Omega \times M} u \cdot \frac{\partial \psi}{\partial \nu} \right| &= \left| \int_{\Omega \times M} u \cdot \Delta_\Omega \psi - \int_{\Omega \times M} \langle d_\Omega u, d_\Omega \psi \rangle \right| \\ &\leq (c_u + \|d_\Omega u\|_{L^p(\Omega \times M)}) \|\psi\|_{W^{1,p^*}(\Omega \times M)}. \end{aligned}$$

For any given $g \in \mathcal{C}^\infty(\partial\Omega \times M)$ one now finds $\psi \in \mathcal{C}^\infty(\Omega \times M)$ with $\psi|_{\partial\Omega} = 0$ and $\frac{\partial \psi}{\partial \nu} = g$, and these can be chosen such that $\|\psi\|_{W^{1,p^*}}$ becomes arbitrarily small. Then one obtains $\int_{\partial\Omega \times M} u g = 0$ and thus $u|_{\partial\Omega} = 0$. Thus in the case (iii) one finds a constant C' such that

$$\|u\|_{W^{1,p}(\Omega, Z)}^p = \int_M \|u\|_{W^{1,p}(\Omega)}^p \leq C' \int_M \|d_\Omega u\|_{L^p(\Omega)}^p = C' \|d_\Omega u\|_{L^p(\Omega \times M)}^p,$$

which finishes the proof of (iii).

In case (iv) with the additional assumption $\int_\Omega u = 0$ one also has a constant C' such that $\|u(\cdot, y)\|_{W^{1,p}(\Omega)} \leq C' \|d_\Omega u(\cdot, y)\|_{L^p(\Omega)}$ for all $y \in M$ and thus

$$\|u\|_{W^{1,p}(\Omega, Z)} \leq C' \|d_\Omega u\|_{L^p(\Omega \times M)} \leq C' C c_u.$$

In the general case (iv) one similarly has

$$\|u\|_{W^{1,p}(\Omega,Z)}^p = \|d_\Omega u\|_{L^p(\Omega \times M)}^p + \|u\|_{L^p(\Omega \times M)}^p \leq (Cc_u + \|u\|_{L^p(\Omega,Z)})^p.$$

□

The proof of theorem 1.2 will moreover use the following quantitative version of the implicit function theorem. This is proven e.g. in [MS, Proposition A.3.4] by a Newton-Picard method. (Here we only need the special case $x_0 = x_1 = 0$.)

Proposition 2.3 *Let X and Y be Banach spaces and let $U \subset Y$ be a neighbourhood of 0. Suppose that $f : U \rightarrow X$ is a continuously differentiable map such that $d_0 f : Y \rightarrow X$ is bijective. Then choose constants $c \geq \|(d_0 f)^{-1}\|$ and $\delta > 0$ such that $B_\delta(0) \subset U$ and*

$$\|d_y f - d_0 f\| \leq \frac{1}{2c} \quad \forall y \in B_\delta(0).$$

Now if $\|f(0)\| \leq \frac{\delta}{4c}$ then there exists a unique solution $y \in B_\delta(0)$ of $f(y) = 0$. Moreover, this solution satisfies

$$\|y\| \leq 2c\|f(0)\|.$$

Proof of theorem 1.2 :

Let $z_0 \in \mathcal{L}$ and let $J_0 \in \text{End } X$ be a complex structure with respect to which \mathcal{L} is totally real. Choose a Banach manifold chart $\phi : V \rightarrow \mathcal{L}$ from a neighbourhood $V \subset Y$ of 0 to a neighbourhood of $\phi(0) = z_0$. Then one obtains a Banach submanifold chart of $\mathcal{L} \subset X$ from a neighbourhood $\mathcal{W} \subset Y \times Y$ of zero to a ball $B_\varepsilon(z_0) \subset X$ around z_0 ,

$$\Theta : \begin{array}{ccc} \mathcal{W} & \xrightarrow{\sim} & B_\varepsilon(z_0) \\ (v_1, v_2) & \longmapsto & \phi(v_1) + J_0 d_{v_1} \phi(v_2). \end{array}$$

To see that this is indeed a diffeomorphism for sufficiently small \mathcal{W} and $\varepsilon > 0$ we just check that $D := d_{(0,0)}\Theta = d_0\phi \oplus J_0 d_0\phi$ is an isomorphism. This is since $d_0\phi : Y \rightarrow T_{z_0}\mathcal{L}$ is an isomorphism and so is the map $T_{z_0}\mathcal{L} \times T_{z_0}\mathcal{L} \rightarrow X$ given by the splitting $X = T_{z_0}\mathcal{L} \oplus J_0 T_{z_0}\mathcal{L}$. The size $\varepsilon > 0$ of the chart can be quantified by proposition 2.3 as follows. For the maps $f = \Theta - x : V \times V \rightarrow X$ one finds constants $c = \|D^{-1}\|$ and $\delta > 0$ independently of $x \in X$ such that $B_\delta(0) \subset V \times V$ and

$$\|d_y f - d_0 f\| = \|d_y \Theta - d_0 \Theta\| \leq \frac{1}{2c} \quad \forall y \in B_\delta(0).$$

Then for $\|x - z_0\| \leq \frac{\delta}{4c} =: \varepsilon$ one obtains a unique $y = \Theta^{-1}(x)$ in $B_\delta(0)$.

Next, if one replaces z_0 and J_0 by $z \in \mathcal{L}$ and a complex structure $J \in \text{End } X$ in sufficiently small neighbourhoods of z_0 and J_0 respectively, then one still obtains a Banach submanifold chart $\Theta : \mathcal{W} \rightarrow B_\varepsilon(z)$ with $\Theta(0) = z$. Here \mathcal{W} varies with (z, J) , but one can choose a uniform $\varepsilon > 0$. This is since one can find uniform constants c and δ in proposition 2.3. (The map Θ varies with z via the

chart map $\phi_z : V \rightarrow \mathcal{L}$, $v \mapsto \phi(\phi^{-1}(z) + v)$ that is defined for sufficiently small V and that satisfies $\phi_z(0) = z$.) Moreover, one obtains the uniform estimate

$$\|\Theta^{-1}(x)\|_{Z \times Z} \leq C \|x - z\|_X \quad \forall x \in B_\varepsilon(z). \quad (8)$$

(Recall that Y is a closed subspace of the Banach space Z , so the norm on Y is induced by the norm on Z .)

Now consider a solution $u \in W^{k,q}(\Omega, X)$ of (2) for some $G \in W^{k,q}(\Omega, X)$ and $J \in W^{k+1,\infty}(\Omega, \text{End } X)$ as in theorem 1.2 (i). Fix any $(s_0, 0) \in K$ and let $z \equiv u(s_0, 0) \in \mathcal{L}$. Then the above construction of the coordinates Θ can be done for all $J = J_{s,t}$ with $(s, t) \in U$ for a neighbourhood $U \subset \Omega$ of $(s_0, 0)$. Thus one obtains a $W^{k+1,\infty}$ -family of chart maps for $(s, t) \in \bar{U}$,

$$\Theta_{s,t} : Y \times Y \supset \mathcal{W}_{s,t} \xrightarrow{\sim} B_\varepsilon(z_{s,t}).$$

Recall that u is either of class $W^{1,2p}$ or of class $W^{k,p}$ with $k \geq 2$ and $p > 1$. On the 2-dimensional domain Ω , the Sobolev embeddings thus ensure that u is continuous. So on a possibly even smaller neighbourhood U of $(s_0, 0)$ the map u can be expressed in local coordinates,

$$u(s, t) = \Theta_{s,t}(v(s, t)) \quad \forall (s, t) \in U,$$

where $v \in W^{k,q}(U, Z \times Z)$. This follows from the fact that the composition of the $W^{k+1,\infty}$ -map Θ^{-1} with a $W^{k,q}$ -map u is again $W^{k,q}$ -regular if $kq > 2$ (see e.g. [W1, Lemma B.8]). Moreover, v actually takes values in $\mathcal{W} \subset Y \times Y$.

In order to obtain the estimate in (ii), the map Θ has to be constructed independently of u and J , using the fixed u_0 and J_0 . In that case let $z_{s,t} := u_0(s, 0)$, which is welldefined on a small neighbourhood U of $(s_0, 0) \in K \subset \text{int } \Omega$. Then the coordinates $\Theta_{s,t}$ are defined for all $(s, t) \in U$ and for all complex structures in a sufficiently small neighbourhood of $J_0(s_0, 0)$. In particular, $\Theta_{s,t}$ is defined for all $J = J_{s,t}$ with $(s, t) \in U$, provided that $J \in W^{k+1,\infty}(\Omega, \text{End } X)$ satisfies the assumption $\|J - J_0\|_{L^\infty} \leq \delta$. Here one again makes sufficiently small choices of U and $\delta > 0$. Thus one obtains a $W^{k+1,\infty}$ -family of chart maps $\Theta_{s,t}$ as above that now also satisfy the uniform estimate (8) for all $(s, t) \in U$, where the constant C only depends on u_0 and J_0 . Now in order to again express u in local coordinates, choose U even smaller such that $u_0(s, t) \in B_{\frac{\varepsilon}{2}}(x_0)$ for all $(s, t) \in U$ and let $\delta \leq \frac{\varepsilon}{2}$. Then every $u \in W^{k,q}(\Omega, X)$ that satisfies $\|u - u_0\|_{L^\infty(\Omega, X)} \leq \delta$ can be written $u = \Theta \circ v$ as above. Now integration of (8) together with the fact that all derivatives of Θ^{-1} up to order k are bounded (due to the $W^{k,\infty}$ -bound on J) yields the estimate

$$\|v\|_{W^{k,q}(U, Z \times Z)} \leq C \|u - u_0\|_{W^{k,q}(U, X)}.$$

Here and in the following C denotes any constant that is independent of the specific choices of J and u in the fixed neighbourhoods of J_0 and u_0 , however, it may depend on c and k .

In the coordinates constructed above, the boundary value problem (7) now becomes

$$\begin{cases} \partial_s v + I \partial_t v = f, \\ v_2(s, 0) = 0 \quad \forall s \in \mathbb{R}. \end{cases} \quad (9)$$

with $v = (v_1, v_2)$ and

$$\begin{aligned} f &= (d_v \Theta)^{-1} (G - \partial_s \Theta(v) - J \partial_t \Theta(v)) \in W^{k,q}(U, Y \times Y), \\ I &= (d_v \Theta)^{-1} J d_v \Theta \in W^{k,q}(U, \text{End}(Y \times Y)). \end{aligned}$$

Note the following difficulty: The complex structure I now explicitly depends on the solution v of the equation (9) and thus is only $W^{k,q}$ -regular. This cannot be avoided when straightening out the Lagrangian boundary condition. However, one obtains one more simplification of the boundary value problem: Θ was constructed such that one obtains the standard complex structure along \mathcal{L} . Indeed, for all $(s, 0) \in U$ using that $J^2 = -\mathbf{1}$

$$\begin{aligned} I(s, 0) &= (d_{(v_1, 0)} \Theta)^{-1} J d_{(v_1, 0)} \Theta = (d_{v_1} \phi \oplus J d_{v_1} \phi)^{-1} J (d_{v_1} \phi \oplus J d_{v_1} \phi) \\ &= \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} =: I_0. \end{aligned}$$

Moreover, in case (ii) one has the following estimates on U :

$$\begin{aligned} \|I\|_{W^{k,q}} &\leq C, \\ \|f\|_{W^{k,q}} &\leq C(\|G\|_{W^{k,q}} + \|u - u_0\|_{W^{k,q}}). \end{aligned}$$

So for every boundary point $(s_0, 0) \in K \cap \partial\mathbb{H}$ we have rewritten the boundary value problem (7) over some neighbourhood $U \subset \Omega$. Now for the compact set $K \subset \Omega$ one finds a covering $K \subset V \cup \bigcup_{i=1}^N U_i$ by finitely many such neighbourhoods U_i at the boundary and a compact domain $V \subset \Omega \setminus \partial\Omega$ away from the boundary. Note that the U_i can be replaced by interior domains \tilde{U}_i (that intersect ∂U_i only on $\partial\mathbb{H}$) that together with V still cover K . We will establish the regularity and estimate for u on all domains \tilde{U}_i near the boundary and on the remaining domain V separately. So firstly consider a domain U_i near the boundary and drop the subscript i . After possibly replacing U by a slightly smaller domain one can assume that U is a manifold with smooth boundary and still $\tilde{U} \cap \partial U \subset \partial\mathbb{H}$. The task is now to prove the regularity and estimate for $u = \Theta \circ v$ on \tilde{U} from (9).

Since $\Theta_{s,t} : Y \times Y \rightarrow X$ are smooth maps in $W^{k+1,\infty}$ -dependence on $(s, t) \in U$, it suffices to prove that $v \in W^{k+1,p}(\tilde{U}, Z \times Z)$ with the according estimate. (One already knows that v takes values – almost everywhere – in $Y \times Y$, so one automatically also obtains $v \in W^{k+1,p}(\tilde{U}, Y \times Y)$.) For that purpose fix a cutoff function $h \in C^\infty(\mathbb{H}, [0, 1])$ with $h \equiv 1$ on \tilde{U} and $h \equiv 0$ on $\mathbb{H} \setminus U$. Moreover, this function can be chosen such that $\partial_t h|_{t=0} = 0$. Note that $h \equiv 0$ on $\partial U \setminus \partial\mathbb{H}$, so $h v_2$ satisfies the Dirichlet boundary condition on ∂U . Indeed, we will see that $h v_2 \in W^{k,p}(U, Z)$ solves a weak Dirichlet problem.

In the following, Y^* denotes the dual space of Y and we write $\langle \cdot, \cdot \rangle$ for both the pairings between Y and Y^* and between $Y \times Y$ and $Y^* \times Y^*$. We obtain $\Delta = -(\partial_s + \partial_t I^*)(\partial_s - I^* \partial_t) + (\partial_t I^*) \partial_s - (\partial_s I^*) \partial_t$ for $I^* \in W^{k,q}(\Omega, \text{End}(Y^* \times Y^*))$ the pointwise dual operator of I . Thus for all $\phi \in C^\infty(\Omega, Y^* \times Y^*)$

$$\begin{aligned} h \Delta \phi &= -(\partial_s + \partial_t I^*)(\partial_s - I^* \partial_t)(h \phi) - (\Delta h) \phi + 2(\partial_s h) \partial_s \phi + 2(\partial_t h) \partial_t \phi \\ &\quad + (\partial_t I^*) \partial_s (h \phi) - (\partial_s I^*) \partial_t (h \phi). \end{aligned}$$

Hence (9) and partial integration (for smooth approximations of v, f, I) yields

$$\begin{aligned}
& \int_U \langle hv, \Delta\phi \rangle \\
&= \int_U \langle \partial_s v + I\partial_t v, (\partial_s - I^* \partial_t)(h\phi) \rangle \\
&\quad - \int_U \langle (\Delta h)v + 2(\partial_s h)\partial_s v + 2(\partial_t h)\partial_t v + h(\partial_t I)\partial_s v - h(\partial_s I)\partial_t v, \phi \rangle \\
&\quad + \int_{\partial U \cap \partial \mathbb{H}} \langle Iv, (\partial_s - I^* \partial_t)(h\phi) \rangle + \langle h(\partial_s I)v - 2(\partial_t h)v, \phi \rangle \\
&= \int_U \langle h(-\partial_s f + I\partial_t f + (\partial_t I)f - (\partial_t I)\partial_s v + (\partial_s I)\partial_t v) \\
&\quad \quad \quad - (\Delta h)v - 2(\partial_s h)\partial_s v - 2(\partial_t h)\partial_t v, \phi \rangle \\
&\quad + \int_{\partial U \cap \partial \mathbb{H}} \langle h \cdot If, \phi \rangle + \int_{\partial U \cap \partial \mathbb{H}} \langle v, \partial_t(h\phi) \rangle + \langle Iv, \partial_s(h\phi) \rangle \\
&= \int_U \langle F, \phi \rangle + \int_{\partial U} \langle H, \phi \rangle + \int_{\partial U \cap \partial \mathbb{H}} \langle v_1, \partial_t(h\phi_1) + \partial_s(h\phi_2) \rangle. \tag{10}
\end{aligned}$$

This uses the notation $\phi = (\phi_1, \phi_2)$, the boundary condition $v_2|_{t=0} = 0$, and the fact that $I|_{t=0} \equiv I_0$. One then reads off $F = (F_1, F_2) \in W^{k-1,p}(U, Y \times Y)$, $H = (H_1, H_2) \in W^{k,p}(U, Y \times Y)$, and that in case (ii) for some constants C

$$\begin{aligned}
\|F\|_{W^{k-1,p}} + \|H\|_{W^{k,p}} &\leq C(\|f\|_{W^{k,q}} + \|I\|_{W^{k,q}}\|f\|_{W^{k,q}} + \|I\|_{W^{k,q}}\|v\|_{W^{k,q}}) \\
&\leq C(\|G\|_{W^{k,q}} + \|u - u_0\|_{W^{k,q}}).
\end{aligned}$$

We point out that the crucial terms here are $(\partial_s I)\partial_t v$ and $(\partial_t I)\partial_s v$. In the case $k \geq 3$ the estimate holds with $q = p$ due to the Sobolev embedding $W^{k-1,p} \cdot W^{k-1,p} \hookrightarrow W^{k-1,p}$. In the case $k = 1$ one only has $L^{2p} \cdot L^{2p} \hookrightarrow L^p$ and hence one needs $q = 2p$ in the above estimate. **In the case $k = 2$ the Sobolev embedding $W^{1,q} \cdot W^{1,q} \hookrightarrow W^{1,p}$ holds with $q = p$ for $p > 2$, with any $q > 2$ for $p = 2$, and requires $q \geq \frac{4p}{2+p}$ for $p < 2$.**

Now in order to obtain a weak Laplace equation for v_2 we test the weak equation (10) with $\phi = (\phi_1, \phi_2) = (0, \pi \circ \psi)$ for $\psi \in \mathcal{C}_\delta^\infty(U, Z^*)$ and where $\pi : Z^* \rightarrow Y^*$ is the canonical embedding. In that case, both boundary terms vanish and one obtains for all $\psi \in \mathcal{C}_\delta^\infty(U, Z^*)$

$$\int_U \langle hv_2, \Delta\psi \rangle = \int_U \langle F_2, \psi \rangle.$$

By lemma 2.1 (i) this weak equation for $hv_2 \in W^{k,p}(U, Z)$ now implies that $hv_2 \in W^{k+1,p}(U, Z)$ and thus $v_2 \in W^{k+1,p}(\bar{U}, Z)$. Moreover, one obtains the estimate

$$\begin{aligned}
\|v_2\|_{W^{k+1,p}(\bar{U}, Z)} &\leq \|hv_2\|_{W^{k+1,p}(U, Z)} \leq C\|F_2\|_{W^{k-1,p}(U, Z)} \\
&\leq C(\|G\|_{W^{k,q}(\Omega, X)} + \|u - u_0\|_{W^{k,q}(\Omega, X)}).
\end{aligned}$$

To obtain a weak Laplace equation for v_1 we test the weak equation (10) with $\phi = (\phi_1, \phi_2) = (\pi \circ \psi, 0)$, where $\psi \in \mathcal{C}^\infty(U, Z^*)$ such that $\partial_t \psi|_{t=0} = 0$. This makes the second boundary term vanish, so we obtain for all $\psi \in \mathcal{C}_\nu^\infty(U, Z^*)$

$$\int_{\Omega} \langle hv_1, \Delta \psi \rangle = \int_{\Omega} \langle F_1, \psi \rangle + \int_{\partial\Omega} \langle H_1, \psi \rangle.$$

So we have established a weak Laplace equation with Neumann boundary condition for hv_1 . Now lemma 2.1 (ii) implies that $hv_1 \in W^{k+1,p}(U, Z)$, hence $v_1 \in W^{k+1,p}(\tilde{U}, Z)$. Moreover, one obtains the estimate

$$\begin{aligned} \|v_1\|_{W^{k+1,p}(\tilde{U}, Z)} &\leq \|hv_1\|_{W^{k+1,p}(U, Z)} \\ &\leq C(\|F_1\|_{W^{k-1,p}(U, Z)} + \|H_1\|_{W^{k,p}(U, Z)} + \|hv_1\|_{W^{k,p}(U, Z)}) \\ &\leq C(\|G\|_{W^{k,q}(\Omega, X)} + \|u - u_0\|_{W^{k,q}(\Omega, X)}). \end{aligned}$$

This now provides the regularity and the estimate for $u = \Theta \circ v$ on \tilde{U} as follows. We have established that $v : \tilde{U} \rightarrow Z \times Z$ is a $W^{k+1,p}$ -map that takes values in $\mathcal{W} \subset Y \times Y$. All derivatives of $\Theta : \Omega \times \mathcal{W} \rightarrow X$ up to order $k+1$ are uniformly bounded on Ω . Hence $u \in W^{k+1,p}(\tilde{U}, X)$ and⁵

$$\begin{aligned} \|u - u_0\|_{W^{k+1,p}(\tilde{U}, X)} &\leq C(\|v\|_{W^{k+1,p}(\tilde{U}, X)}) \\ &\leq C(\|G\|_{W^{k,q}(\Omega, X)} + \|u - u_0\|_{W^{k,q}(\Omega, X)}). \end{aligned}$$

For the regularity of u on the domain $V \subset \Omega \setminus \partial\Omega$ away from the boundary one does not need any special coordinates. As for U , one replaces Ω by a possibly smaller domain with smooth boundary. Moreover, one chooses a cutoff function $h \in \mathcal{C}^\infty(\mathbb{H}, [0, 1])$ such that $h|_V \equiv 1$ and that vanishes outside of $\Omega \subset \mathbb{H}$ and in a neighbourhood of $\partial\Omega$. Then in the same way as for (10) one obtains a weak Dirichlet equation. For all $\phi \in \mathcal{C}_\delta^\infty(\Omega, X^*)$

$$\begin{aligned} \int_{\Omega} \langle hu, \Delta \phi \rangle &= \int_{\Omega} \langle h(-\partial_s G + J\partial_t G + (\partial_t J)G - (\partial_t J)\partial_s u + (\partial_s J)\partial_t u) \\ &\quad - (\Delta h)u - 2(\partial_s h)\partial_s u - 2(\partial_t h)\partial_t u, \phi \rangle. \end{aligned}$$

Note that $X \cong Y \times Y \subset Z \times Z$ also is bounded isomorphic to a closed subspace of an L^p -space. So by lemma 2.1 this weak equation implies that $hu \in W^{k+1,p}(\Omega, X)$, and thus $u \in W^{k+1,p}(V, X)$ with the estimate

$$\|u\|_{W^{k+1,p}(V, X)} \leq C(\|G\|_{W^{k,q}(\Omega, X)} + \|u - u_0\|_{W^{k,q}(\Omega, X)}).$$

(Note that here it suffices to have a $W^{k,\infty}$ -bound on J .) Thus we have proven the regularity and estimates of u on all parts of the finite covering $K \subset V \cup \bigcup_{i=1}^N U_i$, which finishes the proof of the theorem. \square

⁵Here and in the following we dropped a constant term $C(1 + \dots)$ from the estimate.

Proof of theorem 1.3 :

The Banach manifold charts near the path $x : \mathbb{R} \rightarrow \mathcal{L}$ give rise to a $W^{1,\infty}$ -path of isomorphisms $\phi_s : Y \xrightarrow{\sim} T_{x(s)}\mathcal{L}$ for all $s \in \mathbb{R}$. Together with the family of complex structures $J \in W^{1,\infty}(\overline{\Omega}, \text{End } X)$ these give rise to a family $\Theta \in W^{1,\infty}(\overline{\Omega}, \text{Hom}(Y \times Y, X))$ of bounded isomorphisms

$$\Theta_{s,t} : \begin{array}{ccc} Y \times Y & \xrightarrow{\sim} & X \\ (z_1, z_2) & \longmapsto & \phi_s(z_1) + J_{s,t}\phi_s(z_2). \end{array}$$

The inverses of the dual operators of $\Theta_{s,t}$ give a family of bounded isomorphisms $\Theta' \in W^{1,\infty}(\overline{\Omega}, \text{Hom}(Y^* \times Y^*, X^*))$,

$$\Theta'_{s,t} := (\Theta_{s,t}^*)^{-1} : Y^* \times Y^* \xrightarrow{\sim} X^*.$$

One checks that for all $(s, t) \in \overline{\Omega}$

$$\Theta_{s,t}^{-1} J_{s,t} \Theta_{s,t} = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} =: I_0 \in \text{End}(Y \times Y).$$

Next, after possibly replacing Ω by a slightly smaller domain that still contains K in its interior, one can assume that Ω is a manifold with smooth boundary. Then fix a cutoff function $h \in C^\infty(\mathbb{H}, [0, 1])$ such that $h|_K \equiv 1$ and $\text{supp } h \subset \Omega$, i.e. $h \equiv 0$ near $\partial\Omega \setminus \partial\mathbb{H}$. Now let $u \in L^p(\Omega, X)$ be given as in the theorem and express it in the above coordinates as $u = \Theta \circ v$, where $v \in L^p(\Omega, Y \times Y)$. We will show that v satisfies a weak Laplace equation. For all $\phi \in C^\infty(\Omega, Y^* \times Y^*)$ we introduce $\psi := \Theta'((\partial_s + I_0\partial_t)\phi) \in W^{1,\infty}(\overline{\Omega}, X^*)$ and calculate

$$\begin{aligned} \partial_s(h\psi) + \partial_t(J^*h\psi) &= h\Theta'((\partial_s + I_0^*\partial_t)(\partial_s + I_0\partial_t)\phi) \\ &\quad + (\partial_s h)\psi + \partial_t(hJ^*)\psi + h(\partial_s\Theta' + J^*\partial_t\Theta')\Theta'^{-1}(\psi). \end{aligned}$$

If $\psi(s, 0) \in (J(s, 0)T_{x(s)}\mathcal{L})^\perp$ for all $(s, 0) \in \partial\Omega \cap \partial\mathbb{H}$, then $h\psi$ is an admissible test function in the given weak estimate for u in the theorem and we obtain, denoting all constants by C and using $\Theta^*\Theta' = \text{id}$,

$$\begin{aligned} \left| \int_\Omega \langle hv, \Delta\phi \rangle \right| &= \left| \int_\Omega \langle \Theta(v), h\Theta'((-\partial_s + I_0\partial_t)(\partial_s + I_0\partial_t)\phi) \rangle \right| \\ &= \left| \int_\Omega \langle u, \partial_s(h\psi) + \partial_t(J^*h\psi) \rangle \right| \\ &\quad + \left| \int_\Omega \langle u, (\partial_s h)\psi + \partial_t(hJ^*)\psi + h(\partial_s\Theta' + J^*\partial_t\Theta')\Theta'^{-1}(\psi) \rangle \right| \\ &\leq (c_u + C\|u\|_{L^p(\Omega, X)})\|\psi\|_{L^{p^*}(\Omega, X^*)} \\ &\leq C(c_u + \|u\|_{L^p(\Omega, X)})\|\phi\|_{W^{1,p^*}(\Omega, Y^* \times Y^*)}. \end{aligned}$$

Here we used the fact that J^* and Θ' as well as their first derivatives and inverses are bounded linear operators between $Y^* \times Y^*$ and X^* . This inequality then holds for all $\phi = (\phi_1, \phi_2)$ with $\phi_1 \in C_\nu^\infty(\Omega, Y^*)$ and $\phi_2 \in C_\delta^\infty(\Omega, Y^*)$ since in

that case ψ is admissible. Indeed, $\psi|_{t=0} = \Theta'(\partial_s \phi_1 - \partial_t \phi_2, 0) \in (JT_x \mathcal{L})^\perp$ due to $\Theta'(Y^* \times \{0\}) = \Theta'(I_0(Y \times \{0\}))^\perp = (JT_x \mathcal{L})^\perp$.

Recall that $Y \subset Z$ is a closed subset of the Banach space Z with the induced norm. So one has $v \in L^p(\Omega, Z \times Z)$. Let $\pi : Z^* \rightarrow Y^*$ be the natural embedding, then above inequality holds with $\phi = (\pi \circ \psi_1, \pi \circ \psi_2)$ for all $\psi_1 \in \mathcal{C}_\nu^\infty(\Omega, Z^*)$ and $\psi_2 \in \mathcal{C}_\delta^\infty(\Omega, Z^*)$. Since $\|\pi \circ \psi_i\|_{Y^*} \leq \|\psi_i\|_{Z^*}$ one then obtains for all such $\Psi = (\psi_1, \psi_2) \in \mathcal{C}^\infty(\Omega, Z^* \times Z^*)$

$$\left| \int_{\Omega} \langle hv, \Delta \Psi \rangle \right| \leq C(c_u + \|u\|_{L^p(\Omega, X)}) \|\Psi\|_{W^{1,p^*}(\Omega, Z^* \times Z^*)}.$$

Now lemma 2.1 (iii) and (iv) asserts the $W^{1,p}$ -regularity of hv and hence one obtains $v \in W^{1,p}(\Omega, Z \times Z)$ with the estimate

$$\|v\|_{W^{1,p}(K, Z \times Z)} \leq \|hv\|_{W^{1,p}(\Omega, Z \times Z)} \leq C(c_u + \|u\|_{L^p(\Omega, X)}) + \|v\|_{L^p(\Omega, Z \times Z)}.$$

For the first factor of $Z \times Z$, this follows from lemma 2.1 (iv), in the second factor one uses (iii). Since it was already known that v takes values in $Y \times Y$ (almost everywhere), one in fact has $v \in W^{1,p}(\Omega, Y \times Y)$ with the same estimate as above. Finally, recall that $u = \Theta \circ v$ and use the fact that all derivatives up to first order of Θ and Θ^{-1} are bounded to obtain $u \in W^{1,p}(K, X)$ with the claimed estimate (using again [W1, Lemma B.8])

$$\|u\|_{W^{1,p}(K, X)} \leq C\|v\|_{W^{1,p}(K, Z \times Z)} \leq C(c_u + \|u\|_{L^p(\Omega, X)}).$$

□

Proof of corollary 1.4:

Let $u \in W^{1,p}(\Omega, X)$ and $\psi \in W^{1,\infty}(\Omega, X^*)$ such that $\text{supp } \psi \subset \text{int } \Omega$ and with the boundary conditions $u(s, 0) \in T_{x(s)} \mathcal{L}$ and $\psi(s, 0) \in (J(s, 0)T_{x(s)} \mathcal{L})^\perp$ for all $(s, 0) \in \partial\Omega$. Then one obtains the weak estimate, where the boundary term vanishes,

$$\begin{aligned} \left| \int_{\Omega} \langle u, \partial_s \psi + \partial_t(J^* \psi) \rangle \right| &= \left| \int_{\Omega} \langle \partial_s u + J \partial_t u, \psi \rangle - \int_{\partial\Omega \cap \partial\mathbb{H}} \langle Ju, \psi \rangle \right| \\ &\leq \|\partial_s u + J \partial_t u\|_{L^p(\Omega, X)} \|\psi\|_{L^{p^*}(\Omega, X^*)}. \end{aligned}$$

This holds for all ψ as above, so the estimate follows from theorem 1.3. □

3 Weakly flat connections

In this section we consider the trivial G -bundle over a closed manifold Σ of dimension $n \geq 2$. Here G is a compact Lie group with Lie algebra \mathfrak{g} . We recall that \mathfrak{g} is equipped with a Lie bracket $[\cdot, \cdot]$ and a G -invariant inner product $\langle \cdot, \cdot \rangle$ that moreover satisfy the relation

$$\langle [\xi, \eta], \zeta \rangle = \langle \xi, [\eta, \zeta] \rangle \quad \forall \xi, \zeta, \eta \in \mathfrak{g}.$$

A (smooth) connection on this bundle is a \mathfrak{g} -valued 1-form $A \in \Omega^1(\Sigma; \mathfrak{g})$. The exterior derivative d_A associated to it is given by $d_A \eta = d\eta + [A \wedge \eta]$ for all \mathfrak{g} -valued differential forms η . Here the Lie bracket indicates how the values of the differential forms are paired. Now $d_A \circ d_A$ vanishes if and only if the connection is flat, that is its curvature $F_A = dA + \frac{1}{2}[A \wedge A]$ vanishes.

Now fix $p > n$ and consider the space $\mathcal{A}^{0,p}(\Sigma) = L^p(\Sigma, T^*\Sigma \otimes \mathfrak{g})$ of L^p -regular connections. Their curvature is not welldefined, but the flatness condition can also be formulated weakly: A connection $A \in \mathcal{A}^{0,p}(\Sigma)$ is called **weakly flat** if

$$\int_{\Sigma} \langle A, d^* \omega - \frac{1}{2}(-1)^n * [A \wedge * \omega] \rangle = 0 \quad \forall \omega \in \Omega^2(\Sigma; \mathfrak{g}). \quad (11)$$

For sufficiently regular connections one sees by partial integration that (11) is equivalent to the connection being flat. We denote the space of weakly flat L^p -connections over Σ by

$$\mathcal{A}_{\text{flat}}^{0,p}(\Sigma) := \{A \in \mathcal{A}^{0,p}(\Sigma) \mid A \text{ satisfies (11)}\}.$$

One can check that this space is invariant under the action of the gauge group $\mathcal{G}^{1,p}(\Sigma) = W^{1,p}(\Sigma, G)$,

$$u^* A = u^{-1} A u + u^{-1} du \quad \forall A \in \mathcal{A}^{0,p}(\Sigma), u \in \mathcal{G}^{1,p}(\Sigma).$$

Note that (11) is welldefined for $p \geq 2$, but $\mathcal{G}^{1,p}(\Sigma)$ and its action on $\mathcal{A}^{0,p}(\Sigma)$ are only welldefined for $p > n$, see e.g. [W1, Appendix B]. The next theorem shows that the quotient $\mathcal{A}_{\text{flat}}^{0,p}(\Sigma)/\mathcal{G}^{1,p}(\Sigma)$ can be identified with the usual moduli space of flat connections $\mathcal{A}_{\text{flat}}(\Sigma)/\mathcal{G}(\Sigma)$ – smooth flat connections modulo smooth gauge transformations.

Theorem 3.1 *For every weakly flat connection $A \in \mathcal{A}_{\text{flat}}^{0,p}(\Sigma)$ there exists a gauge transformation $u \in \mathcal{G}^{1,p}(\Sigma)$ such that $u^* A \in \mathcal{A}_{\text{flat}}(\Sigma)$ is smooth.*

The proof will be based on the following L^p -version of the local slice theorem, a proof of which can be found in [W1, Theorem 8.3].

Proposition 3.2 *Fix a reference connection $\hat{A} \in \mathcal{A}^{0,p}(\Sigma)$. Then there exists a constant $\delta > 0$ such that for every $A \in \mathcal{A}^{0,p}(\Sigma)$ with $\|A - \hat{A}\|_p \leq \delta$ there exists a gauge transformation $u \in \mathcal{G}^{1,p}(\Sigma)$ such that*

$$\int_{\Sigma} \langle u^* A - \hat{A}, d_{\hat{A}} \eta \rangle = 0 \quad \forall \eta \in \mathcal{C}^{\infty}(\Sigma, \mathfrak{g}). \quad (12)$$

Equivalently, one has for $v = u^{-1} \in \mathcal{G}^{1,p}(\Sigma)$

$$\int_{\Sigma} \langle v^* \hat{A} - A, d_A \eta \rangle = 0 \quad \forall \eta \in \mathcal{C}^{\infty}(\Sigma, \mathfrak{g}).$$

The weak flatness together with the weak Coulomb gauge condition (12) form an elliptic system, so theorem 3.1 is then a consequence of the regularity theory for the Laplace operator, or the Hodge decomposition of L^p -regular 1-forms.

Proof of theorem 3.1 :

Consider a weakly flat connection $A \in \mathcal{A}_{\text{flat}}^{0,p}(\Sigma)$. Let $\delta > 0$ be the constant from proposition 3.2 for the reference connection A and choose a smooth connection $\tilde{A} \in \mathcal{A}(\Sigma)$ such that $\|\tilde{A} - A\|_p \leq \delta$. Then by proposition 3.2 there exists a gauge transformation $u \in \mathcal{G}^{1,p}(\Sigma)$ such that

$$\int_{\Sigma} \langle u^* A - \tilde{A}, d_{\tilde{A}} \eta \rangle = 0 \quad \forall \eta \in \mathcal{C}^{\infty}(\Sigma, \mathfrak{g}).$$

Now [W1, Theorem 5.3(ii)] asserts that $\alpha := u^* A - \tilde{A} \in L^p(\Sigma, T^* \Sigma \otimes \mathfrak{g})$ is in fact smooth. (By the definition of Sobolev spaces via coordinate charts it suffices to prove the regularity and estimate for $\alpha(X)$, where $X \in \Gamma(T\Sigma)$ is any smooth vector field on Σ . Alternatively to this lemma – a consequence of the regularity theory for the Laplace operator – one can also deduce the regularity of α directly from the regularity of the Hodge decomposition.) This is due to the weak equations

$$\begin{aligned} \int_{\Sigma} \langle \alpha, d\eta \rangle &= - \int_{\Sigma} \langle *[\alpha \wedge * \tilde{A}], \eta \rangle & \forall \eta \in \mathcal{C}^{\infty}(\Sigma, \mathfrak{g}), \\ \int_{\Sigma} \langle \alpha, d^* \omega \rangle &= - \int_{\Sigma} \langle d\tilde{A} + \frac{1}{2}[u^* A \wedge u^* A], \omega \rangle & \forall \omega \in \Omega^2(\Sigma; \mathfrak{g}). \end{aligned}$$

Firstly, the inhomogeneous terms are of class $L^{\frac{p}{2}}$, hence the lemma asserts $W^{1, \frac{p}{2}}$ -regularity of α and $u^* A$. Now if $p \leq 2n$, then the Sobolev embedding gives L^{p_1} -regularity of $u^* A$ with $p_1 := \frac{np}{2n-p}$ (in case $p = 2n$ one can choose any $p_1 > 2n$). This is iterated to obtain L^{p_j} -regularity for the sequence $p_{j+1} = \frac{np_j}{2n-p_j}$ (or any $p_{j+1} > 2n$ in case $p_j \geq 2n$) with $p_0 = p$. One checks that $p_{j+1} \geq \theta p_j$ with $\theta = \frac{n}{2n-p} > 1$ due to $p > n$. So after finitely many steps this yields $W^{1,q}$ -regularity for some $q = \frac{pN}{2} > n$. The same is the case if $p > 2n$ at the beginning. Next, if $u^* A$ is of class $W^{k,q}$ for some $k \in \mathbb{N}$, then the inhomogeneous terms also are of class $W^{k,q}$ and the lemma asserts the $W^{k+1,q}$ -regularity of α and hence $u^* A$. Iterating this argument proves the smoothness of $u^* A = \tilde{A} + \alpha$. \square

Weakly flat connections over a Riemann surface

Now we consider more closely the special case when Σ is a Riemann surface. Theorem 3.1 shows that the injection $\mathcal{A}_{\text{flat}}(\Sigma)/\mathcal{G}(\Sigma) \hookrightarrow \mathcal{A}_{\text{flat}}^{0,p}(\Sigma)/\mathcal{G}^{1,p}(\Sigma)$ in fact is a bijection. These moduli spaces are identified and denoted by M_{Σ} . Furthermore, the holonomy induces an injection from M_{Σ} to the space of conjugacy classes of homomorphisms from $\pi_1(\Sigma)$ to G (see e.g. [DK, Proposition 2.2.3]),

$$M_{\Sigma} := \mathcal{A}_{\text{flat}}^{0,p}(\Sigma)/\mathcal{G}^{1,p}(\Sigma) \cong \mathcal{A}_{\text{flat}}(\Sigma)/\mathcal{G}(\Sigma) \hookrightarrow \text{Hom}(\pi_1(\Sigma), G)/\sim.$$

If G is connected and simply connected, then every G -bundle over a Riemann surface is automatically trivial and the holonomy in fact induces a bijection. If there exist nontrivial G -bundles with flat connections, then $\text{Hom}(\pi_1(\Sigma), G)/\sim$ is identified with the union of the moduli spaces for all such bundles. From this one sees that M_Σ is a finite dimensional singular manifold.

For $G = \text{SU}(2)$ for example, $M_\Sigma \cong \text{Hom}(\pi_1(\Sigma), \text{SU}(2))/\sim$ has singularities at the product connection and at the further reducible connections⁶ – corresponding to the connections for which the holonomy group is not $\text{SU}(2)$ but only $\{\mathbf{1}\}$ or is conjugate to the maximal torus $S^1 \subset \text{SU}(2)$.⁷ Away from these singularities, the dimension of M_Σ is $6g - 6$, where g is the genus of Σ . (The arguments in [DS1, §4] show that $T_{[A]}M_\Sigma \cong \ker d_A/\text{im } d_A = h_A^1$ has dimension $3 \cdot (2g - 2)$ at irreducible connections A .)

For the same reasons, the space of weakly flat connections $\mathcal{A}_{\text{flat}}^{0,p}(\Sigma)$ is in general not a Banach submanifold of $\mathcal{A}^{0,p}(\Sigma)$ but a principal bundle over a singular base manifold. To be more precise fix a point $z \in \Sigma$ and consider the space of based gauge transformations, defined as

$$\mathcal{G}_z^{1,p}(\Sigma) := \{u \in \mathcal{G}^{1,p}(\Sigma) \mid u(z) = \mathbf{1}\}.$$

This Lie group acts freely on $\mathcal{A}_{\text{flat}}^{0,p}(\Sigma)$. The quotient space $\mathcal{A}_{\text{flat}}^{0,p}(\Sigma)/\mathcal{G}_z^{1,p}(\Sigma)$ can be identified with $\text{Hom}(\pi_1(\Sigma), G)$ (or a subset thereof if there exist nontrivial G -bundles over Σ) via the holonomy based at z . This based holonomy map $\rho_z : \mathcal{A}_{\text{flat}}^{0,p}(\Sigma) \rightarrow \text{Hom}(\pi_1(\Sigma), G)$ is defined by first choosing a based gauge transformation that makes the connection smooth and then computing the holonomy around loops based at z . Now ρ_z gives $\mathcal{A}_{\text{flat}}^{0,p}(\Sigma)$ the structure of a principal bundle with fibre $\mathcal{G}_z^{1,p}(\Sigma)$ over the finite dimensional singular manifold $\text{Hom}(\pi_1(\Sigma), G)$ (or a subset thereof)

$$\mathcal{G}_z^{1,p}(\Sigma) \hookrightarrow \mathcal{A}_{\text{flat}}^{0,p}(\Sigma) \xrightarrow{\rho_z} \text{Hom}(\pi_1(\Sigma), G).$$

Note that this discussion does not require the Riemann surface Σ to be connected. Only when fixing a base point for the holonomy map and the based gauge transformations one has to adapt the definition. Whenever $\Sigma = \bigcup_{i=1}^n \Sigma_i$ has several connected components Σ_i , then ‘fixing a point $z \in \Sigma$ ’ implicitly means that one fixes a point $z_i \in \Sigma_i$ in each connected component. The group of based gauge transformations then becomes

$$\mathcal{G}_z^{1,p}(\bigcup_{i=1}^n \Sigma_i) := \{u \in \mathcal{G}^{1,p}(\Sigma) \mid u(z_i) = \mathbf{1} \quad \forall i = 1, \dots, n\}.$$

4 Lagrangians in the space of connections

Consider the trivial G -bundle over a (possibly disconnected) Riemann surface Σ of (total) genus g , where G is a compact Lie group with Lie algebra \mathfrak{g} . There

⁶A connection $A \in \mathcal{A}_{\text{flat}}(\Sigma)$ is called reducible if its isotropy subgroup of $\mathcal{G}(\Sigma)$ (the group of gauge transformations that leave A fixed) is not discrete.

⁷The holonomy group of a connection is given by the holonomies of all loops in Σ . Now the isotropy subgroup of $\mathcal{G}(\Sigma)$ of the connection is isomorphic to the centralizer of the holonomy group, see [DK, Lemma 4.2.8].

is a gauge invariant symplectic form ω on the space of connections $\mathcal{A}^{0,p}(\Sigma)$ for $p > 2$ defined as follows. For tangent vectors $\alpha, \beta \in L^p(\Sigma, T^*\Sigma \otimes \mathfrak{g})$ to the affine space $\mathcal{A}^{0,p}(\Sigma)$

$$\omega(\alpha, \beta) = \int_{\Sigma} \langle \alpha \wedge \beta \rangle. \quad (13)$$

The action of the infinite dimensional gauge group $\mathcal{G}^{1,p}(\Sigma)$ on the symplectic Banach space $(\mathcal{A}^{0,p}(\Sigma), \omega)$ is Hamiltonian with moment map $A \mapsto *F_A$ (more precisely, the equivalent weak expression in $(W^{1,p^*}(\Sigma, \mathfrak{g}))^*$). So the moduli space of flat connections $M_{\Sigma} = \mathcal{A}_{\text{flat}}^{0,p}(\Sigma)/\mathcal{G}^{1,p}(\Sigma)$ can be viewed as the symplectic quotient $\mathcal{A}^{0,p}(\Sigma)//\mathcal{G}^{1,p}(\Sigma)$ as was first observed by Atiyah and Bott [AB]. However, 0 is not a regular value of the moment map, so M_{Σ} is a singular symplectic manifold. Due to these singularities at the reducible connections the infinite dimensional setting suggests itself.

Note that for any metric on Σ the Hodge $*$ operator is an ω -compatible complex structure since $\omega(\cdot, *)$ is the L^2 -metric: For all $\alpha, \beta \in L^p(\Sigma, T^*\Sigma \otimes \mathfrak{g})$

$$\omega(\alpha, *\beta) = \int_{\Sigma} \langle \alpha \wedge *\beta \rangle = \langle \alpha, \beta \rangle_{L^2}. \quad (14)$$

Next, we call a Banach submanifold $\mathcal{L} \subset \mathcal{A}^{0,p}(\Sigma)$ Lagrangian if it is isotropic, i.e. $\omega|_{\mathcal{L}} \equiv 0$, and if $T_A\mathcal{L}$ is maximal for all $A \in \mathcal{L}$ in the following sense: If $\omega(T_A\mathcal{L}, \alpha) = \{0\}$ for some $\alpha \in \mathcal{A}^{0,p}(\Sigma)$, then $\alpha \in T_A\mathcal{L}$. In general, this condition does not imply that \mathcal{L} is also totally real with respect to any ω -compatible complex structure. However, we will only consider Lagrangian submanifolds $\mathcal{L} \subset \mathcal{A}^{0,p}(\Sigma)$ that are gauge invariant and contained in the space of weakly flat connections. These are automatically totally real with respect to the Hodge $*$ operator, as lemma 4.2 will show. It is based on the following twisted Hodge decomposition.

Lemma 4.1 *Fix a metric on Σ and let $A \in \mathcal{A}_{\text{flat}}^{0,p}(\Sigma)$. Then*

$$L^p(\Sigma, T^*\Sigma \otimes \mathfrak{g}) = d_A W^{1,p}(\Sigma, \mathfrak{g}) \oplus *d_A W^{1,p}(\Sigma, \mathfrak{g}) \oplus h_A^1,$$

with the finite dimensional space $h_A^1 = \ker d_A \cap \ker d_A^* \subset W^{1,q}(\Sigma, T^*\Sigma \otimes \mathfrak{g})$ for $\frac{1}{q} = \frac{1}{2} + \frac{1}{p}$.

Proof: Recall that $p > 2$, hence $d_A W^{1,2}(\Sigma, \mathfrak{g}) \subset L^2(\Sigma, T^*\Sigma \otimes \mathfrak{g})$ due to the Sobolev embedding $W^{1,2}(\Sigma) \hookrightarrow L^r(\Sigma)$ for any $r < \infty$. The weak flatness of A then implies that $d_A W^{1,2}(\Sigma, \mathfrak{g})$ and $*d_A W^{1,2}(\Sigma, \mathfrak{g})$ are L^2 -orthogonal. The orthogonal complement of their direct sum then exactly is h_A^1 . (To see that every L^2 -regular 1-form that is orthogonal to $\text{im } d_A$ and to $*\text{im } d_A$ is automatically $W^{1,q}$ -regular, one can use the regularity theory for the Laplace operator or the Hodge decomposition, or see e.g. [W1, Theorem 5.3].) Next, note that $h_A^1 \subset L^p(\Sigma, T^*\Sigma \otimes \mathfrak{g})$ due to the Sobolev embedding $W^{1,q}(\Sigma) \hookrightarrow L^p(\Sigma)$. Now the same regularity arguments as above show that the orthogonal decomposition

$$L^2(\Sigma, T^*\Sigma \otimes \mathfrak{g}) = d_A W^{1,2}(\Sigma, \mathfrak{g}) \oplus *d_A W^{1,2}(\Sigma, \mathfrak{g}) \oplus h_A^1 \quad (15)$$

restricts to the claimed decomposition of $L^p(\Sigma, T^*\Sigma \otimes \mathfrak{g})$. Finally, to see that h_A^1 is finite dimensional note that it is isomorphic to the cokernel of the operator $d_A \oplus *d_A : W^{1,p}(\Sigma, \mathfrak{g}) \times W^{1,p}(\Sigma, \mathfrak{g}) \rightarrow L^p(\Sigma, T^*\Sigma \otimes \mathfrak{g})$. Now this operator is a compact perturbation of the Fredholm operator $d \oplus *d$, hence its cokernel is finite dimensional. \square

Lemma 4.2 *Let $\mathcal{L} \subset \mathcal{A}^{0,p}(\Sigma)$ be a Lagrangian submanifold. Suppose that \mathcal{L} is invariant under the action of $\mathcal{G}^{1,p}(\Sigma)$ and that $\mathcal{L} \subset \mathcal{A}_{\text{flat}}^{0,p}(\Sigma)$. Then \mathcal{L} is totally real with respect to the Hodge $*$ operator for any metric on Σ .*

Proof: Pick any $A \in \mathcal{L}$ and denote $L := T_A \mathcal{L}$. Then we have to show that $\mathcal{A}^{0,p}(\Sigma) = L \oplus *L$. Firstly, the only element $\alpha \in L \cap *L$ in the intersection is $\alpha = 0$ since $*\alpha \in **L = L$ and thus $\|\alpha\|_{L^2}^2 = \omega(\alpha, *\alpha) = 0$.

Secondly, to see that the direct sum $L \oplus *L$ exhausts all of $\mathcal{A}^{0,p}(\Sigma)$, assume the contrary. Then there exists a nonzero linear functional ϕ on $\mathcal{A}^{0,p}(\Sigma)$ that vanishes on $L \oplus *L$. Due to the gauge invariance of \mathcal{L} one has $d_A W^{1,p}(\Sigma, \mathfrak{g}) \subset L$, so ϕ vanishes in particular on $d_A W^{1,p}(\Sigma, \mathfrak{g}) \oplus *d_A W^{1,p}(\Sigma, \mathfrak{g})$. Now recall the Hodge decomposition in lemma 4.1 and (15). It implies that ϕ has to be nonzero on h_A^1 and hence can be extended to a nonzero linear functional on $\mathcal{A}^{0,2}(\Sigma)$ that vanishes on $d_A W^{1,2}(\Sigma, \mathfrak{g}) \oplus *d_A W^{1,2}(\Sigma, \mathfrak{g})$. Thus the extended functional can be written as $\phi = \langle \alpha, \cdot \rangle_{L^2}$ for some $\alpha \in L^2(\Sigma, T^*\Sigma \otimes \mathfrak{g})$. But now the orthogonal decomposition (15) implies that $\alpha \in h_A^1 \subset \mathcal{A}^{0,p}(\Sigma)$. Now for all $\beta \in L = T_A \mathcal{L}$ one has

$$\omega(\beta, \alpha) = \langle \alpha, *\beta \rangle_{L^2} = \phi(*\beta) = 0.$$

The Lagrangian property of \mathcal{L} then implies that $\alpha \in L$ and hence

$$\|\alpha\|_{L^2}^2 = \langle \alpha, \alpha \rangle_{L^2} = \phi(\alpha) = 0.$$

This proves $\alpha = 0$ in contradiction to the assumption $\phi \neq 0$. Hence \mathcal{L} is indeed totally real with respect to the complex structure $*$, i.e. for all $A \in \mathcal{L}$

$$L^p(\Sigma, T^*\Sigma \otimes \mathfrak{g}) = T_A \mathcal{L} \oplus *T_A \mathcal{L}. \quad (16)$$

\square

The assumption $\mathcal{L} \subset \mathcal{A}_{\text{flat}}^{0,p}(\Sigma)$ directly implies that \mathcal{L} is gauge invariant if G is connected and simply connected. On the other hand, the gauge invariance of \mathcal{L} implies $\mathcal{L} \subset \mathcal{A}_{\text{flat}}^{0,p}(\Sigma)$ if the Lie bracket on G is nondegenerate (i.e. the center of G is discrete). So for example in the case $G = \text{SU}(2)$ both conditions are equivalent. We will always assume both conditions. Then moreover, \mathcal{L} descends to a (singular) submanifold of the (singular) moduli space of flat connections,

$$L := \mathcal{L}/\mathcal{G}^{1,p}(\Sigma) \subset \mathcal{A}_{\text{flat}}^{0,p}(\Sigma)/\mathcal{G}^{1,p}(\Sigma) =: M_\Sigma.$$

This submanifold is obviously isotropic, i.e. the symplectic structure induced by (13) on M_Σ vanishes on L . Moreover, its tangent spaces have half of the dimension of those of M_Σ , so $L \subset M_\Sigma$ is a Lagrangian submanifold. Indeed,

in the Hodge decomposition, lemma 4.1, $*d_A W^{1,p}(\Sigma, \mathfrak{g})$ is the complement of $\ker d_A = T_A \mathcal{A}_{\text{flat}}^{0,p}(\Sigma)$, $d_A W^{1,p}(\Sigma, \mathfrak{g})$ is the tangent space to the orbit of $\mathcal{G}^{1,p}(\Sigma)$ through A , and so $h_A^1 \cong T_{[A]} M_\Sigma$. Now compare this with the decomposition (16). Here $T_A \mathcal{L} = d_A W^{1,p}(\Sigma, \mathfrak{g}) \oplus V$, where the complement $V \subset \mathcal{A}^{0,p}(\Sigma)$ is finite dimensional and $V \oplus *V$ can replace h_A^1 in the Hodge decomposition. Thus $T_{[A]} \mathcal{L} \cong T_A \mathcal{L} / d_A W^{1,p}(\Sigma, \mathfrak{g}) \cong V$ must have half the dimension of h_A^1 .

Moreover, our assumptions on the Lagrangian submanifold ensure that the holonomy map $\rho_z : \mathcal{L} \rightarrow \text{Hom}(\pi_1(\Sigma), G)$ based at $z \in \Sigma$ is welldefined and invariant under the action of the based gauge group $\mathcal{G}_z^{1,p}(\Sigma)$. (The holonomy map and based gauge group are introduced in section 3.) Note that $\text{Hom}(\pi_1(\Sigma), G)$ naturally embeds into $\text{Hom}(\pi_1(\Sigma \setminus \{z\}), G)$, which is a smooth manifold diffeomorphic to G^{2g} . This gives $\text{Hom}(\pi_1(\Sigma), G)$ a differentiable structure (that is in fact independent of $z \in \Sigma$), however, it is a manifold with singularities. In the following lemma we list some crucial properties of the Lagrangian submanifolds. Here we use the notation

$$W_z^{1,p}(\Sigma, \mathfrak{g}) := \{\xi \in W^{1,p}(\Sigma, \mathfrak{g}) \mid \xi(z) = 0\}$$

for the Lie algebra $T_{\mathbf{1}} \mathcal{G}_z^{1,p}(\Sigma)$ of the based gauge group. (If Σ is not connected then as before one fixes a base point in each connected component and modifies the definition of $W_z^{1,p}(\Sigma, \mathfrak{g})$ accordingly.) Moreover, we will denote the differential of a map ϕ at a point x by $T_x \phi$ in order to distinguish it from the exterior differential on differential forms, d_A , associated with a connection A .

Lemma 4.3 *Let $\mathcal{L} \subset \mathcal{A}^{0,p}(\Sigma)$ be a Lagrangian submanifold and fix $z \in \Sigma$. Suppose that $\mathcal{L} \subset \mathcal{A}_{\text{flat}}^{0,p}(\Sigma)$ and that \mathcal{L} is invariant under the action of $\mathcal{G}^{1,p}(\Sigma)$. Then the following holds:*

- (i) $L := \mathcal{L} / \mathcal{G}_z^{1,p}(\Sigma)$ is a smooth manifold of dimension $m = g \cdot \dim G$ and the holonomy induces a diffeomorphism $\rho_z : L \rightarrow M$ to a submanifold $M \subset \text{Hom}(\pi_1(\Sigma), G)$.
- (ii) \mathcal{L} has the structure of a principal $\mathcal{G}_z^{1,p}(\Sigma)$ -bundle over M ,

$$\mathcal{G}_z^{1,p}(\Sigma) \hookrightarrow \mathcal{L} \xrightarrow{\rho_z} M.$$

- (iii) Fix $A \in \mathcal{L}$. Then there exists a local section $\phi : V \rightarrow \mathcal{L}$ over a neighbourhood $V \subset \mathbb{R}^m$ of 0 such that $\phi(0) = A$ and $\rho_z \circ \phi$ is a diffeomorphism to a neighbourhood of $\rho_z(A)$. This gives rise to Banach submanifold coordinates for $\mathcal{L} \subset \mathcal{A}^{0,p}(\Sigma)$, namely a smooth embedding

$$\Theta : \mathcal{W} \rightarrow \mathcal{A}^{0,p}(\Sigma)$$

defined on a neighbourhood $\mathcal{W} \subset W_z^{1,p}(\Sigma, \mathfrak{g}) \times \mathbb{R}^m \times W_z^{1,p}(\Sigma, \mathfrak{g}) \times \mathbb{R}^m$ of zero by

$$\Theta(\xi_0, v_0, \xi_1, v_1) := \exp(\xi_0)^* \phi(v_0) + *d_A \xi_1 + *T_0 \phi(v_1).$$

Moreover, if A is smooth, then the local section can be chosen such that the image is smooth, $\phi : V \rightarrow \mathcal{L} \cap \mathcal{A}(\Sigma)$. Now the same map Θ is a diffeomorphism between neighbourhoods of zero in $W_z^{1,p}(\Sigma, \mathfrak{g}^2) \times \mathbb{R}^{2m}$ and neighbourhoods of A in $\mathcal{A}^{0,p}(\Sigma)$ for all $p > 2$.

We postpone the proof and first note that this lemma shows that the Lagrangian submanifolds considered here all satisfy the crucial assumption for theorem 1.2 and 1.3.

Corollary 4.4 *Let $\mathcal{L} \subset \mathcal{A}^{0,p}(\Sigma)$ be as in lemma 4.3, then it satisfies (H_p) , i.e. \mathcal{L} is modelled on a closed subspace of an L^p -space.*

Proof of corollary 4.4:

The bundle structure of \mathcal{L} in lemma 4.3 (ii) shows that \mathcal{L} is modelled on $W_z^{1,p}(\Sigma, \mathfrak{g}) \times \mathbb{R}^m$. This is since the Banach manifold $\mathcal{G}_z^{1,p}(\Sigma)$ is modelled on $W_z^{1,p}(\Sigma, \mathfrak{g})$, which is a closed subspace of $W^{1,p}(\Sigma, \mathfrak{g})$. Now recall example 1.1 to see that \mathcal{L} indeed satisfies (H_p) . \square

The Banach submanifold charts Θ in lemma 4.3 (iii) are essentially the same as the charts Θ in the proof of theorem 1.2. In this special case, we have more detailed information on the structure of Θ , which is the main point in the proof of the following approximation result for $W^{1,p}$ -connections with Lagrangian boundary values.

Corollary 4.5 *Let $\mathcal{L} \subset \mathcal{A}^{0,p}(\Sigma)$ be as in lemma 4.3 and let*

$$\Omega \subset \mathbb{H} := \{(s, t) \in \mathbb{R}^2 \mid t \geq 0\}$$

be a compact submanifold. Suppose that $A \in \mathcal{A}^{1,p}(\Omega \times \Sigma)$ satisfies the boundary condition

$$A|_{(s,0) \times \Sigma} \in \mathcal{L} \quad \forall (s, 0) \in \partial\Omega. \quad (17)$$

Then there exists a sequence of smooth connections $A^\nu \in \mathcal{A}(\Omega \times \Sigma)$ that satisfy (17) and converge to A in the $W^{1,p}$ -norm.

Proof of corollary 4.5:

We decompose $A = \Phi ds + \Psi dt + B$ into two functions $\Phi, \Psi \in W^{1,p}(\Omega \times \Sigma, \mathfrak{g})$ and a family of 1-forms $B \in W^{1,p}(\Omega \times \Sigma, T^*\Sigma \otimes \mathfrak{g})$ on Σ such that $B(s, 0) \in \mathcal{L}$ for all $(s, 0) \in \partial\Omega$. Then it suffices to find an approximating sequence for B with Lagrangian boundary conditions on a neighbourhood of $\Omega \cap \partial\mathbb{H}$. This can be patched together with any smooth $W^{1,p}$ -approximation of B on the rest of Ω and can be combined with standard approximations of the functions Φ and Ψ to obtain the required approximation of A .

So fix any $(s_0, 0) \in \Omega \cap \partial\mathbb{H}$ and use theorem 3.1 to find $u_0 \in \mathcal{G}^{1,p}(\Sigma)$ such that $A_0 := u_0^* B(s_0, 0)$ is smooth. Lemma 4.3 (iii) gives a diffeomorphism $\Theta : \mathcal{W} \rightarrow \mathcal{V}$ between neighbourhoods $\mathcal{W} \subset W_z^{1,p}(\Sigma, \mathfrak{g}^2) \times \mathbb{R}^{2m}$ of zero and $\mathcal{V} \subset \mathcal{A}^{0,p}(\Sigma)$ of A_0 . This was constructed such that $\mathcal{C}^\infty(\Sigma, \mathfrak{g}^2) \times \mathbb{R}^{2m}$ is mapped to $\mathcal{A}(\Sigma)$ and such that $\Theta : \mathcal{W} \cap W_z^{2,p}(\Sigma, \mathfrak{g}^2) \times \mathbb{R}^{2m} \rightarrow \mathcal{V} \cap \mathcal{A}^{1,p}(\Sigma)$ also is

a diffeomorphism. Now note that $B \in \mathcal{C}^0(\Omega, \mathcal{A}^{0,p}(\Sigma))$. Hence there exists a neighbourhood $U \subset \Omega$ of $(s_0, 0)$ and one can choose a smooth gauge transformation $u \in \mathcal{G}(\Sigma)$ that is $W^{1,p}$ -close to u_0 such that $u^*B(s, t) \in \mathcal{V}$ for all $(s, t) \in U$. Now we define $\xi = (\xi_0, \xi_1) : U \rightarrow W_z^{1,p}(\Sigma, \mathfrak{g}^2)$ and $v = (v_0, v_1) : U \rightarrow \mathbb{R}^{2m}$ by $\Theta(\xi(s, t), v(s, t)) = u^*B(s, t)$. Recall that B is of class $W^{1,p}$ on $U \times \Sigma$, hence it lies in both $W^{1,p}(U, \mathcal{A}^{0,p}(\Sigma))$ and $L^p(U, \mathcal{A}^{1,p}(\Sigma))$. Thus $\xi \in W^{1,p}(U, W_z^{1,p}(\Sigma, \mathfrak{g}^2)) \cap L^p(U, W_z^{2,p}(\Sigma, \mathfrak{g}^2))$ and $v \in W^{1,p}(U, \mathbb{R}^{2m})$, and these satisfy the boundary conditions $\xi_1|_{t=0} = 0$ and $v_1|_{t=0} = 0$ due to the Lagrangian boundary condition for B . Now there exist $\xi^\nu \in \mathcal{C}^\infty(U \times \Sigma, \mathfrak{g}^2)$ and $v^\nu \in \mathcal{C}^\infty(U, \mathbb{R}^{2m})$ such that $\xi^\nu \rightarrow \xi$ and $v^\nu \rightarrow v$ in all these spaces, $\xi^\nu(\cdot, z) \equiv 0$, $\xi_1^\nu|_{t=0} = 0$, and $v_1^\nu|_{t=0} = 0$. (These are constructed with the help of mollifiers as in lemma 2.2. One first reflects ξ at the boundary and mollifies it with respect to U to obtain approximations in $\mathcal{C}^\infty(U, W_z^{2,p}(\Sigma, \mathfrak{g}^2))$ with zero boundary values. Next, one mollifies on Σ , and finally one corrects the value at z .) It follows that $B^\nu(s, t) := (u^{-1})^*\Theta(\xi^\nu(s, t), v^\nu(s, t))$ is a sequence of smooth maps from U to $\mathcal{A}(\Sigma)$ which satisfies the Lagrangian boundary condition and converges to B in the $W^{1,p}$ -norm.

Now $\Omega \cap \partial\mathbb{H}$ is compact, so it is covered by finitely many such neighbourhoods U_i on which there exist smooth $W^{1,p}$ -approximations of B with Lagrangian boundary values. These can be patched together in a finite procedure since the above construction allows to interpolate in the coordinates between ξ^ν, v^ν and other smooth approximations ξ', v' (arising from approximations of B on another neighbourhood U' in different coordinates) of ξ and v respectively. This gives the required approximation of B in a neighbourhood of $\Omega \cap \partial\mathbb{H}$ and thus finishes the proof. \square

Proof of lemma 4.3:

Fix $A \in \mathcal{L}$ and consider the following two decompositions:

$$\begin{aligned} L^p(\Sigma, T^*\Sigma \otimes \mathfrak{g}) &= T_A \mathcal{L} \oplus *T_A \mathcal{L} \\ &= d_A W_z^{1,p}(\Sigma, \mathfrak{g}) \oplus *d_A W_z^{1,p}(\Sigma, \mathfrak{g}) \oplus \tilde{h}_A. \end{aligned} \tag{18}$$

The first direct sum is due to lemma 4.2. In the second decomposition, \tilde{h}_A is a complement of the image of the following Fredholm operator:

$$D_A : \begin{array}{ccc} W_z^{1,p}(\Sigma, \mathfrak{g}) \times W_z^{1,p}(\Sigma, \mathfrak{g}) & \longrightarrow & L^p(\Sigma, T^*\Sigma \otimes \mathfrak{g}) \\ (\xi, \zeta) & \longmapsto & d_A \xi + *d_A \zeta. \end{array}$$

To see that D_A is Fredholm note that for every $A \in \mathcal{A}_{\text{flat}}^{0,p}(\Sigma)$ the operator D_A is injective and is a compact perturbation of D_0 . Hence the dimension of coker D_A (and thus of \tilde{h}_A) is the same as that of coker D_0 . In the case $A = 0$ one can choose the space of \mathfrak{g} -valued harmonic 1-forms $h^1 = \ker d \cap \ker d^*$ as complement \tilde{h}_0 . So \tilde{h}_A must always have the dimension $\dim \tilde{h}_A = \dim h^1 = 2g \cdot \dim \mathbb{G} = 2m$. (Note that in general one can choose \tilde{h}_A to contain h_A^1 , but this might not exhaust the whole complement.)

Due to the $\mathcal{G}_z^{1,p}(\Sigma)$ -invariance of \mathcal{L} the splittings (18) now imply that there exists an m -dimensional subspace $L_A \subset \tilde{h}_A$ such that

$$T_A \mathcal{L} = d_A W_z^{1,p}(\Sigma, \mathfrak{g}) \oplus L_A.$$

So $T_A \mathcal{L}$ is isomorphic to the Banach space $W_z^{1,p}(\Sigma, \mathfrak{g}) \times \mathbb{R}^m$ via $d_A \oplus F$ for some isomorphism $F : \mathbb{R}^m \rightarrow L_A$. Here we have used the fact that d_A is injective when restricted to $W_z^{1,p}(\Sigma, \mathfrak{g})$. Now choose a coordinate chart $\Phi : T_A \mathcal{L} \rightarrow \mathcal{L}$ defined near $\Phi(0) = A$, then the following map is defined for a sufficiently small neighbourhood $V \subset \mathbb{R}^m$ of 0,

$$\Psi : \begin{array}{ccc} \mathcal{G}_z^{1,p}(\Sigma) \times V & \longrightarrow & \mathcal{L} \\ (u, v) & \longmapsto & u^*(\Phi \circ (T_0 \Phi)^{-1} \circ F(v)). \end{array}$$

We will show that this is an embedding and a submersion (and thus a diffeomorphism to its image). Firstly, $T_{(\mathbb{1},0)} \Psi : (\xi, w) \mapsto d_A \xi + Fw$ is an isomorphism. Next, note that $\Psi(u, v) = u^* \Psi(\mathbb{1}, v)$ and use this to calculate for all $u \in \mathcal{G}_z^{1,p}(\Sigma)$, $\xi \in W_z^{1,p}(\Sigma, \mathfrak{g})$, and $v, w \in \mathbb{R}^m$

$$T_{(u,v)} \Psi : (\xi u, w) \mapsto u^{-1} (d_{\Psi(\mathbb{1},v)} \xi + T_{(\mathbb{1},v)} \Psi(0, w)) u.$$

One sees that $u(T_{(u,v)} \Psi)u^{-1}$ is a small perturbation of $T_{(\mathbb{1},0)} \Psi$, hence one can choose V sufficiently small (independently of u) such that $T_{(u,v)} \Psi$ also is an isomorphism for all $v \in V$. So it remains to check that Ψ in fact is globally injective.

Suppose that $u, u' \in \mathcal{G}_z^{1,p}(\Sigma)$ and $v, v' \in V$ such that $\Psi(u, v) = \Psi(u', v')$. Rewrite this as $\Psi(\mathbb{1}, v) = \Psi(\tilde{u}, v')$ with $\tilde{u} := u'u^{-1} \in \mathcal{G}_z^{1,p}(\Sigma)$. Now by the choice of a sufficiently small V the norm $\|\Psi(\mathbb{1}, v) - \Psi(\mathbb{1}, v')\|_p$ can be made arbitrarily small. Then the identity $\Psi(\mathbb{1}, v) = \tilde{u}^* \Psi(\mathbb{1}, v')$ automatically implies that \tilde{u} is \mathcal{C}^0 -close to $\mathbb{1}$. (Otherwise one would find a sequence of L^p -connections $A^\nu \rightarrow A$ and $u^\nu \in \mathcal{G}_z^{1,p}(\Sigma)$ such that $\|u^\nu * A^\nu - A^\nu\|_p \rightarrow 0$ but $d_{\mathcal{C}^0}(u^\nu, \mathbb{1}) \geq \Delta > 0$. However, from $(u^\nu)^{-1} du^\nu = u^\nu * A^\nu - (u^\nu)^{-1} A^\nu u^\nu$ one obtains an L^p -bound on du^ν and thus finds a weakly $W^{1,p}$ -convergent subsequence of the u^ν . Its limit $u \in \mathcal{G}_z^{1,p}(\Sigma)$ would have to satisfy $u^* A = A$, hence $u \equiv \mathbb{1}$ in contradiction to $d_{\mathcal{C}^0}(u, \mathbb{1}) \geq \Delta > 0$.) So one can write $\tilde{u} = \exp(\xi)$ where $\xi \in W_z^{1,p}(\Sigma, \mathfrak{g})$ is small in the L^∞ -norm. Next, the identity

$$\tilde{u}^{-1} d\tilde{u} = \Psi(\mathbb{1}, v) - \tilde{u}^{-1} \Psi(\mathbb{1}, v') \tilde{u}$$

shows that $\|\xi\|_{W^{1,p}}$ will be small if V is small (and thus \tilde{u} is \mathcal{C}^0 -close to $\mathbb{1}$). Hence if V is sufficiently small, then (\tilde{u}, v') and $(\mathbb{1}, v)$ automatically lie in a neighbourhood of $(\mathbb{1}, 0)$ on which Ψ is injective, and hence $u = u'$ and $v = v'$.

We have thus shown that $\Psi : \mathcal{G}_z^{1,p}(\Sigma) \times V \rightarrow \mathcal{L}$ is a diffeomorphism to its image. This provides manifold charts $\psi : V \rightarrow \mathcal{L}/\mathcal{G}_z^{1,p}(\Sigma)$, $v \mapsto [\Psi(\mathbb{1}, v)]$ for $L := \mathcal{L}/\mathcal{G}_z^{1,p}(\Sigma)$. Now fix $2g$ generators of the fundamental group $\pi_1(\Sigma)$ based at z , then the corresponding holonomy map $\rho_z : L \rightarrow \mathbb{G} \times \cdots \times \mathbb{G}$ is an embedding, so its image $M \subset \text{Hom}(\pi_1(\Sigma), \mathbb{G})$ is a smooth submanifold. This proves (i).

For (ii) the diffeomorphism Ψ gives a bundle chart over $\mathcal{U} := \rho_z(\psi(V)) \subset M$, namely

$$\Psi \circ (\text{id} \times (\rho_z \circ \psi)^{-1}) : \mathcal{G}_z^{1,p}(\Sigma) \times \mathcal{U} \longrightarrow \mathcal{L}.$$

Furthermore, the local section for (iii) is given by $\phi(v) := \Psi(\mathbb{1}, v)$. However, this is a map $\phi : V \rightarrow \mathcal{L}$; it does not necessarily take values in the smooth connections. Now if $A \in \mathcal{L} \cap \mathcal{A}(\Sigma)$ is smooth, then for a sufficiently small neighbourhood V this section can be modified by gauge transformations such that $\phi : V \rightarrow \mathcal{L} \cap \mathcal{A}(\Sigma)$. To see this, note that the gauge transformations in the local slice theorem are given by an implicit function theorem: One solves $D(v, \xi) = 0$ for $\xi = \xi(v) \in W^{1,p}(\Sigma, \mathfrak{g})$ with the following operator:

$$D : \begin{array}{ccc} V \times W^{1,p}(\Sigma, \mathfrak{g}) & \longrightarrow & \text{im } d'_A \subset (W^{1,p^*}(\Sigma, \mathfrak{g}))^* \\ (v, \xi) & \longmapsto & d'_A(\exp(\xi)^* \phi(v) - A) \end{array} .$$

Here d'_A denotes the dual operator of d_A on $W^{1,p^*}(\Sigma, \mathfrak{g})$. One has $D(0, 0) = 0$ and checks that $\partial_2 D(0, 0) : \xi \rightarrow d'_A d_A \xi$ is a surjective map to $\text{im } d'_A$, see e.g. [W1, Lemma 8.5]. The implicit function theorem [L, XIV, Theorem 2.1] then gives the required gauge transformations $\exp(\xi(v)) \in \mathcal{G}^{1,p}(\Sigma)$ that bring $\phi(v)$ into local Coulomb gauge and thus make it smooth. (By construction $\phi(v)$ is weakly flat, then see the proof of theorem 3.1.) This modification by gauge transformations however does not affect the topological direct sum decomposition $T_A \mathcal{L} = d_A W_z^{1,p}(\Sigma, \mathfrak{g}) \oplus \text{im } T_0 \phi$.

To see that the given map Θ is a diffeomorphism between neighbourhoods of 0 and A just note that the inverse of $T_0 \Theta$ is given by the splitting

$$\begin{aligned} L^p(\Sigma, T^* \Sigma \otimes \mathfrak{g}) &= T_A \mathcal{L} \oplus *T_A \mathcal{L} \\ &= d_A W_z^{1,p}(\Sigma, \mathfrak{g}) \oplus \text{im } T_0 \phi \oplus *d_A W_z^{1,p}(\Sigma, \mathfrak{g}) \oplus *\text{im } T_0 \phi \end{aligned}$$

composed with the inverses of $d_A|_{W_z^{1,p}(\Sigma, \mathfrak{g})}$ and $T_0 \phi$. \square

Now observe that the choice of $p > 2$ for the Lagrangian submanifolds in the above lemma is accidental. All connections $A \in \mathcal{L}$ are gauge equivalent to a smooth connection, and the L^q -completion of $\mathcal{L} \cap \mathcal{A}(\Sigma)$ is a Lagrangian submanifold in $\mathcal{A}^{0,q}(\Sigma)$ for all $q > 2$. In fact, this simply is the restricted ($q > p$) or completed ($q < p$) $\mathcal{G}_z^{1,q}(\Sigma)$ -bundle over M .

The main example

Suppose that G is connected and simply connected and that $\Sigma = \partial Y$ is the boundary of a handlebody Y . (Again, the handlebody and thus its boundary might consist of several connected components.) The crucial property of the handle body Y is that the inclusion $\iota : \Sigma \rightarrow Y$ induces an isomorphism $\pi_1(Y) \cong \pi_1(\Sigma) / \partial \pi_2(Y, \Sigma)$. This is since Y retracts onto its 1-skeleton, which can be chosen to lie in Σ , so we have the exact sequence

$$0 = \pi_2(Y) \rightarrow \pi_2(Y, \Sigma) \xrightarrow{\partial} \pi_1(\Sigma) \xrightarrow{\iota} \pi_1(Y) \rightarrow \pi_1(Y, \Sigma) = 0.$$

The assumptions on G together with the fact that $\pi_2(G) = 0$ for any Lie group G (see e.g. [B, Proposition 7.5]) ensure that the gauge group $\mathcal{G}^{1,p}(\Sigma)$ is connected and that every gauge transformation on Σ can be extended to Y .

Let $p > 2$ and let \mathcal{L}_Y be the $L^p(\Sigma)$ -closure of the set of smooth flat connections on Σ that can be extended to a flat connection on Y ,

$$\mathcal{L}_Y := \text{cl} \{A \in \mathcal{A}_{\text{flat}}(\Sigma) \mid \exists \tilde{A} \in \mathcal{A}_{\text{flat}}(Y) : \tilde{A}|_{\Sigma} = A\} \subset \mathcal{A}^{0,p}(\Sigma).$$

This is an example of a totally real submanifold of $(\mathcal{A}^{0,p}(\Sigma, \mathfrak{g}), *)$ that satisfies the assumption of theorem 1.2 and 1.3. This is due to the lemmata 4.2 and 4.3 and the following properties of \mathcal{L}_Y .

Lemma 4.6

- (i) $\mathcal{L}_Y = \{u^*(A|_{\Sigma}) \mid A \in \mathcal{A}_{\text{flat}}(Y), u \in \mathcal{G}^{1,p}(\Sigma)\}$
- (ii) $\mathcal{L}_Y \subset \mathcal{A}^{0,p}(\Sigma)$ is a Lagrangian submanifold.
- (iii) $\mathcal{L}_Y \subset \mathcal{A}_{\text{flat}}^{0,p}(\Sigma)$ and \mathcal{L}_Y is invariant under the action of $\mathcal{G}^{1,p}(\Sigma)$.
- (iv) Fix any $z \in \Sigma$. Then

$$\mathcal{L}_Y = \{A \in \mathcal{A}_{\text{flat}}^{0,p}(\Sigma) \mid \rho_z(A) \in \text{Hom}(\pi_1(Y), G) \subset \text{Hom}(\pi_1(\Sigma), G)\},$$

where we identify

$$\text{Hom}(\pi_1(Y), G) \cong \{\rho \in \text{Hom}(\pi_1(\Sigma), G) \mid \rho(\partial\pi_2(Y, \Sigma)) = \{\mathbf{1}\}\}.$$

So \mathcal{L}_Y obtains the structure of a $\mathcal{G}_z^{1,p}(\Sigma)$ -bundle over the g -fold product $M = G \times \dots \times G \cong \text{Hom}(\pi_1(Y), G)$,

$$\mathcal{G}_z^{1,p}(\Sigma) \hookrightarrow \mathcal{L}_Y \xrightarrow{\rho_z} \text{Hom}(\pi_1(Y), G).$$

Proof: Firstly, $\mathcal{L}_Y \subset \mathcal{A}_{\text{flat}}^{0,p}(\Sigma)$ follows from the fact that weak flatness is an L^p -closed condition for $p > 2$. The holonomy $\rho_z : \mathcal{A}_{\text{flat}}^{0,p}(\Sigma) \rightarrow G \times \dots \times G$ is continuous with respect to the L^p -topology. Thus for every $A \in \mathcal{L}_Y$ the holonomy vanishes on those loops in Σ that are contractible in Y . On the other hand, in view of theorem 3.1, every $A \in \mathcal{A}_{\text{flat}}^{0,p}(\Sigma)$ whose holonomy descends to $\text{Hom}(\pi_1(Y), G)$ can be written as $A = u^*\tilde{A}$, where $u \in \mathcal{G}_z^{1,p}(\Sigma)$ and the holonomy of $\tilde{A} \in \mathcal{A}_{\text{flat}}(\Sigma)$ also vanishes along the loops that are contractible in Y . Thus \tilde{A} can be extended to a flat connection on Y and smooth approximation of u proves that $A \in \mathcal{L}_Y$. This proves the alternative definitions of \mathcal{L}_Y in (iv) and (i). Then (iii) is a consequence of (i).

To prove the second assertion in (iv) we explicitly construct local sections of \mathcal{L}_Y . Let the loops $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g \subset \Sigma$ be disjoint from z and represent the standard generators of $\pi_1(\Sigma)$ such that $\alpha_1, \dots, \alpha_g$ generate $\pi_1(Y)$ and such that the only nonzero intersections are $\alpha_i \cap \beta_i$.⁸ One can then modify the α_i

⁸ $\pi_1(\Sigma)$ is the group generated by $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$ with the relation $\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1} \dots \alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1} = \mathbf{1}$, whereas $\pi_1(Y)$ is the free group generated by $\alpha_1, \dots, \alpha_g$.

such that they run through z but still do not intersect the β_j for $j \neq i$. Now fix $A \in \mathcal{L}_Y$. In order to change its holonomy along α_i by some $g \in \mathbf{G}$ close to $\mathbb{1}$, one gauge transforms A in a small neighbourhood of β_i in Σ with a smooth gauge transformation that equals $\mathbb{1}$ and g respectively near the two boundary components of that ring about β_i . That way one obtains a smooth local section $\phi : V \rightarrow \mathcal{L}_Y$ defined on a neighbourhood $V \subset \mathfrak{g}^g$ of 0, such that $\phi(0) = A$ and $\rho_z \circ \phi : V \rightarrow \text{Hom}(\pi_1(Y), \mathbf{G})$ is a bijection onto a neighbourhood of $\rho_z(A)$. This leads to a bundle chart

$$\Psi : \begin{array}{ccc} \mathcal{G}_z^{1,p}(\Sigma) \times V & \longrightarrow & \mathcal{L}_Y \\ (u, v) & \longmapsto & u^* \phi(v). \end{array}$$

Note that for smooth $A \in \mathcal{L}_Y \cap \mathcal{A}(\Sigma)$ the local section ϕ constructed above in fact is a section in the smooth part $\mathcal{L}_Y \cap \mathcal{A}(\Sigma)$ of the Lagrangian. Using these bundle charts one also checks that $\mathcal{L}_Y \subset \mathcal{A}^{0,p}(\Sigma)$ is indeed a Banach submanifold. Now a submanifold chart near $\Psi(u, v) \in \mathcal{A}^{0,p}(\Sigma)$ is given by $(\xi, w) \mapsto \Psi(\exp(\xi)u, v + w) + *T_{(u,v)}\Psi(\xi, w)$. As in lemma 4.3 one checks that this is a local diffeomorphism.

To verify the Lagrangian condition it suffices to consider ω on $T_A \mathcal{A}^{0,p}(\Sigma)$ for smooth $A \in \mathcal{L}_Y$. This is because both ω and \mathcal{L}_Y are invariant under the gauge action. So we pick some $A \in \mathcal{L}_Y \cap \mathcal{A}(\Sigma)$ and find $\tilde{A} \in \mathcal{A}_{\text{flat}}(Y)$ such that $A = \tilde{A}|_{\Sigma}$. Let $\alpha, \beta \in T_A \mathcal{L}_Y$, then by the characterization of \mathcal{L}_Y in (i) we find $\xi, \zeta \in W^{1,p}(\Sigma, \mathfrak{g})$ and paths $\tilde{A}^\alpha, \tilde{A}^\beta : [-1, 1] \rightarrow \mathcal{A}_{\text{flat}}(Y)$ such that $\tilde{A}^\alpha(0) = \tilde{A}^\beta(0) = \tilde{A}$ and

$$\alpha = d_A \xi + \left. \frac{d}{ds} \right|_{s=0} \tilde{A}^\alpha(s)|_{\Sigma}, \quad \beta = d_A \zeta + \left. \frac{d}{ds} \right|_{s=0} \tilde{A}^\beta(s)|_{\Sigma}.$$

Now firstly Stokes' theorem on Σ with $\partial\Sigma = \emptyset$ proves

$$\omega(d_A \xi, d_A \zeta) = \lim_{\nu \rightarrow \infty} \int_{\Sigma} \langle d_A \xi^\nu \wedge d_A \zeta^\nu \rangle = \lim_{\nu \rightarrow \infty} \int_{\Sigma} d \langle \xi^\nu \wedge \zeta^\nu \rangle = 0.$$

Here we have used smooth $W^{1,p}$ -approximations ξ^ν and ζ^ν of ξ and ζ respectively.

Similarly, one obtains $\omega(d_A \xi, \left. \frac{d}{ds} \tilde{A}^\beta \right|_{\Sigma}) = 0$ and $\omega(\left. \frac{d}{ds} \tilde{A}^\alpha \right|_{\Sigma}, d_A \zeta) = 0$ since $d_A \left(\left. \frac{d}{ds} \tilde{A}^\alpha \right|_{\Sigma} \right) = \left. \frac{d}{ds} F_{\tilde{A}^\alpha} \right|_{\Sigma} = 0$. Finally, Stokes' theorem with $\partial Y = \Sigma$ yields due to $F_{\tilde{A}^\alpha(s)} = 0$ for all s

$$\begin{aligned} \omega(\alpha, \beta) &= \int_{\Sigma} \langle \left. \frac{d}{ds} \tilde{A}^\alpha \right|_{\Sigma} \wedge \left. \frac{d}{ds} \tilde{A}^\beta \right|_{\Sigma} \rangle = \int_Y d \langle \left. \frac{d}{ds} \tilde{A}^\alpha \right|_{\Sigma} \wedge \left. \frac{d}{ds} \tilde{A}^\beta \right|_{\Sigma} \rangle \\ &= \int_Y \langle \left. \frac{d}{ds} F_{\tilde{A}^\alpha} \right|_{\Sigma} \wedge \left. \frac{d}{ds} \tilde{A}^\beta \right|_{\Sigma} \rangle - \int_Y \langle \left. \frac{d}{ds} \tilde{A}^\alpha \right|_{\Sigma} \wedge \left. \frac{d}{ds} F_{\tilde{A}^\beta} \right|_{\Sigma} \rangle = 0. \end{aligned}$$

This proves that $\omega|_{T_A \mathcal{L}_Y} = 0$ and recalling (14) one moreover sees that $T_A \mathcal{L}_Y$ and $*T_A \mathcal{L}_Y$ are L^2 -orthogonal. In fact, we even have the topological decomposition $L^p(\Sigma, T^* \Sigma \otimes \mathfrak{g}) = T_A \mathcal{L}_Y \oplus *T_A \mathcal{L}_Y$, and this proves the Lagrangian property of \mathcal{L}_Y . To see that this direct sum indeed exhausts the whole space consider the Hodge type decomposition as in the proof of lemma 4.3,

$$L^p(\Sigma, T^* \Sigma \otimes \mathfrak{g}) = d_A W_z^{1,p}(\Sigma, \mathfrak{g}) \oplus *d_A W_z^{1,p}(\Sigma, \mathfrak{g}) \oplus \tilde{h}_A.$$

Here we have $\dim \tilde{h}_A = 2g \cdot \dim G$, and we have already seen that \mathcal{L}_Y is a $\mathcal{G}_z^{1,p}(\Sigma)$ -bundle over the $(g \cdot \dim G)$ -dimensional manifold $\text{Hom}(\pi_1(Y), G)$. So $d_A W_z^{1,p}(\Sigma, \mathfrak{g}) \subset T_A \mathcal{L}_Y$ is the tangent space to the fibre through A , and then for dimensional reasons $T_A \mathcal{L}_Y \oplus *T_A \mathcal{L}_Y$ also exhausts \tilde{h}_A and thus all of $L^p(\Sigma, T^*\Sigma \otimes \mathfrak{g})$. \square

References

- [Ad] R.A.Adams, *Sobolev Spaces*, Academic Press, 1978.
- [At] M.F. Atiyah, New invariants of three and four dimensional manifolds, *Proc. Symp. Pure Math.* **48** (1988), 285-299.
- [AB] M.F. Atiyah, R. Bott, The Yang Mills equations over Riemann surfaces, *Phil. Trans. R. Soc. Lond. A* **308** (1982), 523–615.
- [B] T. Bröcker, T. tom Dieck, *Representations of Compact Lie Groups*, Springer, 1985.
- [DK] S.K. Donaldson, P.B. Kronheimer, *The Geometry of Four-Manifolds*, Oxford Science Publications, 1990.
- [DS1] S. Dostoglou, D.A. Salamon, Instanton Homology and Symplectic Fixed Points, *Symplectic Geometry*, edited by D. Salamon, Proceedings of a Conference, LMS Lecture Notes Series 192, Cambridge University Press, 1993, 57-94.
- [DS2] S. Dostoglou, D.A. Salamon, Self-dual instantons and holomorphic curves, *Annals of Mathematics* **139** (1994), 581–640.
- [F] K. Fukaya, Floer homology for 3-manifolds with boundary I, Preprint 1997. <http://www.kusm.kyoto-u.ac.jp/~fukaya/fukaya.html>
- [L] S. Lang, *Analysis II*, Addison-Wesley, 1969.
- [MS] D. McDuff, D. Salamon, *J-holomorphic curves and symplectic topology*, American Mathematical Society Colloquium Publications, 2004.
- [S] D.A. Salamon, Lagrangian intersections, 3-manifolds with boundary, and the Atiyah–Floer conjecture, *Proceedings of the ICM, Zürich, 1994*, Birkhäuser, Basel, 1995, Vol. 1, 526–536.
- [W1] K. Wehrheim, *Uhlenbeck Compactness*, EMS Series of Lectures in Mathematics, 2004.
- [W2] K. Wehrheim, Anti-self-dual instantons with Lagrangian boundary conditions I: Elliptic theory, to appear in *Comm. Math. Phys.*