The issues noted here were communicated to the authors 2000-2011 and are discussed in joint work with Dusa McDuff.
The purpose of these notes is to move beyond the politics and help readers with clarifying the mathematics for themselves – at the example of simplified cases in which the main issues seem more evident.
This is no attempt at a complete review though – particularly no comments on gluing are made.

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# ARNOLD CONJECTURE AND GROMOV–WITTEN INVARIANT

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## 1. INTRODUCTION

The method of pseudoholomorphic curves initiated by Gromov [33] has now become the most basic tool in studying the global structure of symplectic manifolds. Its important applications include estimates of the number of fixed points of an exact symplectic diffeomorphism and invariants of symplectic manifolds by counting number of pseudoholomorphic curves.

The application of the method of pseudoholomorphic curves to study the number of fixed points of an exact symplectic diffeomorphism is initiated by Floer [15–20] and leads a homology theory of semi-infinite dimension, which is now called the Floer homology. One of the targets of Floer's work was to prove a celebrated conjecture by Arnold [1] which states that the number of fixed points of an exact symplectic diffeomorphism on $M$ is as many as the number of critical points of functions on $M$. After Floer, Hofer and Salamon [35], and the second named author [52] generalized it and established Arnold conjecture in various cases. One of the main results of this paper proves a version of Arnold conjecture on a general symplectic manifold.

An invariant of symplectic manifold by counting number of pseudoholomorphic curves is related to topological sigma models in mathematical physics and is studied independently from that point of view. Especially Witten in [73] (see also [70]) discussed it. This invariant now is called Gromov–Witten invariant. After Mirror symmetry conjecture was discovered by physicists [8], a number of efforts have been done to give a mathematically rigorous definition of this invariant in full geometry. Let us quote the result by Ruan [58] and Ruan–Tian [60, 61] which establish it under additional assumptions. Another main result of this paper establish it in full generality (over rational coefficients).

For both (and most of other) applications of pseudoholomorphic curves in symplectic geometry, one had to assume that the symplectic manifold is weakly monotone (or semi-positive). The reason one had to do so is related to the compactness and transversality of the moduli space of pseudoholomorphic curves. Gromov and later McDuff established various results on compactness and transversality. Their results are basic for various applications. However, in the case of a general symplectic manifold, one needs additional results to establish relevant compactness and transversality theorems of the moduli space of pseudoholomorphic curves. The difficulty, which is called negative multiple cover problem, is in fact, closely related to the problem of stability in algebraic geometry and was also studied extensively there.

The main purpose of this paper is to show a way to overcome the problem of negative multiple cover in symplectic geometry. Our work is influenced by Kontsevich's paper [38], where Kontsevich (quoting Deligne's letter to Esnault) proposed to use the notion of stack to study Gromov–Witten invariant. Especially his idea to regard obstruction bundle as "super structure sheaf" is basic to our approach. In fact the starting point of this work is the author's effort to understand [38]. We succeed to fill most of what Kontsevich mentioned as "certain gaps in foundation" in [38] and give a rigorous mathematical basis to this beautiful paper. Kontsevich in [38] also suggested a possibility to use his idea to prove Arnold conjecture.

We now state our results. Let $(M, \omega)$ be a $2n$-dimensional symplectic manifold. Namely we assume that $d\omega = 0$ and $\omega^n$ vanishes nowhere. We assume that $M$ is compact. We consider a smooth function $H: M \times S^1 \to \mathbf{R}$. We put $H_t(x) = H(x, t)$, $t \in S^1$. Let $X_{H_t}$ be the Hamilton vector field associated to it, which is defined by

$$i(X_{H_t})\omega = dH_t.$$

We consider a one-parameter group of diffeomorphisms $\phi: M \times \mathbf{R} \to M$ such that

$$\frac{d}{dt}\phi(x, t) = X_{H_t}\phi(x, t)$$

$$\phi(x, 0) = x.$$

We put $\phi(x) = \phi(x, 1)$. $\phi$ is called an exact symplectic diffeomorphism. We put

$$Fix(\phi) = \{x \mid \phi(x) = x\}.$$

We say that $x_0 \in Fix(\phi)$ is *nondegenerate* if 1 is not an eigenvalue of $D\phi_{x_0}$.

THEOREM 1.1. *Suppose that every element of $Fix(\phi)$ is nondegenerate. Then the number of elements of $Fix(\phi)$, is not smaller than*

$$\sum_{k=0}^{2n} \operatorname{rank} H_k(M; \mathbf{Q}).$$

*Remark* 1.2. Theorem 1.1 was proved by various mathematicians under various additional assumptions. Especially, it was proved in the following cases:

$M = T^{2n}$ (Torus), Conley–Zehnder [9],
$M = \Sigma_g$ (Riemann surface of higher genus), Floer [15] and Sikorav [66],
$M$ is monotone, Floer [19],
$M$ is semi-positive, Hofer and Salamon [35] and Ono [52].

The estimate in terms of the ranks of the torsion parts of the homology is also obtained in the cases quoted above. However, our method does not work to prove it in the general case.

We next turn to the Gromov–Witten invariant. We choose an almost complex structure $J$ on $M$ which is compatible with the symplectic structure in the following sense.

$$\omega(J\mathbf{v}, J\mathbf{w}) = \omega(\mathbf{v}, \mathbf{w})$$

$$\omega(\mathbf{v}, J\mathbf{v}) > 0 \text{ for every nonzero } \mathbf{v}.$$

It follows that $g_J(\mathbf{v}, \mathbf{w}) = \omega(\mathbf{v}, J\mathbf{w})$ is a Riemannian metric.

Let $g, m$ be nonnegative integers. We consider an oriented compact 2 manifold $\Sigma_g$ of genus $g$, and $m$ points $z = (z_1, \dots, z_m)$ on it, such that $z_i \neq z_j$ for $i \neq j$. Let $Diff(\Sigma, z)$ be the group of all diffeomorphisms $u$ of $M$ such that $u(z_i) = z_i$. Let $\beta \in H_2(M; \mathbf{Z})$ be a homology class.

We denote by $\hat{\mathscr{M}}_{g,m}(M, \beta)$ the set of all pairs $(J_\Sigma, h)$ such that $J_\Sigma$ is a complex structure of $\Sigma_g$ and $h: \Sigma \to M$ is a pseudoholomorphic map. Namely it satisfies

$$Dh \circ J_\Sigma = J \circ Dh.$$

We also assume $h_*[\Sigma] = \beta$. The group $Diff(\Sigma, z)$ acts on $\hat{\mathscr{M}}_{g,m}(M, \beta)$ by $u(J_\Sigma, h) = (u^* J_\Sigma, h \circ u^{-1})$. Let $\mathscr{M}_{g,m}(M, \beta)$ be the quotient space.

Let $\mathscr{M}_{g,m}$ be the moduli space of all complex structures on $(\Sigma, z)$. In other words, $\mathscr{M}_{g,m}$ is a quotient of the set of all $J_\Sigma$ (complex structure on $\Sigma_g$) by the action of $Diff(\Sigma, z)$. There is a natural projection $\pi: \mathcal{M}_{g,m}(M,\beta) \to \mathcal{M}_{g,m}$, $\pi[J_\Sigma, h] = [J_\Sigma]$. We also consider an evaluation map $ev: \mathcal{M}_{g,m}(M,\beta) \to M^m$ defined by

$$ev(J_\Sigma, h) = (h(z_1), \ldots, h(z_m)).$$

Let $C\mathcal{M}_{g,m}$ be the Deligne–Mumford compactification [11] of $\mathcal{M}_{g,m}$. (We assume $3g + m \geqslant 3$. Otherwise $\mathcal{M}_{g,m}$ is empty. The case $g = 1$, $m = 0$ is exceptional also. Namely $\dim_{\mathbf{C}} \mathcal{M}_{1,0} = 1$ and is different from the usual dimension, i.e. $\dim_{\mathbf{C}} \mathcal{M}_{g,m} = 3g - 3 + m$.)

We remark that a symplectic structure determines a homotopy class of compatible almost complex structure hence the Chern classes of its tangent bundle is well defined, which we denote by $c_i$.

The next theorem looks rather complicated. Roughly speaking it means that the fundamental class over $\mathbf{Q}$ of our moduli space $\mathcal{M}_{g,m}(M,\beta)$ is well defined. The precise meaning of it will be clear from the discussion of later sections.

THEOREM 1.3. *Suppose* $3g + m \geqslant 3$. *There exists a perturbation and compactification of* $\mathcal{M}_{m,g}(M,\beta)$, *which we denote by* $\mathcal{P}C\mathcal{M}_{m,g}(M,\beta)$. *The map* $\pi \times ev: \mathcal{M}_{m,g}(M,\beta) \to \mathcal{M}_{g,m} \times M^m$ *extends to a map* $\mathcal{P}C\mathcal{M}_{m,g}(M,\beta) \to C\mathcal{M}_{g,m} \times M^m$. *This space has a fundamental class* $[\mathcal{P}C\mathcal{M}_{m,g}(M,\beta)]$ *over* $\mathbf{Q}$ *in the following sense.*

$\mathcal{P}C\mathcal{M}_{m,g}(M,\beta)$ *is a simplicial complex of dimension* $2m + 2\beta c_1 + 2(3-n)(g-1)$. *The map* $\mathcal{P}C\mathcal{M}_{m,g}(M,\beta) \to C\mathcal{M}_{g,m} \times M^m$ *is smooth on each simplex. We regard top dimensional simplexes together with the restriction of the map* $\mathcal{P}C\mathcal{M}_{m,g}(M,\beta) \to C\mathcal{M}_{g,m} \times M^m$ *as a singular simplex. We can define a coefficient to each of such singular simplex and define a singular chain on* $C\mathcal{M}_{g,m} \times M^m$. *It is a singular cycle and gives an element* $[\mathcal{P}C\mathcal{M}_{m,g}(M,\beta)] \in H_{2m+2\beta c_1+2(3-n)(g-1)}(C\mathcal{M}_{g,m} \times M^m; \mathbf{Q})$, *which depends only on the symplectic manifold* $M$, $m$, $g$ *and* $\beta$. *Moreover, for every piecewise smooth cycle* $C$ *in* $C\mathcal{M}_{g,m} \times M^m$, *we may take* $\mathcal{P}C\mathcal{M}_{m,g}(M,\beta)$ *so that the restriction of* $\mathcal{P}C\mathcal{M}_{m,g}(M,\beta) \to C\mathcal{M}_{g,m} \times M^m$ *to each simplex is transversal to* $C$.

We remark that the class $[\mathcal{P}C\mathcal{M}_{m,g}(M,\beta)]$ is not defined as an element of homology group of $C\mathcal{M}_{m,g}(M,\beta)$, but we define its image in $H_*(C\mathcal{M}_{g,m} \times M^m; \mathbf{Q})$.

We mention the following consequence.

COROLLARY 1.4. *There exists a Gromov–Witten class satisfying all the axioms in Kontsevich–Manin* [39] *except possibly Motivic one.*

The precise statement of Corollary 1.4 will be given in Section 23 as Theorems 23.1.1–23.1.7.

*Remark* 1.5. Theorem 1.3 and Corollary 1.4 are proved by Ruan–Tian in the case when $M$ is semi-positive, $g = 0$, $m \geqslant 3$ in [60] and $M$ is semi-positive, $2g + m \geqslant 3$ in [61]. Corollary 1.4 is proved by Kontsevich–Manin [39] and Behrend–Manin [7] in the case when $M$ is algebraic and convex. Ruan–Tian proved their result over $\mathbf{Z}$.

Our method also works in the case when $3g + m < 3$. But we need to change the statement since $C\mathcal{M}_{g,m}$ is empty in that case. See Section 17 for the statement in that case.

Let us sketch the main idea of the proof of Theorems 1.1 and 1.3. The basic idea of them are the same. The difficulty we need to overcome is "negative multiple cover problem", which we review briefly here.

This problem is on the transversality and compactness of the moduli space of pseudoholomorphic curves. Compactness in this case is a consequence of "transversality at infinity" and dimension counting. So the problem is the transversality. That is the problem whether the actual (geometric) dimension of the moduli space coincides with the index of the linearized operator.

One typical case where it appears is as follows. Let us consider a homology class $\beta = \beta_1 + N\beta_2 \in H_2(M, \mathbf{Z})$. We assume that

$$3 - n \leqslant c_1(M)\beta_2 < 0. \tag{1.6}$$

(Here we remark that semi-positivity is the condition which asserts that there are no pseudoholomorphic curve satisfying (1.6).) Using Riemann–Roch's theorem we find that the virtual dimension of the moduli space $\mathscr{M}_{0,0}(M, \beta_2)$ is given by

$$\text{vir}\dim_{\mathbf{R}} \mathscr{M}_{0,0}(M, \beta_2) = 2c_1\beta_2 + 2(n-3) \geqslant 0$$

Hence, we cannot prove, by dimension counting, that this space is empty. On the other hand, we consider the virtual dimensional of the moduli space $\mathscr{M}_{0,0}(M, N\beta_2)$ and find that

$$\text{vir}\dim_{\mathbf{R}} \mathscr{M}_{0,0}(M, N\beta_2) = 2Nc_1\beta_2 + 2(n-3) < 0$$

if $N$ is large. However, if $h \in \mathscr{M}_{0,0}(M, \beta_2)$ and if $\varphi_N : \mathbf{C}P^1 \to \mathbf{C}P^1$ is a holomorphic map of degree $N$, then $h \circ \varphi_N \in \mathscr{M}_{0,0}(M, N\beta_2)$. Hence, if $\mathscr{M}_{0,0}(M, \beta_2)$ is nonempty, then $\mathscr{M}_{0,0}(M, N\beta_2)$ is nonempty. (Moreover $\dim \mathscr{M}_{0,0}(M, \beta_2) \leqslant \dim \mathscr{M}_{0,0}(M, N\beta_2)$ if we define the dimension in an appropriate way, say the topological dimension.)

Thus we have $\text{vir}\dim \mathscr{M}_{0,0}(M, N\beta_2) \neq \dim \mathscr{M}_{0,0}(M, N\beta_2)$. Namely the space $\mathscr{M}_{0,0}(M, N\beta_2)$ cannot be transversal for any choice of compatible almost complex structure.

A similar trouble will be induced to the compactification $C\mathscr{M}_{0,0}(M, \beta)$ of the moduli space $\mathscr{M}_{0,0}(M, \beta)$. Namely we consider a pair $(h_1, h_2 \circ \varphi_N)$ such that $h_1 \in \mathscr{M}_{0,1}(M, \beta_1)$, $h_2 \in \mathscr{M}_{0,1}(M, \beta_2)$, and that $h_1(z_0) = h_2\varphi_N(z_1)$, where $z_0, z_1$ are marked points. This element is regarded as a stable map of genus 0 which we define in Section 7 and is an element of a compactification $C\mathscr{M}_{0,0}(M, \beta)$. When we fix $\varphi_N$, the space of such pairs $(h_1, h_2 \circ \varphi_N)$ is a codimension $2n$ submanifold of $\mathscr{M}_{0,1}(M, \beta_1) \times \mathscr{M}_{0,1}(M, \beta_2)$. (The codimension $= 2n$ is the number of conditions, $h_1(z_0) = h_2\varphi_N(z_1)$ we assumed.) We find that:

$$\text{vir}\dim(\mathscr{M}_{0,0}(M, \beta_1) \times \mathscr{M}_{0,0}(M, \beta_2)) = 2c_1\beta_1 + 2(n-2) + 2c_1\beta_2 + 2(n-2).$$

Hence the dimension of the space of such pairs $(h_1, h_2 \circ \varphi_N)$ may be as large as

$$2c_1\beta_1 + 2c_1\beta_2 + 2(n-4). \tag{1.8}$$

(In fact we need to take into account the moduli space of $\varphi_N$. Hence the dimension may be larger than (1.8).)

On the other hand, the virtual dimension of the moduli space $\mathscr{M}_{0,0}(M, \beta)$ is

$$\text{vir}\dim_{\mathbf{R}} \mathscr{M}_{0,0}(M, \beta) = 2c_1\beta_1 + 2Nc_1\beta_2 + 2(n-3). \tag{1.9}$$

In case when $N$ is large, the right-hand side of (1.9) is smaller than (1.8). Namely the "boundary" of $C\mathscr{M}_{0,0}(M, \beta)$ is of larger dimension than $\mathscr{M}_{0,0}(M, \beta)$ itself. This causes trouble to define a fundamental class of $C\mathscr{M}_{0,0}(M, \beta)$.

The map $h_2 \circ \varphi_N$ does not satisfy the condition "somewhere injective" established by McDuff [45]. McDuff proved that, for generic almost structure, the virtual dimension of the subset of the moduli space $\mathscr{M}_{m,g}(M, \beta)$ which consists of somewhere injective elements, coincides with its actual (geometric) dimension. The above argument shows that one needs to consider also maps which is not somewhere injective, to study the moduli space $\mathscr{M}_{0,0}(M, \beta)$ in case $M$ is not semi-positive.

For higher genus, there is also a similar problem. As a typical example, let us consider $N_i, g_i, g_i', g, \beta_i, \beta$ such that $N_1\beta_1 + N_2\beta_2 = \beta$, $g_i' \geqslant 1 + N_i(g_i - 1)$. We remark that there exists a holomorphic map $\varphi_{N_i}: \Sigma_{g_i'} \to \Sigma_{g_i}$ of degree $N_i$ between Riemann surfaces of genus $g_i', g_i$ if and only if $g_i' \geqslant 1 + N_i(g_i - 1)$. We consider a pair $(h_1 \circ \varphi_{N_1}, h_2 \circ \varphi_{N_2})$ where $h_1 \in \mathscr{M}_{g_1,1}(M, \beta_1)$, $h_2 \in \mathscr{M}_{g_2,1}(M, \beta_2)$. If $h_1(z_0) = h_2(z_1)$ then such a pair $(h_1 \circ \varphi_{N_1}, h_2 \circ \varphi_{N_2})$ may be regarded as an element of $C\mathscr{M}_{g_i'+g_2',0}(M, \beta)$. The virtual dimension of $\mathscr{M}_{g_i'+g_2',0}(M, \beta)$ is

$$\text{vir}\dim \mathscr{M}_{g_i'+g_2',0}(M, \beta) = 2\beta c_1 + 2(3-n)(g_1' + g_2' - 1). \tag{1.10}$$

On the other hand, if we fix $\varphi_{N_i}: \Sigma_{g_i'} \to \Sigma_{g_i}$, the set of pairs $(h_1 \circ \varphi_{N_1}, h_2 \circ \varphi_{N_2})$ such that $h_1(z_0) = h_2(z_0)$ is of codimension $2n$ in $C\mathscr{M}_{g_1,1}(M, \beta_1) \times C\mathscr{M}_{g_2,1}(M, \beta_2)$. The dimension of it is

$$2\beta_1 c_1 + 2(3-n)(g_1 - 1) + 2\beta_2 c_1 + 2(3-n)(g_2 - 1) - 2n. \tag{1.11}$$

In case $g_i' \quad 1 + N_i(g_i - 1)$ and $n > 3$, we can easily find examples such that (1.11) is much larger than (1.10) (and $2\beta_1 c_1 + 2(3-n)(1-g_1) \geqslant 0$, $2\beta_2 c_1 + 2(3-n)(1-g_2) \geqslant 0$.)

Thus the 'boundary' of $C\mathscr{M}_{g_i'+g_2',0}(M, \beta)$ is of larger dimension than $\mathscr{M}_{g_i'+g_2',0}(M, \beta)$ itself, in some cases. This problem (in the higher genus case) was handled by Ruan and Tian [61], under additional assumptions. Their method is to use inhomogeneous perturbation, which was first introduced by Gromov [33].

However, the method to use inhomogeneous perturbation alone is not enough to settle negative multiple cover problem in general. Our method in this paper may be regarded as using inhomogeneous and *multivalued* perturbation (though we define our perturbation in more abstract way). Example 7.12 shows that in general the order counted with sign of $\mathscr{M}_{m,g}(M, \beta)$ (in the case its virtual dimension is 0) is a rational number. Hence, we need multivalued perturbation to achieve transversality. It works to settle both problems (over rational coefficient).

Let us now go back to the sketch of the ideas of the proofs of Theorems 1.1 and 1.3.

For each point $\sigma$ in our moduli space $C\mathscr{M}_{g,m}(M, \beta)$, we find its neighborhood diffeomorphic to

$$f^{-1}(0)/\Gamma_0$$

where $f: \mathbf{R}^k \to \mathbf{R}^\ell$ and $\Gamma_0$ is a finite group acting on $\mathbf{R}^k, \mathbf{R}^\ell$ such that $f$ is $\Gamma_0$-invariant. This is a general principle which applies to a moduli space of solutions of an elliptic partial differential equation with automorphism groups. Kuranishi [40] first applied this method to the study of deformation theory of complex structures. This map $f$ is, in general, called the Kuranishi map.

This description is used extensively in Gauge theory (anti-self-dual equation) by Donaldson and Taubes.

One idea to achieve the transversality is then to perturb the map $f$ so that it is transversal to 0. This was the way taken by Donaldson in his paper [12] to study anti-self-dual equation. Ruan used it in [56] for pseudoholomorphic curves. (However, the description there is rather confusing, since the problem arising from the presence of $\Gamma_0$ is not discussed carefully.)

However, in our case we cannot do it because there is no $\Gamma_0$-invariant perturbation of $f$ which is transversal to 0 in general.

At this point we need to leave the general theory and use properties of our equation.

First by using the idea of Kontsevich to introduce stable maps, we may assume that $\Gamma_0$ is a finite group. In our case, $\Gamma_0$ is the group of automorphisms of the pair $(\Sigma, h)$ representing $\sigma \in \mathscr{M}_{g,m}(M, \beta)$. Kontsevich's definition of stable map is designed so that $\Gamma_0$ is always finite.

Second we work out Kuranishi theory in the case of pseudoholomorphic curves and describe our moduli space as $f^{-1}(0)/\Gamma_0$, where $f: \mathbf{R}^k \to \mathbf{R}^\ell$. One important point is that the difference $k - \ell$ is constant. It means that the virtual dimension of our moduli space is $k - \ell$

and is constant. This is a consequence of the fact that the linearization of the pseudo-holomorphic curve equation gives a two-step elliptic complex (the Dolbeault complex on the curve tensored with the pull back of the tangent bundle of the target space).

We regard such a description as charts and call such a structure the Kuranishi structure.

Now we use the fact that $\Gamma_0$ is finite to find a multivalued perturbaton of $f$ such that each branch is transversal to 0. More precisely we construct and use a multisection of an orbibundle.

The zero set of a multivalued function (or more precisely a multisection) gives a cycle over $\mathbb{Q}$ hence we prove Theorem 1.3. (In fact we need to perform those constructions at infinity as well.)

The proof of Theorem 1.1 is a combination of arguments in [19, 35, 52] and the discussion above on the compactification of the moduli space of pseudoholomorphic curves.

The organization of this paper is as follows.

In Chapter 1, we review basic facts on orbifolds and then define Kuranishi structures, multisections and prove a general transversality theorem for multisections. The contents of this chapter is elementary. However, we give rather detailed description because we do not find an appropriate reference and because such a method may look strange to some workers of symplectic geometry. (Using orbifolds to study the moduli space of curves was initiated by Mumford [50] and is familiar to algebraic geometers.)

In Chapter 2, we define the moduli space of stable maps and its topology. We include discussions on Deligne–Mumford compactification of the moduli space of curves in this chapter, since we use it frequently and since there seems to be no reference describing it in the way we need. We also prove the compactness of the moduli space of stable maps. It seems that this fact is, in principle, known to Gromov already and there are papers [53, 54, 74] published on this topic. But we give a proof of it, since we do not find any reference discussing the stability and unstability (which are quite essential here) enough carefully. Especially it seems that there is no reference which gives a proof that moduli space of the stable maps is Hausdorff. Kontsevich [38] seems to be the first person who observed that the notion of stable maps allows to obtain a moduli space which is Hausdorff.

In Chapter 3, we construct Kuranishi structure on the moduli space of stable maps. The main part of the construction is Taubes' type gluing argument with obstruction bundles. One needs some more arguments to glue Kuranishi maps. Also the definition and the construction of the orientation needs some more arguments including the definition of $K$-group over Kuranishi structure. The analytic part of the construction is a minor modification of the case of weakly monotone symplectic manifolds. We here follow the approach by McDuff–Salamon [47] for analysis.

In Chapter 4, we use those machineries and prove Theorems 1.1 and 1.3.

# CHAPTER 1. ORBIFOLD, MULTISECTION AND KURANISHI STRUCTURE

## 2. ORBIFOLD AND ORBIBUNDLE

The definition of orbifold (or -manifold) and orbibundle is due to Satake [63] and is now standard. However, to fix our notation, we recall their definitions here. The experienced reader may skip this section and come back only to check our notations when necessary.

*Definition* 2.1. *A local model of an n-dimensional orbifold* is a pair $( \ , \Gamma)$ where $\Gamma$ is a finite group which has a linear representation of $\mathbf{R}^n$, and is a $\Gamma$-invariant open neighborhood of $\mathbf{0} \in \mathbf{R}^n$. We assume that the action of $\Gamma$ on is effective.

Let $( \ , \Gamma)$ be a local model of an $n$-dimensional orbifold. We put $U = \ /\Gamma$ and let $\pi\colon \ \to U$ be the projection. For each $q \in U$, we obtain a local model of an $n$-dimensional orbifold $( \ _q, \Gamma_q)$ as follows. We take $\tilde{q} \in$ such that $\pi(\tilde{q}) = q$. We put

$$\Gamma_q = \{g \in \Gamma \mid g\tilde{q} = \tilde{q}\}.$$

We take a sufficiently small $\Gamma_q$-invariant neighborhood $_q$ of $\tilde{q}$. We may regard the pair $( \ _q, \Gamma_q)$ as a local model of an $n$-dimensional orbifold. There is a map $\pi_q\colon \ _q \to U$ such that $\pi_q(gx) = \pi_q(x)$.

The germ of the triple $( \ _q, \Gamma_q, \pi_q)$ is well defined in the following sense. If $( \ '_q, \Gamma'_q, \pi'_q)$ is another such triple then there exists a $\Gamma_q$-invariant neighborhood $\bar{\ }_q$ of $\pi_q^{-1}(q)$, a $\Gamma'_q$-invariant neighborhood $\bar{\ }'_q$ of $\pi_q'^{-1}(q)$, an isomorphism $\psi\colon \Gamma_q \to \Gamma'_q$, and a diffeomorphism $\varphi\colon \bar{\ }_q \to \bar{\ }'_q$ such that $\varphi$ is $\psi$-equivariant and $\pi'_q \circ \varphi = \pi_q$.

We call $( \ _q, \Gamma_q, \pi_q)$ an *induced chart.*

*Definition* 2.2. Let $X$ be a paracompact Hausdorff space. An *n-dimensional orbifold structure* on $X$ is an open covering $X = \bigcup_i U_i$, local models $( \ _i, \Gamma_i)$ of an $n$-dimensional orbifold for each $i$, and homeomorphisms

$$\varphi_i\colon \ _i/\Gamma_i \to U_i$$

with the following properties. Let $q \in U_i \cap U_j$. We have induced charts $( \ _{q,i}, \Gamma_{q,i}, \pi_{q,i})$ and $( \ _{q,j}, \Gamma_{q,j}, \pi_{q,j})$ respectively. Here, $\pi_{q,i}\colon \ _{q,i} \to \ _i/\Gamma_i$ etc. Then, replacing $_{q,i}$ and $_{q,j}$ by smaller ones if necessary there exists an isomorphism $\psi_{i,j,q}\colon \Gamma_{q,i} \to \Gamma_{q,j}$, and a diffeomorphism $\varphi_{i,j,q}\colon \ _{q,i} \to \ _{q,j}$ such that $\varphi_{i,j;q}$ is $\psi_{i,j;q}$-equivariant and

$$\varphi_j \circ \pi_{j,q} \circ \varphi_{i,j;q} = \varphi_i \circ \pi_{i,q}.$$

We call $\{( \ _i, \Gamma_i, \pi_i)\}$ an *orbifold structure*, and $X$ an *orbifold*.

*Definition* 2.3. Let $X$ be an orbifold. Let $( \ , \Gamma)$ be a local model of $n$-dimensional orbifold and $\pi\colon \ \to U \subseteq X$ be a map inducing a homeomorphism $\ /\Gamma \cong U$ onto its image. We call $( \ , \Gamma, \pi)$ a *chart* if $\{( \ _i, \Gamma_i, \pi_i) \mid i\} \cup \{( \ , \Gamma, \pi)\}$ is an orbifold structure.

Hereafter we identify $\ /\Gamma$ with $U$ and omit $\pi$ when no confusion can occur.

*Definition* 2.4. Let $X$, $Y$ be orbifolds and $f\colon X \to Y$ be a continuous map. We say that $f$ is a *smooth map* if for each $p \in X$ there exists a chart $( \ _p, \Gamma_p, \pi_p)$ of $X$ and a chart $( \ _{f(p)}, \Gamma_{f(p)}, \pi_{f(p)})$ of $Y$, a smooth map $f_p\colon \ _p \to \ _{f(p)}$ and a homomorphism $\psi_p\colon \Gamma_p \to \Gamma_{f(p)}$

(2.4.1) $p \in \ _p/\Gamma_p, f(p) \in \ _{f(p)}/\Gamma_{f(p)}.$

(2.4.2) $f_p$ is $\psi_p$-equivariant.

(2.4.3) $\pi_{f(p)} \circ f_p = f \circ \pi_p.$

We say that $f$ is a *smooth embedding* if $f_p$: $_p \to$ $_{f(p)}$ is an embedding of smooth manifolds and $\psi_p : \Gamma_p \to \Gamma_{f(p)}$ is an isomorphism.

A smooth map whose inverse is also a smooth map is called a diffeomorphism. A manifold can be regarded also as an orbifold. So the set of all smooth maps $C^\infty(X)$ from an orbifold $X$ to $\mathbf{R}$ is well defined. It is easy to see that $C^\infty(X)$ is a ring.

*Remark* 2.5. Let $X = \mathbf{C} \times \mathbf{R}/\mathbf{Z}_2$ where the action is $(z, t) \mapsto (-z, t)$. Then the map $f : \mathbf{R} \to X$, $f(t) = [0, t]$ is a smooth map in our sense but is not a smooth embedding.

We next define an orbibundle. We proceed in the same way and start with the definition of local model.

*Definition* 2.6. Let ( , $\Gamma$) be a local model of an $n$-dimensional orbifold. Suppose that we have a linear representation of $\Gamma$ on $\mathbf{R}^k$. We say that a pair ( $\times \mathbf{R}^k, \Gamma, pr$) is a *local model of smooth orbibundle of rank* $k$ over ( , $\Gamma$). Here $pr$: $\times \mathbf{R}^k/\Gamma \to$ $/\Gamma$ is the projection.

Let ( $\times \mathbf{R}^k, \Gamma, pr$) be a local model of smooth orbibundle of rank $k$ and ( $_q, \Gamma_q, \pi_q$) be an induced chart of an orbifold ( , $\Gamma$). We then obtain a local model of smooth orbibundle of rank $k$ over ( $_q, \Gamma_q$) by restricting ( $\times \mathbf{R}^k, \Gamma, pr$) to $_q$. Let ( $_q \times \mathbf{R}^k, \Gamma_q, pr$) denote it. We say that ( $_q \times \mathbf{R}^k, \Gamma_q, pr$) is an *induced chart*. We remark that it is well defined in the sense of a germ.

*Definition* 2.7. Let $E$ and $X$ be orbifolds and $pr : E \to X$ be a smooth map. A structure of smooth orbibundle on $pr : E \to X$ is the following collections of objects.

(2.7.1) Family of charts ( $_i, \Gamma_i, \pi_i$) of $X$ such that $\bigcup_i$ $_i/\Gamma_i = X$.
(2.7.2) A local model of smooth orbibundle of rank $k$( $_i \times \mathbf{R}^k, \Gamma_i, pr$) over ( $_i, \Gamma_i$) for each $i$.
(2.7.3) Maps $\tilde{\pi}_i$: $_i \times \mathbf{R}^k \to E$ such that ( $_i \times \mathbf{R}^k, \Gamma_i, \tilde{\pi}_i$) are charts of $E$ and $\bigcup_i$ $_i \times \mathbf{R}^k/\Gamma_i = E$.

These objects should satisfy the following compatibility conditions.

(2.8) Let $q \in$ $_i/\Gamma_i \cap$ $_j/\Gamma_j$, and ( $_{q,i} \times \mathbf{R}^k, \Gamma_{q,i}, pr$) be induced charts. Then, by shrinking $_i$, $_j$ if necessary, there exists a diffeomorphism $\tilde{\varphi}_{i,j;q}$: $_{q,i} \times \mathbf{R}^k \to$ $_{q,j} \times \mathbf{R}^k$ such that
(2.8.1) $\tilde{\varphi}_{i,j;q}$ is $\psi_{i,j;q}$-equivariant, where $\psi_{i,j;q} : \Gamma_{q,i} \to \Gamma_{q,j}$ is an isomorphism as in Definition 2.2.
(2.8.2) $\tilde{\varphi}_{i,j;q}$: $_{q,i} \times \mathbf{R}^k \to$ $_{q,j} \times \mathbf{R}^k$ commutes with the projections $_{q,i} \times \mathbf{R}^k \to$ $_{q,i}$, $_{q,j} \times \mathbf{R}^k \to$ $_{q,j}$ and is a linear isomorphism on each fibre.
(2.8.3) $\tilde{pr} \circ \tilde{\varphi}_{i,j;q} = \tilde{pr}$, where $\tilde{pr}$: $\times \mathbf{R}^k \to$ is a lift of $pr$.

We define a chart of orbibundle in a way similar to Definition 2.3.

If $X$ is an orbifold then its tangent bundle $TX$ is well defined as an orbibundle. If $f : X \to Y$ is an embedding of orbifolds then the normal bundle $N_Y X$ is well defined as an orbibundle.

One defines in an obvious way the notion of bundle map covering a smooth map.

We can define a Whitney sum, subbundle, quotient bundle, tensor product, etc. of orbibundle in an obvious way. Also if there is a smooth map $f : X \to Y$ of orbifolds and an orbibundle $E \to Y$ then the pull back $f^*E$ is well defined as an orbibundle. There is a bundle map $\tilde{f}: f^*E \to E$ covering $f: X \to Y$.

A section $s: X \to E$ to an orbibundle is a continuous map such that $pr \circ s = id$. We say $s$ is smooth if it is a smooth map of orbifold. Let $C^\infty(\Omega, E)$ be the set of all smooth sections on $\Omega \subseteq X$. It is easy to find a $C^\infty(\Omega)$-module structure on $C^\infty(\Omega, E)$.

### 3. MULTISECTION

For a space $Z$, let $\mathscr{S}^k(Z)$ be the $k$th symmetric power of $Z$. Namely we put

$$\mathscr{S}^k(Z) = Z^k / \mathscr{S}^k.$$

Here $\mathscr{S}_k$ is the symmetric group of order $k!$ which acts on $Z^k$ as permutations of the factors. If $Z$ is an orbifold then $\mathscr{S}^k(Z)$ is an orbifold. If there is a smooth action of $\Gamma$ on $Z$, then it induces a smooth action of $\Gamma$ on $\mathscr{S}^k(Z)$.

*Definition* 3.1. Let $( \;\; \times \mathbf{R}^k, \Gamma, pr)$ be a local model of smooth orbibundle of rank $k$ over $( \;\; , \Gamma)$ and $\ell$ be a positive integer. An *$\ell$-multisection* of $( \;\; \times \mathbf{R}^k, \Gamma, pr)$ is a continuous map $s: \;\; \to \mathscr{S}^\ell(\mathbf{R}^k)$ which is $\Gamma$-variant.

We define the smoothness of a multisection later. (Definition 3.8).

We remark that there is a canonical map $tm_{\ell'}: \mathscr{S}^\ell(Z) \to \mathscr{S}^{\ell\ell'}(Z)$ for each $\ell, \ell'$. Namely we define

$$tm_{\ell'}[x_1, \dots, x_\ell] = [\underbrace{x_1, \dots, x_1}_{\ell' \text{ times}}, \dots, \underbrace{x_\ell, \dots, x_\ell}_{\ell' \text{ times}}].$$

If $s: \;\; \to \mathscr{S}^\ell(\mathbf{R}^k)$ is an $\ell$-multisection then $tm_{\ell'} \circ s$ is an $\ell\ell'$-multisection.

If $s: \;\; \to \mathscr{S}^\ell(\mathbf{R}^k)$ is an $\ell$-multisection and if $( \;\; _q \times \mathbf{R}^k, \Gamma_q, pr)$ is an induced chart then the restriction of $s$ is an $\ell$-multisection over $( \;\; _q \times \mathbf{R}^k, \Gamma_q, pr)$.

*Definition* 3.2. Let $pr: E \to X$ be an orbibundle. A *multisection* is an isomorphism class of the following objects $(\{( \;\; _i \times \mathbf{R}^k, \Gamma_i, pr)\}, \{s_i\})$ such that

(3.2.1) $\{( \;\; _i \times \mathbf{R}^k, \Gamma_i, pr)\}$ is a family of charts of $E$ such that $\bigcup_i \;\; _i/\Gamma_i = X$.

(3.2.2) $s_i$ is an $\ell_i$-multisection of $( \;\; _i \times \mathbf{R}^k, \Gamma_i, pr)$.

(3.2.3) Let $q \in \;\; _i/\Gamma_i \cap \;\; _j/\Gamma_j$. We have an $\ell_i$-multisection $s_{i,q}$ on $( \;\; _{q,i} \times \mathbf{R}^k, \Gamma_{q,i}, pr)$ and an $\ell_j$-multisection $s_{j,q}$ on $( \;\; _{q,j} \times \mathbf{R}^k, \Gamma_{q,j}, pr)$ on induced charts. Then

$$\hat{\tilde{\varphi}}_{i,j;q} \circ tm_{\ell_j} \circ s_{i,q} = tm_{\ell_i} \circ s_{j,q} \circ \varphi_{i,j;q}.$$

Here $\varphi_{i,j;q}$ is a map in Definition 2.2 and $\hat{\tilde{\varphi}}_{i,j;q}: S^{\ell_i \ell_j}(\mathbf{R}^k) \to S^{\ell_i \ell_j}(\mathbf{R}^k)$ is a map induced by the restriction of the map $\tilde{\varphi}_{i,j,q}$ in Definition 2.7 to each fibre.

We say that $(\{( \;\; _i \times \mathbf{R}^k, \Gamma_i, pr)\}, \{s_i\})$ is equivalent to $(\{( \;\; '_i \times \mathbf{R}^k, \Gamma'_i, pr)\}, \{s'_i\})$ if the

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$$q \in \frac{\;\; _i}{\Gamma_i} \cap \frac{\;\; '_j}{\Gamma'_j}.$$

We have the $\ell_i$-multisection $s_{i,q}$ on $( \ _{q,i}\times \mathbf{R}^k, \Gamma_{q,i}, pr)$ and the $\ell'_j$-multisection $s'_{j,q}$ on $( \ '_{q,j}\times \mathbf{R}^k, \Gamma'_{q,j}, pr)$ on induced charts. Then

$$\hat{\hat{\varphi}}_{i,j;q} \circ tm_{\ell'_j} \circ s_{i,q} = tm_{\ell_i} \circ s'_{j,q} \circ \varphi_{i,j;q}$$

Here the notation is as above.

Let $s$ be a multisection represented by $(\{( \ _i \times \mathbf{R}^k, \Gamma_i, pr)\}, \{s_i\})$. If $q \in \ _i/\Gamma_i$, then we say that $(( \ _i \times \mathbf{R}^k, \Gamma_i, pr), s_i)$ (or simply $s_i$) is a *local representative around* $q$. Using the set of germs of multisections we can define a sheaf. It might be shorter to use sheaf theory. However, to keep the exposition as elementary as possible, we do not take that way.

*Definition* 3.3. For an open subset $\Omega$ of $X$, we let $C^0_m(\Omega; E)$ be the set of all continuous multisections over $\Omega$.

It is easy to see that the ring $C^0(\Omega)$ acts on $C^0_m(\Omega; E)$. It might be slightly less obvious to define the sum $s^{(1)} + s^{(2)}$ of two multisections. To define it, it is enough to define a local representative around each $q$. Let $(( \ _i \times \mathbf{R}^k, \Gamma_i, pr), s_i^{(1)})$ and $(( \ _i \times \mathbf{R}^k, \Gamma_i, pr), s_j^{(2)})$ be the local representations. By shrinking $\ _i$, $\ _j$ if necessary, we may assume that $\ _i = \ _j = \ $, $\Gamma_i = \Gamma_j = \Gamma$. Now let us consider the following map

$$+ : \mathcal{S}^{\ell}(\mathbf{R}^k) \times \mathcal{S}^{\ell'}(\mathbf{R}^k) \to \mathcal{S}^{\ell\ell'}(\mathbf{R}^k)$$

$$+([x_1, \dots, x_l], [y_1, \dots, y_{\ell'}]) = [x_a + y_b : a = 1 \dots \ell, b = 1 \dots \ell'].$$

*Definition* 3.4. Let the *sum* of two multisections $s^{(1)} + s^{(2)}$ be the multisection whose local representative around $q$ is $(( \ \times \mathbf{R}^k, \Gamma, pr), +(s_i^{(1)}, s_j^{(2)}))$.

It is straightforward to verify the compatibility condition so we omit it. We remark that the sum is associative and commutative. Hence, it defines a structure of commutative monoid. (However, it does not give the structure of abelian group.)

We thus defined a $C^0(\Omega)$ "module structure" on $C^0_m(\Omega; E)$. Therefore, we can use partition of unity to glue multisections. However we must be careful to apply it because of the following trouble. Even if every $s_i$ is $C^0$-close to $t$, the sum $\sum \chi_i s_i$ may not be $C^0$-close to $t$. $\sum \chi_i s_i$ is $C^0$-close to $t$ if $t$ is single valued. (We remark also that $(f + g)s \neq fs + gs$ in general for $s \in C^0_m(\Omega; E), f, g \in C^0(\Omega)$.)

We next discuss the transversality of a multisection. To do so it is convenient to have a notion of branches of multisections. Unfortunately, it is not always well defined for continuous multisections.

*Definition* 3.5. A multisection $s$ is said to be *locally liftable* if for each point $q$ there exists a local representative $(( \ _q \times \mathbf{R}^k, \Gamma_q, pr), s_q)$ such that the map $s_q\colon \ _q \to \mathcal{S}^{\ell}(\mathbf{R}^k)$ is lifted to a map $\tilde{s}_q\colon \ _q \to (\mathbf{R}^k)^{\ell}$.

*Remark* 3.6. We do *not* require $\tilde{s}_q\colon \ _q \to (\mathbf{R}^k)^{\ell}$ to be $\Gamma_q$ equivariant.

*Example* 3.7. Let $X = \mathbf{C}$ and $E = \mathbf{C} \times \mathbf{C}$ be the trivial bundle. We consider the 2-multisection $s(z) = [\sqrt{z}, -\sqrt{z}]$. This multisection is not locally liftable. However, we can approximate it by a locally liftable one in the following way.

We choose a smooth function $\chi : [0, \infty) \to [0, \infty)$ such that $\chi(t) = 0$ for $t < \varepsilon$, $\chi(t) = t$ for $t > 2\varepsilon$ and is strictly increasing for $t > \varepsilon$. Let $s'_\varepsilon(z) = s(\chi(|z|)z)$. $s'_\varepsilon$ can be chosen arbitrary close to $s$ in the $C^0$-topology. We now claim that $s'_\varepsilon$ is locally liftable. In fact it is obvious that $s'_\varepsilon$ is locally liftable at $z$ such that $|z| < \varepsilon$ since it is identically zero around it. On the other hand at a point $|z| = \varepsilon$, we can take a branch of $\sqrt{z}$ in its neighborhood. So $s'_\varepsilon$ is locally liftable.

So it may not be so natural to restrict ourselves to locally liftable multisections. However, since we use only locally liftable multisections, we consider them only. We remark that usual (single valued) section is locally liftable and the sum of locally liftable multisections is locally liftable.

*Definition* 3.8. A multisection $s$ is said to be *smooth* (resp. of class $C^k$) if it is locally liftable and each branch of it is smooth (resp. of class $C^k$). Let $C^k_m(\Omega; E)$ be the set of all multisections of $E$ on $\Omega$ of class $C^k$ ($k = 1, 2, \dots, \infty$).

If $s$ is a locally liftable multisection then a *branch* of $s$ at $q$ is a component of $\tilde{s}_q\colon \ _q \to (\mathbf{R}^k)^\ell$ where $\tilde{s}_q$ is as in Definition 3.5.

*Definition* 3.9. A multisection $s$ is said to be *transversal* to 0 if it is locally liftable and if for each $q$, each branch of $s$ at $q$ is transversal to 0.

We next state the transversality theorem for multisection. For this purpose, we define the $C^k$-topology ($k = 0, \dots, \infty$) on the set of all multisections.

*Definition* 3.10. Let $s_n, s \in C^k_m(\Omega, E)$. We say that $s_n$ converges to $s$ in the $C^k$-topology if the following is satisfied. For each compact set $K$, there exists a covering of it by charts $(\ _i \times \mathbf{R}^k, \Gamma_i, pr)$ of $E$ which is independent of $n$, and $s_n, s$ has a local representatives on $(\ _i \times \mathbf{R}^k, \Gamma_i, pr)$ as $\ell_i$-multisections $s_n^{(i)}$, $s^{(i)}$, such that $\ell_i$ is independent of $n$ and that each branch of $s_n^{(i)}$ converges to a branch of $s^{(i)}$ in the $C^k$-topology.
(Here and hereafter the running index $n$ is not the same as the dimension of the spaces.)

We now state our transversality theorem.

Theorem 3.11. *Let $s \in C^\infty_m(X; E)$ be a locally liftable smooth multisection on a compact orbifold $X$. Then there exists a sequence $s_n \in \Gamma_m(X; E)$, such that $s_n$ converges to $s$ in the $C^\infty$-topology and that $s_n$ is transversal to* 0.

*Proof.* We take a finite open covering

$$X = \bigcup_i \frac{\ _i}{\Gamma_i}$$

by charts and local lifts $\tilde{s}_i\colon \ _i \to (\mathbf{R}^k)^{\ell_i}$ representing $s$. Let $\chi_i$ be a partition of unity subordinate to $X = \bigcup_i \ _i/\Gamma_i$. For an element $v_i \in C^\infty(\ _i, \mathbf{R}^k)$, we define elements $av(v_i) \in C^\infty_m(\ _i/\Gamma_i)$ as follows. $av(v_i)$ is a $\#\Gamma_i$-multisection such that

$$av(v_i)(x) = [\gamma^{-1}(v_i(\gamma x)) \colon \gamma \in \Gamma_i].$$

Now for $\mathbf{v} = (v_i) \in \ _i C^\infty(\ _i, \mathbf{R}^k)$, we put

$$Q(\mathbf{v}) = s + \sum_i \chi_i \, av(v_i).$$

We will prove that for $\mathbf{v} = (v_i)$ in a Baire subset $\ _i C^\infty(\ _i, \mathbf{R}^k)$, $Q(\mathbf{v})$ is transversal to 0.

Let $q \in X$. Then in a neighborhood of $q$, branches of $Q(\boldsymbol{v})$ are given by

$$s_j(x) + \sum_i \chi_i(x)\gamma_i^{-1} v_i(\gamma_i x).$$

Here $s_j$ is a branch of $s$, we take $\gamma_i \in \Gamma_i$ for each $i$, and $\chi_i$ is a partition of unity such that at least one of $\chi_i$ is nonzero. (We remark that we are not taking the average with respect to $\gamma \in \Gamma$.) It follows that this branch is transversal to 0 in a neighborhood of $q$ for each $\boldsymbol{v} = (v_i)$ in a Baire subset $\ _i C^\infty(\ _i, \mathbf{R}^k)$.

Since there are only finitely many branches and we can cover our orbifolds by finitely many such neighborhoods, it follows that the set of all $\boldsymbol{v}$ such that $Q(\boldsymbol{v})$ is transversal to 0 is dense.

Hence we have a sequence of $\boldsymbol{v}_n$ converging to 0 in the $C^\infty$-topology such that $Q(\boldsymbol{v}_n)$ is transversal to 0. It is easy to see that $Q(\boldsymbol{v}_n)$ converges to $s$ in the $C^\infty$-topology we defined above. The proof of Theorem 3.11 is now complete. □

Let us recall the following well known lemma.

LEMMA 3.12. *For any continuous single valued section $s \in C^0(X; E)$ on a compact orbifold $X$, there exists a sequence of smooth single valued sections $s_n$ which converges to $s$ in $C^0$-topology.*

*Remark* 3.13. It seems possible to show that any continuous multisection can be approximated by smooth multisections in the $C^0$-topology. We do not try to prove it since we do not need it and since a problem of pathology makes the proof cumbersome.

*Proof of Lemma* 3.12. We cover $X$ by charts $X = \bigcup_i \ _i/\Gamma_i$ and let $s_i\colon \ _i \to \mathbf{R}^k$ be the $\Gamma_i$-equivariant map representing $s$. We find a sequence of smooth maps $s_{i,n}\colon \ _i \to \mathbf{R}^k$ converging to $s_i$ in the $C^0$-topology. We then obtain a smooth and $\Gamma_i$-equivariant map $s'_{i,a}\colon \ _i \to \mathbf{R}^k$ by putting $s'_{i,n} = \sum_{\gamma \in \Gamma_i} \gamma s_{i,n} / \#\Gamma_i$. Choose a partition of unity $\chi_i\colon U_i \to [0,1]$ and put $s_n = \sum_i \chi_i s_{i,n}$. Clearly $s_n$ has the required properties. The proof of Lemma 3.12 is complete. □

Finally we state the relative version of Theorem 3.11. The proof is the same as Theorem 3.11 and is omitted.

LEMMA 3.14. *Let $s \in C_m^\infty(X; E)$ be a locally liftable smooth multisection on a compact orbifold $X$. Let $K \subseteq X$ be a compact set and assume that $s$ is transversal to* 0 *on a neighborhood of $K$. Then there exists a sequence $s_n \in C_m^\infty(X; E)$, such that $s_n$ converges to $s$ in the $C^\infty$-topology, $s_n$ is transversal to* 0 *and that $s_n = s$ on $K$.*

## 4. THE EULER CLASS

In this section, we define the Euler class of an orbibundle using multisections, which we introduced in Section 3. We remark that one can define it by using Chern–Weil theory and orbiconnections. (See [64].) However, our approach here can be generalized directly in later sections when we study the perturbation of the moduli space of pseudoholomorphic curves.

Let $pr: E \to X$ be an orbibundle and $s_0 \in C^0(X;E)$. We consider its locally liftable smooth perturbation which is transversal to 0 and is constructed in Section 3.

*Definition* 4.1. $s^{-1}(0)_{\mathrm{set}}$ is the set of all points $q$ of $X$ such that $s_{q,i}(q) = 0$ for some branch $s_{q,i}$ of $s$ around $q$.

LEMMA 4.2. *If $s$ is generic, then $s^{-1}(0)_{\mathrm{set}}$ has a smooth triangulation of dimension* $\dim X - \operatorname{rank} E$.

Here we say that $s^{-1}(0)_{\mathrm{set}} = \bigcup \Delta_a$ is a smooth triangulation if it is a triangulation and if the maps $\Delta_a \to X$ are smooth.

*Proof.* First we may assume that $s$ is transversal to 0 by Theorem 3.11. Let $q \in s^{-1}(0)_{\mathrm{set}}$, and $s_{q,i}\colon {}_q \to \mathbf{R}^k$, $i = 1, \dots, \ell$, be the branches of $s$ around $q$. Then a neighborhood of $q$ in $s^{-1}(0)_{\mathrm{set}}$ is diffeomorphic to

$$\pi_q\left(\bigcup_{i=1}^{\ell} s_{q,i}^{-1}(0)\right).$$

Here $\pi_q : {}_q \to X$. Since $s$ is transversal to 0, it follows that $s_{q,i}^{-1}(0)$ is a smooth manifold. The lemma follows immediately in the case when $\dim X - \operatorname{rank} E$ is 0 to 1, (that is the case we need to prove Theorem 1.1.)

In general we need some more technical argument to exclude the case when the set where two different branches begin to bifurcate is pathological.

For $p \in X$ we define $val_s(p)$ as the number of branches which have different values at $p$. This number is independent of the choice of local lift.

We also consider the order $\# I_p$ of the isotropy groups $I_p = \{\gamma \in \Gamma \mid \gamma\tilde{p} = \tilde{p}\}$. Here $/\Gamma$ is a chart containing $p$. It is independent of the choice of $\tilde{p}$.

We first prove that for generic $s$ the set

$$X_{v,m} = \{p \in X \mid val_s(p) = v,\ \# I_p = m\}.$$

is a smooth orbifold with smooth boundary. To see this we go back to the proof of Theorem 3.11 and use the notation there. It is well known and obvious that the set $\{p \in X \mid \# I_p = m\}$ is smooth. We may chose the partition of unity $\chi_i$ so that the domains $\chi_i^{-1}(0)$ have smooth boundaries which are transversal to each other and to $\{p \in X \mid \# I_p = m\}$. Therefore, we are only to work in the set

$$Y_{m,w} = \{p \in X \mid \# I_p = m,\ \#\{i \mid \chi_i(p) \neq 0\} = w\}\ .$$

We consider a small neighborhood $W(p)$ of $p \in Y_{m,w}$ in $Y_{m,w}$. Now the difference of the two branches (say $s_{n,\alpha}$ and $s_{n,\beta}$), of the multisection $s_n$, we constructed in the proof of Theorem 3.11, is

$$\sum_{i \in I_0} \chi_i(x)(\gamma_{i,\alpha}^{-1} v_i(\gamma_{i,\alpha}(x)) - \gamma_{i,\beta}^{-1} v_i(\gamma_{i,\beta}(x)))$$

on $x \in Y_{m,w}$. (Note that we start from a single valued section.) Here $I_0$ is a set of order $w$. We find that $\chi_i(x) \neq 0$ for any $i \in I_0$ and any $x \in Y_{m,k}$ in a neighborhood of a given point

$p \in Y_{m,w}$. For each pair $\alpha$ and $\beta$ either $\gamma_{i,\alpha}(x) \neq \gamma_{i,\beta}(x)$ for every $x \in W(p)$ or $\gamma_{i,\alpha}(x) = \gamma_{i,\beta}(x)$ for every $x \in W(p)$.

If there exists $i \in I_0$ such that $\gamma_{i,\alpha}(x) \neq \gamma_{i,\beta}(x)$ for every $x \in W(p)$, then we can take $v_i$ generic such that the set $\{x \in W(p) \mid s_{n,\alpha}(x) = s_{n,\beta}(x)\}$ is smooth.

If $\gamma_{i,\alpha}(x) = \gamma_{i,\beta}(x)$ for every $x \in W(p)$ then

$$s_{n,\alpha}(x) - s_{n,\beta}(x) = \sum_{i \in I_0} \chi_i(x)(\gamma_{i,\alpha}^{-1} v_i - \gamma_{i,\beta}^{-1} v_i)(x).$$

In this case again we can take $v_i$ generic that $\{x \in W(p) \mid s_{n,\alpha}(x) = s_{n,\beta}(x)\}$ is smooth.

Thus we have proved that $X_{v,m} \cap Y_{m,w}$ is smooth for any $w$. Therefore, $X_{v,m}$ is smooth. It is easy to see that $s^{-1}(0)_{\text{set}} \cap X_{v,m}$ is a disjoint union of smooth finitely many orbifolds (with boundary). Lemma 4.2 follows immediately. □

We remark that by the proof we may take triangulation $s^{-1}(0)_{\text{set}} = \bigcup \Delta_a$ such that the map $\Delta_a \to X$ has a lift to $\Delta_a \to {}_{q_s}$ for each $a$ and also we may assume that $val_s$ and $\# I_p$ is constant in the interior $Int\,\Delta_a$ of each simplex.

To go further, we need to define orientation of orbifolds and orbibundles.

*Definition* 4.3. A local model of an $n$-dimensional orbifold $( \ ,\Gamma)$ is said to be *oriented* if we have an orientation on which is preserved by the action of $\Gamma$.

An orbifold is *oriented* if it has an open covering $X = \bigcup_i U_i$, local models $( {}_i, \Gamma_i)$ and charts ${}_i \to U_i$ such that $( {}_i, \Gamma_i)$ is oriented and that the diffeomorphism $\varphi_{i,j;q} \colon {}_{q,i} \to {}_{q,j}$ in Definition 2.2 is orientation preserving.

A local model of smooth orbibundles of rank $k$, $( \ \times \mathbf{R}^k . \Gamma, pr)$ over $( \ ,\Gamma)$ is *oriented* if the action of $\Gamma$ on $\mathbf{R}^k$ is orientation preserving.

An orbibundle $pr : E \to X$ is *oriented*, if there exist charts $( {}_i \times \mathbf{R}^k, \Gamma_i, \tilde{\pi}_i)$ such that $( {}_i \times \mathbf{R}^k, \Gamma_i, pr)$ is oriented and $\tilde{\varphi}_{i,j;q} \colon {}_{q,i} \times \mathbf{R}^k \to {}_{q,j} \times \mathbf{R}^k$ in Definition 2.7 is fiberwise orientation preserving.

We are going to define the Poincaré dual to the Euler class of an oriented orbibundle over oriented orbifolds. In fact we can work under a bit weaker assumption. Let $pr : E \to X$ be an orbibundle over an orbifold $X$. The determinant bundle $\det TX$ of the tangent orbibundle of $X$ and determinant bundle $\det E$ of $E$ are well defined as orbibundles. Hereafter in this section, we *assume* that we have a trivialization of $\det TX \otimes \det E$. Note that if $X$ is oriented and $E$ is oriented then a trivialization of $\det TX \otimes \det E$ is induced by the orientations.

Now let $s$ be a multisection which is transversal to 0. Let $s^{-1}(0)_{\text{set}} = \bigcup \Delta_a$ be a smooth triangulation. We fix an orientation for each simplex $\Delta_a$ of dimension $n - k = \dim X - \operatorname{rank} E$. We then define multiplicity of each simplex $\Delta_a$ of dimension $n - k = \dim X - \operatorname{rank} E$ as follows. We may assume that $Int\,\Delta_a \subseteq X_{\ell_a,1}$. (Namely we may assume that it is in the regular part of the orbifold.) We take a lift $h_a : \Delta_a \to {}_{q_a}$. Let $s_{q_a,i} \colon {}_{q_a} \to \mathbf{R}^k$, $i = 1, \ldots, \ell_a$ be the branches of $s$ around $q_a$. Let $i_1, \ldots, i_{m_a}$ be the set of all indices $i$ such that $s_{q_a,i}(h_a(x)) = 0$ for $x \in Int\,\Delta_a$. (This is independent of $x$ since $Int\,\Delta_a \subseteq X_{\ell_a,0}$.) For each $j = 1, \ldots, m_a$ we can assign sign $\varepsilon_j = \pm 1$ as follows. We have an

$$\overline{0 \longrightarrow T_x\Delta_a} \overline{\xrightarrow{Dh_a} T \ {}_{q_a}} \overline{\xrightarrow{Ds_{q_a,i_j}} \mathbf{R}^k \longrightarrow}$$

By assumption we have a trivialization of $\det T_x\Delta_a$, $\det T \quad_{q_a} \otimes \det \mathbf{R}^k$. We put $\varepsilon_j = +1$ if exact sequence 4.4 is compatible with the trivializations and we put $\varepsilon_j = -1$ otherwise.

*Definition* 4.5. $mul_{\Delta_a} = \sum_{j=1}^{m_a} \varepsilon_j/\ell_a$. Here $\ell_a$ is the number of branches of $s$ in a neighborhood of $q_a$.

We remark that $mul_{\Delta_a}$ is independent of the choice of local representatives of $s$. To see this, we need only to check that it does not change when we replace $s_{q_a}$ by $tm_\ell \circ s_{q_a}$. In that case, the independence is obvious since the denominator and the numerator are both multiplied by $\ell$.

*Definition* 4.6. We define a singular chain $s^{-1}(0)$ with $\mathbf{Q}$ coefficient on $X$ by

$$s^{-1}(0) = \sum_a mul_{\Delta_a} \cdot \Delta_a.$$

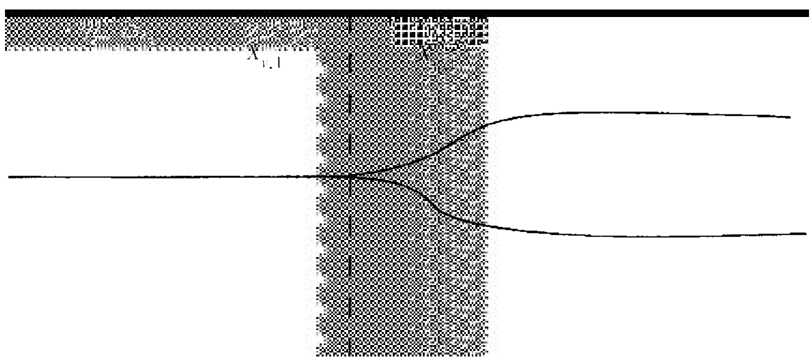
If we change the orientation of $\Delta_a$ then the sign of $mul_{\Delta_a}$ changes. Hence $s^{-1}(0)$ is independent of the orientation of $\Delta_a$. As a singular chain it depends on the choice of the triangulation. However, we are going to show that it is a cycle and its homology class depends only on the orbibundle.

Lemma 4.7. *$\partial s^{-1}(0) = 0$ as a singular chain.*

*Proof.* Let $\Delta_b$ be an $n-k-1$ dimensional simplex of $s^{-1}(0)_{\rm set}$. We are going to show that the coefficient of $\Delta_b$ in $\partial s^{-1}(0)$ vanishes. We first assume that $Int\,\Delta_b \subseteq X_{v,1}$. Let $y \in Int\,\Delta_b$. If $val_s$ is locally constant at $y$, then $s^{-1}(0)_{\rm set}$ is a smooth manifold in its neighborhood. Hence it is immediate to see that the coefficient of $\Delta_b$ in $\partial s^{-1}(0)$ vanishes.

If $val_s$ is not locally constant at $y$ then there are two cases. One case is that $Int\,\Delta_b \subseteq Int\,X_{v,1}$ and $\dim X_{v,1} < \dim X$. In this case, in a neighborhood of $y$, our set $s^{-1}(0)_{\rm set}$ is a union of finitely many smooth manifolds which intersects on $Int\,\Delta_b \subseteq Int\,X_{v,1}$. Hence the coefficient of $\Delta_b$ in $\partial s^{-1}(0)$ vanishes also.

The other case is $\dim X_{v,1} = \dim X$ and $Int\,\Delta_b \subseteq \partial X_{v,1}$. In this case there exists $v' > v$ such that $Int\,\Delta_b \subseteq \partial X_{v',1}$. We then find a local lift of $s$ around $y$ such that there are $v'$ different branches as a germ at $\tilde{y}$ (the point in which goes to $y$.) We can then find that, in a neighborhood of $y$, the set $s^{-1}(0)_{\rm set}$ is a union of finitely many smooth submanifolds, though some of them may coincide on $X_{v,1}$. (Fig. 4.8). It also follows that the coefficient of $\Delta_b$ in $\partial s^{-1}(0)$ vanishes.



Next we suppose that $Int\,\Delta_b \not\subset X_{v,1}$. Then $Int\,\Delta_b \subseteq Int\,X_{v,2}$. Let $y \in Int\,\Delta_b$. A neighborhood of $y$ in $X$ is identified to

$$\mathbf{R}^{n-1} \times \frac{\mathbf{R}}{\{\pm 1\}}.$$

The restriction of $E$ there is

$$\mathbf{R}^{n-1} \times \frac{\mathbf{R} \times \mathbf{R}^{2m-1}}{\{\pm 1\}} \times \mathbf{R}^{k-2m+1}.$$

We put $\tau: \mathbf{R}^k \to \mathbf{R}^k$, $\tau(v_1, \ldots, v_k) = (-v_1, \ldots, -v_{2m-1}, v_{2m}, \ldots, v_k)$. Then our multisection is given by

$$s(\mathbf{x}, a) = [s_1(\mathbf{x}, a), \ldots, s_\ell(\mathbf{x}, a), \tau s_1(\mathbf{x}, -a), \ldots, \tau s_\ell(\mathbf{x}, -a)].$$

It follows that the coefficient of $\Delta_b$ in $\partial s^{-1}(0)$ vanishes also in this case. The proof of Lemma 4.7 is now complete. □

THEOREM 4.9. *The homology class* $[s^{-1}(0)] \in H_{\dim X - \operatorname{rank} E}(X; \mathbf{Q})$ *is independent of the choice of multisection and the triangulation of* $s^{-1}(0)_{\text{set}}$ *and depends only on the orbibundle* $E$.

*Proof.* Let $s_0$ and $s_1$ be the two multisections which are transversal to 0. We consider the multisection $s(x, t) = ts_0(x) + (1-t)s_1(x)$ of $E \times \mathbf{R} \to X \times \mathbf{R}$. By using Lemma 3.14, we can perturb $s$ so that it is transversal to 0, and that $s|_{X \times \{0\}} = s_0$, $s|_{X \times \{1\}} = s_1$. Then we have a space $s^{-1}(0)_{\text{set}} \cap (X \times [0,1])$. We can extend given triangulations of $s_0^{-1}(0)_{\text{set}}$ and $s_1^{-1}(0)_{\text{set}}$ to a triangulation of $s^{-1}(0)_{\text{set}} \cap (X \times [0,1])$. Using this triangulation we obtain a $\mathbf{Q}$ chain $s^{-1}(0) \cap (X \times [0,1])$. By the proof of Lemma 4.7 we have

$$\partial s^{-1}(0) \cap (X \times [0,1]) = s_1^{-1}(0) - s_0^{-1}(0).$$

Theorem 4.9 follows. □

We next assume that $X$ is oriented. It follows that $X$ is a $\mathbf{Q}$-homology manifold. Hence we have Poincaré duality over $\mathbf{Q}$

$$H^d(X; \mathbf{Q}) \cong H_{\dim X - d}(X; \mathbf{Q}). \tag{4.10}$$

*Definition* 4.11. We call the Poincaré dual to the element $[s^{-1}(0)] \in H_{\dim X - \operatorname{rank} E}(X; \mathbf{Q})$, the *Euler class* of $E$.

*Remark* 4.12. We can continue in a similar way to define the Chern classes and the Pontrjagin classes. The basic idea is to define a "multivalued classifying map" by taking a finitely many multisections $s_i$ such that, for any choices of branches of $s_i$, the values of $s_i$, $i = 1, 2, \ldots$, at $p$ generates the fiber of the orbibundle of $p$. We omit this construction since

## 5. KURANISHI STRUCTURE

*Definition* 5.1. A *Kuranishi neighborhood* of $p \in X$ is a system $(U_p, E_p, s_p, \psi_p)$ where

(5.1.1) $U_p = {}_p/\Gamma_p$ is an orbifold and $E_p$ is an orbibundle on it.
(5.1.2) $s_p$ is a (single valued) continuous section of $E_p$.
(5.1.3) $\psi_p$ is a homeomorphism from $s_p^{-1}(0)$ to a neighborhood of $p$ in $X$.

? need differentiability ? (see p. 950)

*Definition* 5.2. Let $(U_p, E_p, s_p, \psi_p)$ and $(U'_p, E'_p, s'_p, \psi'_p)$ be Kuranishi neighborhoods of $p$. We write $(U_p, E_p, s_p, \psi_p) \sim (U'_p, E'_p, s'_p, \psi'_p)$ if there exists another Kuranishi neighborhood $(U''_p, E''_p, s''_p, \psi''_p)$ and $I : U''_p \to U_p$, $I' : U''_p \to U'_p$, $J : E''_p \to E_p$, $J' : E''_p \to E'_p$ such that

conjugation relation between charts

(5.2.1) $I, I'$ are diffeomorphisms to their images. $J, J'$ are bundle isomorphisms covering $I$, $I'$ respectively. $\dim U''_p = \dim U_p = \dim U'_p$.

(5.2.2) $J \circ s''_p = s_p \circ I$, $J' \circ s''_p = s'_p \circ I'$.

(5.2.3) $\psi_p \circ I = \psi''_p$, $\psi'_p \circ I' = \psi''_p$.

The equivalence class of Kuranishi neighborhood of $p$ with respect to $\sim$ is called a *germ of Kuranishi neighborhood*. $(U_p, E_p, s_p, \psi_p)$ representing it is called a (representative of) Kuranishi neighborhood.

*Definition* 5.3. A *Kuranishi structure* of dimension $n$ on $X$ assigns a germ of Kuranishi neighborhood to each $p \in X$, And for each representative $(U_p, E_p, s_p, \psi_p)$ of it, and each $q \in \psi_p(s_p^{-1}(0))$ there exists a germ of maps $\varphi_{pq}$ and $\hat\varphi_{pq}$ with the following properties.

fixed choice? or ambiguity? do all or just some satisfy cocycle?

(5.3.1) $\varphi_{pq} : U_q \to U_p$ is an embedding of orbifolds. $\hat\varphi_{pq} : E_q \to E_p$ is an embedding of orbibundles covering $\varphi_{pq} : U_q \to U_p$.

(5.3.2) $s_p \circ \varphi_{pq} = \hat\varphi_{pq} \circ s_q$, $\psi_p \circ \varphi_{pq} = \psi_q$.

(5.3.3) If $r \in \psi_q(s_q^{-1}(0))$, then $\varphi_{pq} \circ \varphi_{qr} = \varphi_{pr}$, $\hat\varphi_{pq} \circ \hat\varphi_{qr} = \hat\varphi_{pr}$.

(5.3.4) $\dim U_p - \operatorname{rank} E_p = n$ is independent of $p$.

cocycle condition unclear
– on what domain?
– meaning in terms of germs?
– compatibility with above conjugation relation:

We call $(U_p, E_p, s_p, \psi_p)$ a *chart* and $(\varphi_{pq}, \hat\varphi_{pq})$ the *coordinate change*.

For each representatives $U_p, U_q, U_r$ there exist $\varphi_{pq}, \varphi_{qr}, \varphi_{pr}$ so that □ holds? Or □ holds for any choice of $\varphi_{\cdot\cdot}$?

Here, by germs of maps $\varphi_{pq}$ and $\hat\varphi_{pq}$, we mean the following. For each sufficiently small representative $(U_q, E_q, s_q, \psi_q)$, we have $\varphi_{pq} : U_q \to U_p$, $\hat\varphi_{pq} : E_q \to E_p$. If $(U'_q, E'_q, s'_q, \psi'_q)$, $\varphi'_{pq} : U'_q \to U_p$, $\hat\varphi'_{pq} : E'_q \to E'_p$ be another representative, and if $I : U''_p \to U_p$, $I' : U''_p \to U'_p$, $J : E''_p \to E_p$, $J' : E''_p \to E'_p$ be as in Definition 5.2, then

(5.4.1) $\varphi_{pq} \circ I = \varphi''_{pq}$, $\varphi'_{pq} \circ I' = \varphi''_{pq}$.

(5.4.2) $\hat\varphi_{pq} \circ J = \hat\varphi''_{pq}$, $\hat\varphi'_{pq} \circ J' = \hat\varphi''_{pq}$.

Seems to indicate that $\varphi_{pq}$'s are fixed and cocycle condition is $[\varphi_{pq}] \circ [\varphi_{qr}] = [\varphi_{pr}]$ for conjugacy classes – note that this composition is generally ill defined,

When we replace $(U_q, E_q, s_q, \psi_q)$ by another representative, we assume a similar compatibility condition as (5.4). Hereafter we omit this kind of remarks.

i.e. cocycle condition for one set of representatives does not imply cocycle for other choices of representatives.

*Remark* 5.5. Since we are using map germs it may be natural to use sheaf theory and maybe etale topology (because the action of a finite group is involved.) But we do not try to do it here to keep the exposition as elementary as possible.

Our purpose in Sections 5, 6, is to define the fundamental class of Kuranishi structure. We need to define an orientation of Kuranishi structure for this purpose. We first start with defining a "tangent bundle". Let $q, p$ be as in (5.3). We have a normal bundle $N_{U_p}U_q$.

*Definition* 5.6. We say that a Kuranishi structure $(U_p, E_p, s_p, \psi_p, \varphi_{pq}, \hat\varphi_{pq})$ *has a tangent bundle* if there exists a family of (germs of) isomorphisms

$$\Phi_{pq} : N_{U_p}U_q \cong E_p/E_q$$

this needs to be $ds_p$ (as pointed out by Joice), thus need $s_p$ differentiable such that the Diagram 5.7 below commutes for $U_r \subseteq U_q \subseteq U_p$

$$
\begin{array}{ccccccccc}
0 & \to & N_{U_q}U_r & \to & N_{U_p}U_r & \to & N_{U_p}U_q|_{U_r} & \to & 0 \\
 & & \downarrow \Phi_{qr} & & \downarrow \Phi_{pr} & & \downarrow \Phi_{pq} & & \\
0 & \to & E_q/E_r & \to & E_p/E_r & \to & E_p/E_q|_{U_r} & \to & 0
\end{array}.
$$

Diagram 5.7.

We next define an orientation of Kuranishi structure. Let $\det TU_p$, $\det E_p$ be the determinant bundles of tangent and obstruction bundles. The isomorphism $\Phi_{pq} : N_{U_p}U_q \cong E_p/E_q$ induces an isomorphism $\det TU_p \otimes \det E_p|_{U_q} \cong \det TU_q \otimes \det E_q$.

*Definition* 5.8. We say that a Kuranishi structure $(U_p, E_p, s_p, \psi_p, \varphi_{pq}, \hat{\varphi}_{pq})$ is *oriented* if it has a tangent bundle, and if there are family of trivializations of $\det TU_p \otimes \det E_p$ which is compatible with the isomorphism $\det TU_p \otimes \det E_p|_{U_q} \cong \det TU_q \otimes \det E_q$ of bundles.

*Example* 5.9. Let $M$ be a compact manifold which is *not* orientable. Let $TM$ be its tangent bundle. We put $E_p = TM$ and obtain a 0-dimensional Kuranishi structure on $M$. (Here $U_p = \ _p = M$ and $s_p = 0$.) Since $\det TU_p \otimes \det E_p = \det TM \otimes \det TM$ is canonically oriented, it follows that this Kuranishi structure is oriented.

In this example, the fundamental class is well defined. In fact it is the "Poincaré dual to the Euler class of $E = TM$." We remark that the Euler class of $E$ is not well defined and Poincaré duality is not well defined either over the integers. However "Poincaré dual to the Euler class of $E = TM$" is well defined and is equal to the Euler number $\in H_0(M; \mathbf{Z}) \neq H^{\dim M}(M; \mathbf{Z})$. Our definition of the Euler class in Section 4 is designed to include this case.

Next we are going to define a stably almost complex structure. For this purpose it is useful to define a $K$-theory over Kuranishi structures. Also in order to construct an orientation of Kuranishi structure it is essential to use it. The $K$-theory we use is a kind of $K$-group of orbibundles. However, we need to modify the usual definition in a couple of places. First of all, we need to have tangent bundle as an element of our $K$-group. So we need to consider Grothendieck group of some kind of systems of pairs $(TU_p, E_p)$ with compatibility condition as Diagram 5.7, in place of taking a Grothendieck group of orbibundles. In fact, in the case of usual $K$-theory, it gives the same result. (One can prove it by using Mayer–Vietoris exact sequence of $K$-cohomology.) However, in our situation, it may define a different group. The reason is as follows. If we have a vector bundle $E$ on an open set $U \subseteq X$, then we can find $E'$ such that $E \oplus E'$ is trivial (and in particular is a restriction of a vector bundle on $X$). In the case of orbibundle it is no longer true. (The same trouble already happens when one tries to define $K$-group of orbibundles over orbifolds.) The experts of $K$-theory may find it routine, but we include it here since there is no appropriate reference.

*Definition* 5.10. Let $X = (X, (U_p, E_p, s_p, \psi_p, \varphi_{pq}, \hat{\varphi}_{pq}))$ is a space with Kuranishi structure. We say the following objects are a *bundle system.*

(5.10.1) For each point $p \in X$ there exists a germ of orbibundles $F_{1,p}$, $F_{2,p}$ on its Kuranishi neighborhood.

(5.10.2) Let $q \in \psi(U_p)$ and $\varphi_{pq}$ is the coordinate change. Then there exist germs of embeddings of orbibundles $\Phi_{1;pq} : F_{1,q} \to F_{1,p}|_{U_q}$, $\Phi_{2;pq} : F_{2,q} \to F_{2,p}|_{U_q}$ and an isomorphism of orbibundles

$$\Phi_{pq} : \frac{F_{1,p}|_{U_q}}{F_{1,q}} \to \frac{F_{2,p}|_{U_q}}{F_{2,q}}.$$

(5.10.3) If $r \in \psi_q(U_q) \subseteq \psi_p(U_p)$, then $\Phi_{1;pq} \circ \Phi_{1;qr} = \Phi_{1;pr}$, $\Phi_{2;pq} \circ \Phi_{2;qr} = \Phi_{2;pr}$.

(5.10.4) The following diagram commutes for $r \in \psi_q(U_q) \subseteq \psi_p(U_p)$.

$$\begin{array}{ccccccccc}
0 & \to \dfrac{F_{1,q}|_{U_r}}{F_{1,r}} & \to & \dfrac{F_{1,p}|_{U_r}}{F_{1,r}} & \to & \dfrac{F_{1,p}|_{U_r}}{F_{1,q}|_{U_r}} & \to & 0 \\
& \downarrow \Phi_{qr} & & \downarrow \Phi_{pr} & & \downarrow \Phi_{pq} & & \\
0 & \to \dfrac{F_{2,q}|_{U_r}}{F_{2,r}} & \to & \dfrac{F_{2,p}|_{U_r}}{F_{2,r}} & \to & \dfrac{F_{2,p}|_{U_r}}{F_{2,q}|_{U_r}} & \to & 0
\end{array}$$

Diagram 5.11.

We say that $(F_{1,p}, F_{2,p})$ is a *chart* of our bundle system and $(\Phi_{pq}, \Phi_{1,pq}, \Phi_{2,pq})$ is its *coordinate change.*

We define an *isomorphism* between two bundle systems as follows. $((F_{1,p}, F_{2,p}), (\Phi_{1,pq}, \Phi_{2,pq}, \Phi_{pq}))$ is *isomorphic* to $((F'_{1,p}, F'_{2,p}), (\Phi'_{1,pq}, \Phi'_{2,pq}, \Phi'_{pq}))$, if for each $p \in X$ there exists a germs of isomorphisms $\Psi_{1,p} : F_{1,p} \cong F'_{1,p}$, $\Psi_{2,p} : F_{2,p} \cong F'_{2,p}$ which commute with $\Phi_{1;pq}$, $\Phi_{2;pq}$, $\Phi_{pq}$ and $\Phi'_{1;pq}$, $\Phi'_{2;pq}$, $\Phi'_{pq}$.

*Example* 5.12. If a Kuranishi structure has a tangent bundle then its tangent bundle is well defined as a bundle system. Namely we take $F_{1,p} = TU_p$, $F_{2,p} = E_p$, $\Phi_{2;pq} = \hat{\varphi}_{pq}$. $\Phi_{1;pq}$ is an inclusion: $TU_q \to TU_p$, that is the differential of the embedding $\varphi_{pq}$. The isomorphism $N_{U_p}U_q \cong E_p/E_q$ induces an isomorphism

$$\Phi_{pq} : \frac{F_{1,p}|_{U_q}}{F_{1,q}} \to \frac{F_{2,p}|_{U_q}}{F_{2,q}}.$$

The commutativity of Diagram 5.11 is a consequence of the commutativity of Diagram 5.7.

*Definition* 5.13. A bundle system is said to be oriented if $F_{1,p}$, $F_{2,p}$ are oriented and

$$\Phi_{pq} : \overline{\frac{F_{1,p}|_{U_q}}{F_{1,q}}} \to \overline{\frac{F_{2,p}|_{U_q}}{F_{2,q}}}$$

is orientation preserving. It is said to be complex if $F_{1,p}$, $F_{2,p}$ are complex and $\Phi_{1;pq}$, $\Phi_{2;pq}$, $\Phi_{pq}$ are complex linear.

One can define Whitney sum, tensor product, etc. of bundle system in an obvious way.

*Definition* 5.14. A bundle system $((F_{1,p}, F_{2,p}), (\Phi_{1,pq}, \Phi_{2,pq}, \Phi_{pq}))$ is said to be trivial if there exist germs of isomorphisms $F_{1,p} \cong F_{2,p}$ which are compatible with $(\Phi_{1,pq}, \Phi_{2,pq}, \Phi_{pq})$.

*Definition* 5.15. We consider the free abelian group generated by the set of all isomorphism classes of bundle systems and divide it by the relations

$$[((F_{1,p}, F_{2,p}), (\Phi_{1,pq}, \Phi_{2,pq}, \Phi_{pq})) \oplus ((F'_{1,p}, F'_{2,p}), (\Phi'_{1,pq}, \Phi'_{2,pq}, \Phi'_{pq}))]$$

$$= [((F_{1,p}, F_{2,p}), (\Phi_{1,pq}, \Phi_{2,pq}, \Phi_{pq}))] + [((F'_{1,p}, F'_{2,p}), (\Phi'_{1,pq}, \Phi'_{2,pq}, \Phi'_{pq}))]$$

$$[((F_{1,p}, F_{2,p}), (\Phi_{1,pq}, \Phi_{2,pq}, \Phi_{pq}))] = 0 \quad \text{if } ((F_{1,p}, F_{2,p}), (\Phi_{1,pq}, \Phi_{2,pq}, \Phi_{pq})) \text{ is trivial.}$$

Let us write $KO(X)$ for the group we obtain and call it the real $K$-group of our Kuranishi structure. By using oriented bundle system and complex bundle system, we define $KSO(X)$ and $K(X)$.

There is an obvious map

$$K(X) \to KSO(X) \to KO(X). \tag{5.16}$$

The tangent bundle system $(TU_p, E_p)$ defines an element of $KO(X)$, which we write $[TX]$.

*Definition* 5.17. A Kuranishi structure $X$ is said to be *stably orientable* if it has a tangent bundle and if $[TX]$ is in the image of $KSO(X)$. It is said to be *stably almost complex* if $[TX]$ is in the image of $K(X)$.

This definition is a generalization of the definition of stably almost complex structure of manifolds [48].

Lemma 5.18. *A Kuranishi structure is stably orientable if and only if it is orientable.*

We remark that we say that a Kuranishi structure is orientable if it has an orientation in the sense of Definition 5.8. There is a result corresponding to Lemma 5.17 in usual $K$-theory, which is obvious.

*Proof.* Let $((F_{1,p}, F_{2,p}), (\Phi_{1,pq}, \Phi_{2,pq}, \Phi_{pq}))$ be a bundle system. It is easy to see that the line bundle $\det F_{1,p} \otimes \det F_{2,p}$ is well defined and depends only on equivalence class of $((F_{1,p}, F_{2,p}), (\Phi_{1,pq}, \Phi_{2,pq}, \Phi_{pq}))$ in $KO(X)$. Furthermore, it is trivial if $((F_{1,p}, F_{2,p}), (\Phi_{1,pq}, \Phi_{2,pq}, \Phi_{pq}))$ is in $KSO(X)$. On the contrary, we suppose that the space $X$ with a Kuranishi structure is orientable. We consider its tangent bundle system $(TX, E)$. By definition, we find that $(TX, E) \oplus (TX, TX)$ is orientable bundle system. Hence $[TX] = [(TX, E) \oplus (TX, TX)]$ is in the image of $KSO(X)$. The proof of Lemma 5.18 is now complete. □

## 6. PERTURBATION OF A SPACE WITH KURANISHI STRUCTURE

Let $X = (X, ( \ _q, \Gamma_q, E_q, s_q, \psi_q))$ be a space with Kuranishi structure and we assume that

$$\Phi_{pq} : N_{U_p} U_q \cong E_p / E_q.$$

*Definition* 6.1. $(P,((U_p,\psi_p,s_p):p\in P),\varphi_{pq},\hat\varphi_{pq})$ is said to be a *good coordinate system* of $X$ if the following conditions are satisfied. $P\subseteq X$ is a finite subset equipped with an order. $(U_p,E_p,s_p)$ is a representative of a chart for each $p\in P$, $U_{pq}\subseteq U_q$ is a subset for each $p,q$ with $\psi_p(s_p^{-1}(0))\cap\psi_q(s_q^{-1}(0))\neq\emptyset$, $q<p$, and $\varphi_{pq}:U_{pq}\to U_p$, $\hat\varphi_{pq}:E_q|_{U_{p,q}}\to E_p$ are embeddings, such that

(6.1.1) $\bigcup_{p\in P}\psi_p(s_p^{-1}(0))=X$.

(6.1.2) $U_{pq}$ is an open neighborhood of $\psi_q^{-1}(\psi_p(s_p^{-1}(0)))$.

(6.1.3) If $x\in U_{pq}$ and $\varphi_{px}:U_x\to U_q$, $\varphi_{qx}:U_x\to U_q$, $\hat\varphi_{px}:E_x\to E_p$, $\hat\varphi_{qx}:E_x\to E_q$ be map germs giving the coordinate change. Then $\varphi_{pq}\varphi_{qx}=\varphi_{px}$, $\hat\varphi_{pq}\hat\varphi_{qx}=\hat\varphi_{px}$ as map germs.

(6.1.4) Suppose $r<q<p$, $\psi_p(s_p^{-1}(0))\cap\psi_q(s_q^{-1}(0))\cap\psi_r(s_r^{-1}(0))\neq\emptyset$. Then $\varphi_{pq}\circ\varphi_{qr}=\varphi_{pr}$, $\hat\varphi_{pq}\circ\hat\varphi_{qr}=\hat\varphi_{pr}$ on $\varphi_{qr}^{-1}(U_{pq})$ and $E_r|_{\varphi_{qr}^{-1}(U_{pq})}$.

(6.1.5) $s_p\circ\varphi_{pq}=\hat\varphi_{pq}\circ s_p$, $\psi_p\circ\varphi_{pq}=\psi_q$.

*Remark* 6.2. Here $(U_p,E_p,s_p)$ is a representative of chart hence it is a Kuranishi neighborhood and is not a germ. Similarly $\varphi_{pq}:U_{p,q}\to U_p$, $\hat\varphi_{pq}:E_q|_{U_{p,q}}\to E_p$ are maps and not germs of maps. (6.1.4), (6.1.5) are equalities of maps and not of map germs. Condition (6.1.3) is added to make sure that our coordinate system is compatible with the Kuranishi structure we start with. If there exists $(P,((U_p,\psi_p,s_p):p\in P),\varphi_{pq},\hat\varphi_{pq})$ satisfying (6.1.1), (6.1.2), (6.1.4), (6.1.5), then we can define a Kuranishi structure on $X$ so that (6.1.3) is satisfied. We omit the proof of this fact since we do not use it.

Lemma 6.3. *For any open covering of the space X, there exists a good coordinate system such that the covering* (6.1.1) *is a subdivision of the given open covering.*

This is rather a technical lemma. We give a proof of it at the end of the section for completeness.

Theorem 6.4. *Let* $(P,((U_p,\psi_p,s_p):p\in P),\varphi_{pq},\hat\varphi_{pq})$ *be a good coordinate system of a space X with Kuranishi structure. Suppose that X has a tangent bundle given by*

$$\Phi_{pq}:N_{U_p}U_q\cong E_p/E_q.$$

*Then, for each* $p\in P$, *there exists a sequence of smooth multisections* $s_{p,n}$ *such that*

(6.4.1) $s_{p,q}\circ\varphi_{pq}=\hat\varphi_{pq}\circ s_{q,n}$,

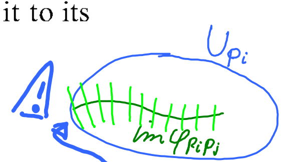
(6.4.2) $\lim_{n\to\infty}s_{p,n}=s_p$ *in the* $C^0$*-topology*

(6.4.3) $s_{p,n}$ *is transversal to* 0.

(6.4.4) *Let* $x\in U_{pq}$. *Then the restriction of the differential of the composition of any branch of* $s_{p,n}$ *and the projection* $E_p\to E_p/E_q$ *coincides with the isomorphism* $\Phi_{pq}:N_{U_p}U_q\cong E_p/E_q$.

*Proof.* We write $P = \{p_1, p_2, \dots\}$ such that $p_i < p_{i+1}$. We may assume that $\operatorname{rank} E_{p_i} \leqslant \operatorname{rank} E_{p_{i+1}}$. We construct $s_{p_i,n}$ by induction on $i$. For $i = 1$, we apply Theorem 3.11 and Lemma 3.12 to obtain $s_{p_i,n}$ satisfying (6.4.2) and (6.4.3). Now let us assume that we have constructed $s_{p_j,n}$ for $j < i$ satisfying eqs (6.4.1)–(6.4.4) and will construct $s_{p_i,n}$. For each $j$ with $\psi_{p_i}(s_{p_i}^{-1}(0)) \cap \psi_{p_j}(s_{p_j}^{-1}(0)) \neq \emptyset$, we have a section $s_{p_j,n}$ on $\varphi_{p_ip_j}(U_{p_ip_j})$. We first extend it to its tubular neighborhood in $U_{p_i}$. For this purpose we use the isomorphism

$$\Phi_{p_ip_j}\colon N_{U_{p_i}} U_{p_i} \cong E_{p_j}/E_{p_j}.$$




Namely we identify $N_{U_{p_i}}(\varphi_{p_ip_j}(U_{p_ip_j}))$ with a tubular neighborhood of $\varphi_{p_ip_j}(U_{p_ip_j})$ in $U_{p_i}$. Let $\pi\colon N_{U_{p_i}}(\varphi_{p_ip_j}(U_{p_ip_j})) \to \varphi_{p_ip_j}(U_{p_ip_j})$. We choose and fix a metric of orbibundles $E_{p_i}$ compatible with $\hat\varphi_{p_ip_j}$. Using it we decompose

$$E_{p_i} \cong E_{p_i}/E_{p_j} \oplus E_{p_j}. \tag{6.5}$$

*should worry about lack of compactness here, resp. boundary of $U_{p_i}$ — "keep geodesics / tub. nbhd inside $U_{p_i}$"*

Then we put for $x \in N_{U_{p_i}}(\varphi_{p_ip_j}(U_{p_ip_j}))$

$$\tilde s_{p_j,n}(x) = I_x(\Phi_{p_ip_j}(x) \oplus s_{q_j,n}(x)).$$

Here $I_x$ is an isomorphism $I_x\colon E_{p_i}(\pi(x)) \to E_{p_i}(x)$ obtained by parallel transport along minimal geodesic. $I_x$ is well defined if $x$ is sufficiently close to $\varphi_{p_ip_j}(U_{p_ip_j})$. Since eq. (6.5) is an isomorphism and $s_{p_j,n}$ are transversal to 0, it follows that $\tilde s_{p_j,n}$ is transversal to 0.

*but also for tubular nbhds?*

Using induction hypothesis we have compatibility condition (6.4.1). Also we have a compatibility condition for $\Phi_{pq}\colon N_{U_p} U_q \cong E_p/E_q$. Thus $\tilde s_{p_j,n}$ for various $j$ together with its first derivatives (on normal direction) coincide on $\bigcup_{j<i} \operatorname{Im}\varphi_{p_ip_j}$. Hence we can glue $\tilde s_{p_j,n}$ for various $j$ and $s_{p_j}$ by partition of unity. (We remark that we did not change the number of branches when we extend the multisection on $\varphi_{p_ip_j}(U_{p_ip_j})$ to its tubular neighborhood. Hence we can add only the branches coming from the same branch to glue them using partition of unity. In other words, gluing by partition of unity here does not mean we use the sum in Definition 3.4.)

We then obtain $s'_{p_i,n}$ which satisfies (6.4.1), (6.4.2) (6.4.4) and which is transversal to 0 in a neighborhood of the union of $\varphi_{p_ip_j}(U_{p_ip_j})$. Thus we can use Theorem 3.11 or Lemma 3.14 again and obtain $s_{p_i,n}$, which is transversal to 0, and which is equal to $s'_{p_i,n}$ in a neighborhood of the union of $\varphi_{p_ip_j}(U_{p_ip_j})$.

We thus complete the proof of Theorem 6.4. □

*⚠ these perturb a compact zero set, but here the complement of the nbhd on which we have transversality is not compact, nor is the set on which we already have transversality*

Now we use Theorem 6.4 and Lemma 6.3 to define the fundamental class of Kuranishi structure. We consider the following situation. Let $Y$ be a topological space, and $X$ be a

*Definition* 6.6. *A strongly continuous map* $f\colon X \to Y$ is a system of map germs $f_p\colon U_p \to Y$ for each $p$ such that $f_p \circ \varphi_{pq} = f_q$.

Suppose that $Y$ is an orbifold. We say that $f$ is *strongly smooth* if each $f_p$ is smooth. We define the rank of $f$ at $p$ by $\operatorname{rank}_p f = \operatorname{rank}(d_p f_p)$.

We say that $f$ is *of maximal rank* if $\operatorname{rank}_p f = \min\{\dim X + \dim E_p, \dim Y\}$ at every $p$.

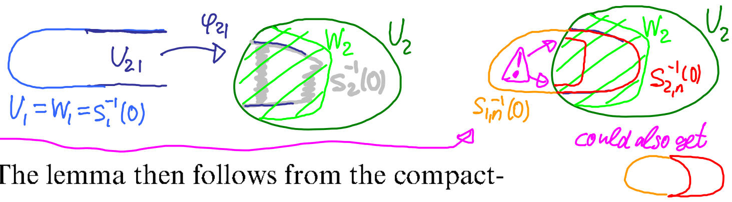
$$f([X]) \in H_n(Y; \mathbf{Q}).$$

*↳ construction of virtual fundamental class*

We choose representatives of map germs $f_p : U_{0,p} \to Y$. We take a good coordinate system finer than $U_{0,p}$. We then get a sequence of multisections $s_{p,n}$ as in Theorem 6.4. For each $p_i$ we consider a small neighborhood $W_i$ of $s_{p_i}^{-1}(0)$. Let $\varepsilon_i = \inf\{\|s_i(x)\| \,|\, x \notin W_i\}$. By taking $n$ large, we may assume that for any branch $s_{p_i,j,n}$ of $s_{q_i,n}$ we have

$$\|s_{p_i,j,n} - s_{p_i}\| \leqslant \frac{\varepsilon_i}{10}. \tag{6.7}$$

We let $s_{p_i,n}^{-1}(0)_{\mathrm{set}}$ be the set of all $x \in U_{p_i}$ such that $s_{p_i,j,n}(x) = 0$ for some branch $s_{p_i,j,n}$ of $s_{p_i,n}$. By (6.4.1), we have $\varphi_{p_ip_j}(x) \in s_{p_i,n}^{-1}(0)_{\mathrm{set}}$ if $x \in s_{p_j,n}^{-1}(0)_{\mathrm{set}}$ and if $\varphi_{p_ip_j}$ is defined at $x$. We can thus glue $s_{p_i,n}^{-1}(0)_{\mathrm{set}}$ to obtain a space $s_n^{-1}(0)_{\mathrm{set}}$.




Lemma 6.8. *$s_n^{-1}(0)_{\mathrm{set}}$ is compact.*

*Proof.* By (6.7) $s_{p_i,n}^{-1}(0)_{\mathrm{set}}$ is contained in $W_i$. The lemma then follows from the compactness of $X$. □

Lemma 6.9. *If $s_n$ is generic, then $s_n^{-1}(0)_{\mathrm{set}}$ has a smooth triangulation.*

The proof is the same as the proof of Lemma 4.2.

Let $\hat{s}_n^{-1}(0)_{\mathrm{set}} = \bigcup_u \Delta_u$ be the triangulation. We may assume that each simplex is contained in some $U_{p_i}$ and can be lifted to $_{p_i}$ and also that $val_{s_n}$ is constant at the interior of each simplex. For each simplex of dimension $\dim X = \dim\ _{p_i} - \operatorname{rank} E_{p_i}$ we can define its multiplicity $mul_{\Delta_u}$ in the same way as in Section 4.

We remark that the multiplicity is well defined. Namely let us regard that $\Delta_u$ is contained either in $U_{p_i}$ or $U_{p_j}$. We then find that the multiplicities we obtained coincides with each other. This is a consequence of (6.4.4).

We now put

$$f_*(s_n^{-1}(0)) = \sum_u mul_{\Delta_u} f_{p_i *}([\Delta_u]). \tag{6.10}$$

$f_*(s_n^{-1}(0))$ is a $\mathbf{Q}$-singular chain in $Y$. Here $f_{p_i} : U_{p_i} \to Y$ where $\Delta_u \in U_{p_i}$. The condition $f_p \circ \varphi_{pq} = f_q$ implies that (6.10) is independent of the choice of $U_{p_i}$ *with* $\Delta_u \in U_{p_i}$.

Lemma 6.11. *If $X$ is oriented then $\partial f_*(s_n^{-1}(0)) = 0$.*

Theorem 6.12. *If $X$ is oriented then $[f_*(s_n^{-1}(0))] \in H_n(Y; \mathbf{Q})$ is independent of choice of $s_n$.*

Again the proof is the same as the proof of Theorem 4.8. We denote the left-hand side by $f_*([X])$.

*Remark* 6.13. Let us assume moreover that $f$ is smooth and of maximal rank. Let $C \subseteq Y$ be a piecewise smooth cycle. Then it follows easily from the proof of Theorem 6.4 that we may choose our multisections $s_n$ such that the restriction of $f_p$ to each simplex $\Delta_a \subset s_n^{-1}(0)_{\mathrm{set}}$ is transversal to $C$.

Finally we prove Lemma 6.3. For each $p \in X$ the rank of $E_p$ is independent of representatives. We denote it $n_p$. We put

$$X_k = \{p \in X \mid n_p \geqslant k\}.$$

It is easy to see that $X_k$ is a closed set. By compactness of $X$, there exists $k_0$, such that the set $X_k$ is empty for $k \geqslant k_0$. We are going to construct a covering $\bigcup_{\ell \geqslant k} \bigcup_{p \in P_\ell} \psi_{\ell,p} s_{\ell,p}^{-1}(0) \supseteq X_k$ by downward induction on $k$.

Namely we assume that we have a covering $\bigcup_{\ell \geqslant k+1} \bigcup_{p \in P_\ell} \psi_{\ell,p} s_{\ell,p}^{-1}(0) \supseteq X_{k+1}$ with the properties (6.14) below and will construct $\bigcup_{\ell \geqslant k} \bigcup_{p \in P_\ell} \psi_{\ell,p} s_{\ell,p}^{-1}(0) \supseteq X_k$ with the same properties (6.14). Hereafter, we write $\bar{U}_{\ell,p} = \psi_{\ell,p} s_{\ell,p}^{-1}(0)$. We remark that $\bar{U}_{\ell,p}$ is an *open* subset of $X$.

(6.14.1) $P_k \subseteq X_k - X_{k+1}$.
(6.14.2) $(U_{i,p}, E_p, s_p)$ is a representative of coordinate chart around $p$.
(6.14.3) Conditions (6.1.2)–(6.1.5) are satisfied.
(6.14.4) $\bigcup_{\ell \geqslant k} \bigcup_{p \in P_\ell} \bar{U}_{\ell,p} \supseteq X_k$ is a subdivision of the given covering of $X$.

We recall that our space $X$ is a metric space. Let $d$ denote the metric. We put

$$D(X_k, r) = \{x \in X \mid d(x, X_k) < r\}$$

$$D_p(r) = \{x \in X \mid d(p, x) < r\}.$$

We have positive number $\varepsilon_1 > 0$ such that

$$D(X_{k+1}, \varepsilon_1) \subseteq \bigcup_{\ell \geqslant k+1} \bigcup_{p \in P_\ell} \bar{U}_{\ell,p}. \tag{6.15}$$

We choose a representative of coordinate chart $(U_p^{(1)}, E_p, s_p)$ around $p \in X_k - X_{k+1}$. We assume

(6.16.1) $\bar{U}_p^{(1)} \cap X_{k+1} = \emptyset$.
(6.16.2) Each of $U_p^{(1)}$ is contained in a member of the given covering.

We take a finite set $Q = \{q_1, \ldots, q_m\} \subseteq X_k - D(X_{k+1}, \varepsilon_1/2)$ such that

$$D(X_{k+1}, \varepsilon_1/2) \cup \bigcup_{q \in Q} \bar{U}_q^{(1)} \supseteq X_k.$$

We can then find $\varepsilon_2 > 0$ such that $\varepsilon_2 < \varepsilon_1/100$ and that

(6.17) If $x \in X_k$, $d(x, X_{k+1}) > \varepsilon_1/2 - \varepsilon_2$, then there exists $q \in Q$ such that $D_x(\varepsilon_2) \subseteq \bar{U}_q^{(1)}$.

We next take $\varepsilon_3$ and an open subset $U_p'$ of $U_p$ for each $p \in \bigcup_{\ell > k} P_\ell$ such that

(6.18.1) If $x \in \bar{U}_p'$ then $D_x(\varepsilon_3) \subseteq \bar{U}_p$.
(6.18.2) $\bigcup_{\ell > k} \bigcup_{p \in P_\ell} \bar{U}_p' \supseteq D(X_{k+1}, \varepsilon_1/2)$.

Here we put $\bar{U}_p' = \psi_p(s_p^{-1}(0) \cap U_p')$.

We put $\varepsilon_4 = \min(\varepsilon_2, \varepsilon_3\}/100$. Now for each $p \in X_k - X_{k+1}$ we take a representative of coordinate chart $(U_p, E_p, s_p)$ around $p$ such that

(6.19.1) Diam $\bar{U}_p < \varepsilon_4$.
(6.19.2) If $D_p(\varepsilon_4) \subseteq \bar{U}_q^{(1)}$ for some $q \in Q$, then there exists a representative of $\varphi_{qp} : U_p \to U_q^{(1)}$ defined on $U_p$ and also there exists $\hat{\varphi}_{qp}$ defined on $E_p$.
(6.19.3) If $D_p(\varepsilon_4) \subseteq \bar{U}_q$ for some $q \in \bigcup_{\ell > k} P_\ell$, then there exists a representative of $\varphi_{qp} : U_p \to U_q$ defined on $U_p$ and also there exists $\hat{\varphi}_{qp}$ defined on $E_p$.
(6.19.4) Compatibility conditions (5.3.2) and (5.3.3) are satisfied as equalities between maps (not only as map germs).

We take a finite set $P_k$ such that

$$D(X_{k+1}, \varepsilon_1/2) \cup \bigcup_{p \in P_k} \bar{U}_p \supseteq X_k. \tag{6.20}$$

Replacing $P_k$ by a subset we may assume that

$$P_k \subseteq X_k - D(X_{k+1}, \varepsilon_1/2 - \varepsilon_2). \tag{6.21}$$

Now we are going to prove that $(U_p, E_p, s_p)$, $p \in P_k$ and $(U'_p, E_p, s_p)$, $p \in P_\ell$, $\ell > k$ has the required properties. We put the order on the set $\bigcup_{\ell \geq k} P_\ell$ such that $q < p$ if $n_q < n_p$.

Now we consider $p_1, p_2$ such that $\bar{U}_{p_1} \cap P_{p_2} \neq \emptyset$ (or $\bar{U}'_{p_1} \cap \bar{U}_{p_2} \neq \emptyset$ or $\bar{U}'_{p_1} \cap \bar{U}_{p_2} \neq \emptyset$) and $p_2 < p_1$. We are going to construct $U_{p_1 p_2}$, $\varphi_{p_1 p_2} : U_{p_1 p_2} \to U_{p_1}$, $\hat{\varphi}_{p_1 p_2} : E_{p_2}|_{U_{p_1 p_2}} \to E_{p_1}$. In case $p_1, p_2 \in \bigcup_{\ell > k} P_\ell$, our charts are restrictions of the charts we constructed in earlier steps. Hence we can use induction hypothesis to obtain $U_{p_1 p_2}, \varphi_{p_1 p_2} : U_{p_1 p_2} \to U_{p_1}$, $\hat{\varphi}_{p_1 p_2} : E_{p_2}|_{U_{p_1 p_2}} \to E_{p_1}$ satisfying required properties.

Hence we need to consider the two cases, $n_{p_1} = n_{p_2} = k$, $n_{p_2} = k < n_{p_1}$.

*Case* 1: $n_{p_1} = n_{p_2} = k$. In this case by (6.21) we have $d(p_1, X_{k+1}) > \varepsilon_1/2 - \varepsilon_2$, $i = 1, 2$. Since $\bar{U}_{p_1} \cap \bar{U}_{p_2} \neq \emptyset$ (6.19.1) implies $d(p_1, p_2) < 2\varepsilon_4$. It follows from (6.17) that there exists $q \in Q$ such that $D_{p_i}(\varepsilon_4) \subseteq \bar{U}_q^{(1)}$, $i = 1, 2$. Thus by (6.19.2) there exists a representative $\varphi_{qp_1} : U_{p_1} \to U_q^{(1)}$, $\varphi_{qp_2} : U_{p_2} \to U_q^{(1)}$. We have also $\hat{\varphi}_{qp_1}, \hat{\varphi}_{qp_2}$. Since $n_{p_1} = n_{p_2} = n_q$, they are diffeomorpisms of orbifolds to its image. We put $U_{p_1 p_2} = \varphi_{qp_2}^{-1} \varphi_{qp_1}(U_{p_1})$, $\varphi_{p_1 p_2} = \varphi_{qp_1}^{-1} \varphi_{qp_2}$, $\hat{\varphi}_{p_1 p_2} = \hat{\varphi}_{qp_1}^{-1} \hat{\varphi}_{qp_2}$. It is straightforward to verify the required properties.

*Case* 2: $n_{p_2} = k < n_{p_1}$. In this case (6.18.1) implies $D_{p_2}(\varepsilon_4) \subseteq \bar{U}_{p_1}$. Therefore by (6.19.3) there exists a representative $\varphi_{p_1 p_2} : U_{p_2} \to U_{p_1}$. We remark that $U'_{p_1}$ is an open subset of $U_{p_1}$. Hence we put $U_{p_1 p_2} = \varphi_{p_1 p_2}^{-1}(U'_{p_1})$, and restrict $\varphi_{p_1 p_2}, \hat{\varphi}_{p_1 p_2}$ there. The required properties are immediate.

The proof of Lemma 6.3 is now complete.

# CHAPTER 2. MODULI SPACE OF STABLE MAPS

## 7. STABLE MAPS

We first recall the notion of stable map due to Kontsevich [38, 39]. Let $(M, \omega)$ be a symplectic manifold and $J : TM \to TM$ be a compatible almost complex structure. Let $g$ and $m$ be nonnegative integers. (We remark that for some of the definitions of this section we do not need symplectic structure but only an almost complex structure. However, to establish compactness of moduli space, we need to assume that there is a symplectic structure. So we restrict ourselves to the case of symplectic manifolds.)

*Definition* 7.1 (*Mumford* [49]). *A semistable curve* with $m$ marked points is a pair $(\Sigma, z)$ of a space $\Sigma = \bigcup \pi_{\Sigma_v}(\Sigma_v)$ where $\Sigma_v$ is a Riemann surface and $\pi_{\Sigma_v} : \Sigma_v \to \Sigma$ is a continuous map, and $z = (z_1, \dots, z_m)$ are $m$ points in $\Sigma$ with the following properties.

(7.1.1) For each $p \in \Sigma_v$ there exists a neighborhood of it such that the restriction of $\pi_{\Sigma_v} : \Sigma_v \to \Sigma$ to this set is a homeomorphism to its image.
(7.1.2) For each $p \in \Sigma$, we have $\sum_v \# \pi_{\Sigma_v}^{-1}(p) \leqslant 2$. Here and hereafter $\#$ means the order of the set.
(7.1.3) $\sum_v \# \pi_{\Sigma_v}^{-1}(z_i) = 1$ for each $z_i$.
(7.1.4) $\Sigma$ is connected.
(7.1.5) $z_i \neq z_j$ for $i \neq j$.
(7.1.6) The number of Riemann surfaces $\Sigma_v$ is finite.
(7.1.7) The set $\{p | \sum_v \# \pi_{\Sigma_v}^{-1}(p) = 2\}$ is of finite order.

We say a point $p \in \Sigma_v$ is *singular* if $\sum_v \# \pi_{\Sigma_v}^{-1}(\pi_{\Sigma_v}(p)) = 2$. We say that $p \in \Sigma_v$ is *marked* if $\pi_{\Sigma_v}(p) = z_j$ for some $j$. We say that $\Sigma_v$ is a *component* of $\Sigma$.

A map $\vartheta : \Sigma \to \Sigma'$ between two semistable curves is called as *isomorphism* if it is homeomorphism and if it can be lifted to biholomorphic isomorphisms $\vartheta_{vw} : \Sigma_v \to \Sigma'_w$ for each component $\Sigma_v$ of $\Sigma$. If $\Sigma$, $\Sigma'$ have marked points $(z_1, \dots, z_m)$, $(z'_1, \dots, z'_m)$ then we require $\vartheta(z_i) = z'_i$ also. Let $Aut(\Sigma, z)$ be the group of all automorphisms of $(\Sigma, z)$.

We next define the genus of a semistable curve $\Sigma = \bigcup \Sigma_v$. For each $\Sigma = \cup \Sigma_v$ we associate a graph $T_\Sigma$ as follows. The vertices of $T_\Sigma$ correspond to the components of $\Sigma$ and we join two vertices by an edge if the corresponding components intersect each other in $\Sigma$. We also add an edge joining the same vertex corresponding to $\Sigma_v$ for each point $p \in \Sigma$ such that $\# \pi_{\Sigma_v}^{-1}(p) = 2$. The graph $T_\Sigma$ is connected since $\Sigma$ is connected.

*Definition* 7.2. The *genus* $g$ of a semistable curve $\Sigma$ is defined by

$$g = \sum_v g_v + \operatorname{rank} H_1(T_\Sigma; \mathbf{Q}),$$

where $g_v$ is the genus of $\Sigma_v$.

*Definition* 7.3. A map $h : \Sigma \to M$ is said to be a *pseudoholomorphic map* if it is continuous and if the composition $h \circ \pi_{\Sigma_v} : \Sigma_v \to M$ is pseudoholomorphic for each $v$.

$$h_*([\Sigma]) = \sum_v (h \circ \pi_v)_* [\Sigma_v] \in H_2(M; \mathbf{Z}).$$

*Definition* 7.4. A pair $((\Sigma, z), h)$ of a semistable curve with $m$-marked points and a pseudoholomorphic map $h : \Sigma \to M$ is said to be *stable* if for each $v$ one of the following conditions holds.

(7.4.1) $h \circ \pi_{\Sigma_v} : \Sigma_v \to M$ is not a constant map.

(7.4.2) Let $m_v$ be the number of points on $\Sigma_v$ which are singular or marked. Then $m_v + 2g_v \geqslant 3$.

*Definition* 7.5. Let $((\Sigma, z), h)$ be a pair of a semistable curve with $m$-marked points and a pseudoholomorphic map $h : \Sigma \to M$. We put

$$Aut(((\Sigma, z), h)) = \left\{ \vartheta : \Sigma \to \Sigma \,\middle|\, \begin{array}{l} \vartheta \text{ is an automorphism} \\ h \circ \vartheta = h \end{array} \right\}.$$

We call it the *automorphism group* of $((\Sigma, z), h)$.

Lemma 7.6. *$((\Sigma, z), h)$ is stable if and only if $Aut(((\Sigma, z), h))$ is a finite group.*

*Proof.* This is an observation by Kontsevich and Manin. We prove only that if $((\Sigma, z), h)$ is stable then $Aut(((\Sigma, z), h))$ is a finite group. (We use this part only.) We first remark that the subgroup of $Aut(((\Sigma, z), h))$ consisting of $\vartheta : \Sigma \to \Sigma$ such that $\vartheta\pi_{\Sigma_v}(\Sigma_v) = \pi_{\Sigma_v}(\Sigma_v)$ is of finite index. Hence it suffices to show that this subgroup is finite. We then find that it suffices to show that the following group is finite for each $v$.

$$\left\{ \vartheta_v : \Sigma_v \to \Sigma_v \,\middle|\, \begin{array}{l} \vartheta_v(p) = p, \text{ for each singular or marked point } p \\ h \circ \pi_{\Sigma_v} \circ \vartheta_v = h \circ \pi_{\Sigma_v} \\ \vartheta_v \text{ is biholomorphic} \end{array} \right\}.$$

If (7.4.2) is satisfied, then the set of all holomorphic automorphisms which fix all singular or marked points is finite.

If (7.4.1) is satisfied, then the set of all holomorphic isomorphisms $\vartheta_v$ satisfying $h \circ \pi_{\Sigma_v} \circ \vartheta_v = h \circ \pi_{\Sigma_v}$ is finite. □

*Definition* 7.7. Let $\beta \in H_2(M; \mathbf{Z})$. We consider the set of all stable maps $((\Sigma, z), h)$ such that $(\Sigma, z)$ is of genus $g$ with $m$ marked points and $h_*([\Sigma]) = \beta$. We divide it by the equivalence relation $\sim$ such that $(\Sigma, z) \sim (\Sigma, z')$ if and only if there exists an isomorphism $\vartheta : (\Sigma, z) \to (\Sigma, z')$ satisfying $h' \circ \vartheta = h$. We let $C\mathscr{M}_{g,m}(M, J, \beta)$ be the quotient. We call it the *moduli space of stable maps of genus $g$, $m$ marked points and of homology class $\beta$.*

We also put, for a positive number $A$,

$$C\mathscr{M}_{g,m}(M, J) = \bigcup_{\beta} C\mathscr{M}_{g,m}(M, J, \beta)$$

$$C\mathscr{M}_{g,m}(M, \omega, J; \leqslant A) = \bigcup_{[\beta] \cap \omega \leqslant A} C\mathscr{M}_{g,m}(M, J, \beta)$$

(We remark that $C\mathscr{M}_{g,m}(M, J, \beta)$ and $C\mathscr{M}_{g,m}(M, J)$ are independent of the symplectic structure $\omega$. Also we can make the definition of $C\mathscr{M}_{g,m}(M, \omega, J; \leqslant A)$ independent of the symplectic structure by using the area of the map in place of $[\beta] \cap \omega \leqslant A$. However, the study of these moduli spaces has an interesting application only in symplectic case. Hereafter, we do not mention these kinds of remarks.)

From now on, we identify $((\Sigma, z), h)$ with its isomorphism class by abuse of notation, when no confusion can occur.

*Definition* 7.8. A semistable curve $(\Sigma, z)$ with $m$-marked points is called *stable* if (7.4.2) holds for each component.

Let $C\mathscr{M}_{g,m}$ be the set of all isomorphism classes of stable curves with $m$ marked points and of genus $g$. $C\mathscr{M}_{g,m}$ is called the *Deligne–Mumford compactification* of the moduli space of curves. It is well known that $C\mathscr{M}_{g,m}$ has a structure of complex orbifold of (complex) dimension $3g - 3 + m$. (See also Section 9.)

The case $g = 1, m = 0$ is exceptional. In that case the moduli space of elliptic curves (with no marked point) is the quotient of upper half-plain by $PSL(2, \mathbf{Z})$. The quotient space is an orbifold and is homeomorphic to $S^2$ minus one point. We compactify it by adding a semistable curve of genus 1 which has one singular point and one irreducible component. We denote this compactification (which is homeomorphic to $S^2$) by $C\mathscr{M}_{1,0}$. We remark that elements of $C\mathscr{M}_{1,0}$ are not stable in the sense above. So in this case the definition of $C\mathscr{M}_{1,0}$ is different to the usual case.

We remark that in Definition 7.8 we assume $3g - 3 + m \geqslant 0$ and otherwise the set $C\mathscr{M}_{g,m}$ is empty. However, in Definition 7.4 (the definition of stable map) we do *not* assume $3g - 3 + m \geqslant 0$. We will give some additional remarks on the case $3g - 3 + m \leqslant 0$ at the end of Section 17.

*Definition* 7.9. Let $(\Sigma, z)$ be a semistable curve with marked points. We say that its component $\Sigma_v$ is *stable* if $2g_v + m_v \geqslant 3$ and we say that it is *unstable* otherwise.

We define a map $\pi: C\mathscr{M}_{g,m}(M, J, \beta) \to C\mathscr{M}_{g,m}$ in case $2g - 3 + m \geqslant 0$ as follows. Let $((\Sigma, z), h) \in C\mathscr{M}_{g,m}(M, J, \beta)$. We shrink $\pi_{\Sigma_v}(\Sigma_v)$ to one point in $\Sigma$ for each unstable component $\Sigma_v$ of $\Sigma$ and obtain $\Sigma'$. Then, we can easily find that $\Sigma'$ together with the composition $\pi_{\Sigma_v}: \Sigma_v \to \Sigma \to \Sigma'$ for stable components is a stable curve of genus $g$ and of $m$ marked points. We let this stable curve $\pi((\Sigma, z), h)$.

In the case when $g = 1, m = 0$ we define $\pi: C\mathscr{M}_{1,0}(M, J, \beta) \to C\mathscr{M}_{1,0}$ as follows. Let $(\Sigma, h) \in C\mathscr{M}_{1,0}(M, J, \beta)$. If there is an irreducible component $\Sigma_v$ of $\Sigma$ such that $g_v = 1$ then we put $\pi(\Sigma, z) = [\Sigma_v]$. We remark that the component $\Sigma_v$ of $\Sigma$ with $g_v = 1$ is unique if it exists. If there is no component $\Sigma_v$ of $\Sigma$ with $g_v = 1$, we define $\pi(\Sigma, h)$ to be the unique point in $C\mathscr{M}_{1,0} - \mathscr{M}_{1,0}$.

We also define a map $ev: \mathscr{M}_{g,m}(M, J, \beta) \to M^m$ by

$$ev((\Sigma, z), h) = (h(z_1), \dots, h(z_m)).$$

Now the main results we are going to prove in Chapters 2 and 3 are the following.

← the compactified moduli space (see above)

Theorem 7.10. *$C\mathscr{M}_{g,m}(M, J, \beta)$ has a Kuranishi structure of dimension $2m + 2\beta c_1(M) + 2(3 - n)(g - 1)$ which is stably complex and is compact.*

Theorem 7.11. *Let $(U_p, E_p, s_p, \psi_p, \varphi_{pq}, \hat{\varphi}_{pq})$ be the Kuranishi structure as in Theorem 7.10. Then there exist strongly smooth maps $\pi_{1,p}: U_p \to C\mathscr{M}_{g,m}$ if $3g + m \geqslant 3$ and $ev_p: U_p \to M^m$ if $m > 0$ with the following properties.*

(7.11.1) *$\pi_q \circ \varphi_{qp} = \pi_p$, and $ev_q \circ \varphi_{qp} = ev_p$.*

(7.11.2) *$\pi\psi_p(x) = \pi_p(x)$, and $ev(\psi_p(x)) = ev_p(x)$ for every $x$ with $s_p(x) = 0$.*

(7.11.3) *$\pi_p \times ev_p$ is of maximal rank if $3g + m \geqslant 3$, and $ev_p: U_p \to M^m$ is of maximal rank if $m > 0$.*

We can prove a family version of them. Namely we have a similar moduli space with Kuranishi structure when we move $\omega$ and $J$ in a finite-dimensional family. This generalization is straightforward so we do not state it. A more interesting problem is to study the case when we consider a family of complex or symplectic structures which is degenerate at some point.

We give an example of Kuranishi structure on the moduli space of pseudoholomorphic curves.

*Example* 7.12. Let $\pi: M \to S$ be an elliptic surface. (Here $S$ is a Riemann surface.) We assume that $\pi: M \to S$ has no singular fiber other than multiple fibers. Let $\pi^{-1}(p_1)$, $i = 1, \ldots, \ell$ be the multiple fibers with multiplicity $n_i$. We regard $S$ as an orbifold by regarding $p_i$ as a singular point which has a local chart $\mathbf{R}^2/\mathbf{Z}_{n_i}$. Let $\beta \in H_2(H; \mathbf{Z})$ be the homology class of general fibre. We consider the moduli space $C\mathcal{M}_{1,0}(M, J, \beta)$ of genus 1 Riemann surfaces representing $\beta$. This space consists of the following components.

(7.13.1) $S$.

(7.13.2) Finitely many points, each of them corresponding to a map $h$ to a multiple fibre $\pi^{-1}(p_i)$, $i = 1, \ldots, \ell$, $h: T^2 \to \pi^{-1}(p_i)$ is an $n_i$-fold covering map, but is not deformable to a map to a regular fiber. (Among the $n_i$-fold covering maps $T^2 \to \pi^{-1}(p_i)$, only one of them is deformed to a map to a regular fiber.)

The virtual dimension of $C\mathcal{M}_{1,0}(M, J, \beta)$ is 0. So we have an obstruction bundle.

We can describe the obstruction bundle $E$ on the component $S$ as follows. First we remark that by definition

$$E_x \cong H^{-1}(\pi^{-1}(x), N_{\pi^{-1}(x)}M) \cong H^{0,1}(\pi^{-1}(x)) \otimes T_x S.$$

If $x \in S$ is a regular point. Hence we can show that $E$ as an orbibundle is isomorphic to the tensor product $H^{0,1}(\pi^{-1}(x)) \otimes TS$. (Here $H^{0,1}(\pi^{-1}(x))$ is an orbibundle over $S$ whose fiber at $x \in S$ is $H^{0,1}(\pi^{-1}(x))$.)

We can prove that the bundle $H^{0,1}(\pi^{-1}(x))$ is flat in the following way.

First, we claim that the holomorphic structure of the regular fiber is constant. This fact is well known and can be proved as follows. Let $\mathfrak{h}/PSL(2;\mathbf{Z})$ be the moduli space of elliptic curves. (Here $\mathfrak{h}$ is the upper half-plain.) $x \mapsto [\pi^{-1}(x)]$ defines a holomorphic map $S - \{p_i | i = 1, \ldots, \ell\} \to \mathfrak{h}/PSL(2;\mathbf{Z})$. At the point $p_i$ corresponding to the multiple fiber, we consider the covering space locally and can prove that $S - \{p_i | i = 1, \ldots, \ell\} \mapsto \mathfrak{h}/PSL(2;\mathbf{Z})$ extends to a holomorphic map $S \to \mathfrak{h}/PSL(2;\mathbf{Z})$. Using the fact that $S$ is compact, we find that this map is constant. Namely the complex structure of the regular fiber is constant.

Therefore the structure group of $\pi: M \to S$ is reduced to the group of biholomorphic map of an elliptic curve. It is an extension of the torus by a finite group. The torus acts trivially on $H^{0,1}(\pi^{-1}(x))$. Hence the structure group of the orbibundle $H^{0,1}(\pi^{-1}(x))$ is finite.

Thus we find that the Euler class of $H^{0,1}(\pi^{-1}(x)) \otimes TS$ coincides with the Euler class of $TS$.

Other components (which are of dimension 0) has isotropy group whose order is $n_i$. Thus the fundamental cycle (which is a rational number) is

$$\chi(S) + \sum_{i=1}^{n_i} \frac{\rho(n_i) - 1}{n_i}, \tag{7.14}$$

here $\rho(n_i)$ is the number of subgroup of $\mathbf{Z}^2$ of index $n_i$ and $\chi(S)$ is the orbifold Euler number

We remark that $S$ may be a bad orbifold. Namely it may not be a global quotient of a manifold. (For example we start with the direct product $S^2 \times T^2$ and perform a logarithmic transformation.)

In the case when there is a singular fiber other than multiple fibers, it seems that a formula similar to (7.14) holds. The authors did not check it yet.

## 8. FINITENESS OF THE NUMBER OF COMBINATORIAL TYPES OF STABLE MAPS

We start the proof of Theorem 7.10. First we need to define a topology on $C\mathcal{M}_{g,m}(M, J, \beta)$ and prove that it is compact. In this section, we prove that there are only finitely many possibilities of the combinatorial types of elements of $C\mathcal{M}_{g,m}(M, J, \beta)$ (Proposition 8.8). The following lemma plays a basic role for it.

LEMMA 8.1. *For each* $(M, \omega, J)$, *there exists* $\delta > 0$ *such that if* $h: \Sigma \to M$ *is a nonconstant pseudoholomorphic map from a closed Riemann surface* $\Sigma$ *to* $M$, *then*

$$\int h^*\omega > \delta.$$

(We remark that Lemma 8.1 holds for almost complex manifold if we replace the assumption $\int h^*\omega > \delta$ by an assumption on the area. The same is true for Lemmata 8.2 and 8.12. In this paper we use only the case of symplectic manifolds.) The lemma is an immediate consequence of the following result due to [33], [53].

LEMMA 8.2. *There exists* $\varepsilon_0$ *and* $C > 0$ *such that the following holds for each* $\varepsilon < \varepsilon_0$ *and each metric ball* $D_p(\varepsilon)$ *centered at* $p$ *and of radius* $\varepsilon$. *Let* $h': \Sigma' \to M$ *be a pseudoholomorphic map from a Riemann surface* $\Sigma'$ *with* (*or without*) *boundary to* $M$. *Suppose* $h'(\Sigma') \subseteq D_p(\varepsilon)$, $h'(\partial\Sigma') \subseteq \partial D_p(\varepsilon)$ *and* $p \in h(\Sigma')$. *Then*

$$\int_{\Sigma} h'^*\omega > C\varepsilon^2.$$

We next define the data parametrizing combinatorial types of stable curves of genus $g$, with $m$ marked points and of homology class $\beta \in H_2(M; \mathbf{Z})$. We consider a connected graph

(8.3.1) $g_v$, a nonnegative integer for each vertex $v$ of $T$.
(8.3.2) $\beta_v \in H_2(M; \mathbf{Z})$ for each vertex $v$ of $T$.
(8.3.3) A map $o: \{1, \ldots, m\} \to \{\text{vertices of } T\}$.

Let $mark(v)$ be the number of $j$ such that $o(j) = v$, let $sing(v)$ be the number of edges containing $v$. (We count twice the edges both of whose vertices are $v$.) We put $m_v = mark(v) + sing(v)$. We assume that one of the following is satisfied for each vertex $v$ of $T$.

(8.4.1) $2g_v + m_v \geqslant$
(8.4.2) $\beta_v \neq 0$.

We assume also that

(8.5) $\sum_v \beta_v = \beta$.
(8.6) $\sum_v g_v + \operatorname{rank} H_1(T; \mathbf{Q}) = g$.
(8.7) For each vertex $v$, there exists a pseudoholomorphic curve $h : \Sigma \to M$ from a Riemann surface of genus $g_v$ such that $h_*([\Sigma]) = \beta_v$.

We let $Comb(g, m, \beta)$ denote the set of all $(T, g_v, \beta_v, o)$ satisfying these conditions. For each element $((\Sigma, \mathbf{z}), h) \in C\mathcal{M}_{g,m}(M, J, \beta)$, we find a unique element $(T, g_v, \beta_v, o)$ of $Comb(g, m, \beta)$ as follows. We take the graph $T = T_\Sigma$ introduced in Section 7. Its vertex $v$ is identified to a component $\Sigma_v$ of $\Sigma$. Let $g_v$ be the genus of $\Sigma_v$. We put $\beta_v = (h \circ \pi_{\Sigma_v})_*([\Sigma_v])$. We put $o(i) = v$ if and only if $z_i \in \pi_{\Sigma_v}(\Sigma_v)$.

(8.4) is then a consequence of stability, (8.5) follows from $h_*([\Sigma]) = \beta$, (8.6) is equivalent to the fact that the genus of $\Sigma$ is $g$. (8.7) holds because $h \circ \pi_{\Sigma_v}$ is a required pseudoholomorphic map.

equivalence classes of,

We let $C\mathcal{M}_{g,m}(M, J, \beta)(T, g_v, \beta_v, o)$ be the set of all stable maps which induce the element $(T, g_v, \beta_v, o)$ of $Comb(g, m, \beta)$. This gives a stratification of $C\mathcal{M}_{g,m}(M, J, \beta)$. We now use Lemma 8.1 to show the following:

Proposition 8.8. *For each $A$, the set $\bigcup_{\beta \cap [\omega] \leqslant A} Comb(g, m, \beta)$ is finite.*

*Proof.* Let $\delta$ be as in Lemma 8.1. We put $K = A/\delta$. Let  be the number of vertices and let $S = \sum g_v$. We remark:

(8.9.1) By (8.6) the number of vertices $v$ with $g_v \geqslant 1$ is smaller than $g$.
(8.9.2) There exists at most $m$ vertices $v$ such that $mark(v) > 0$.
(8.9.3) By Lemma 8.1, (8.5), and (8.7), there exist at most $K$ vertices $v$ with $\beta_v \neq 0$.
(8.9.4) By (8.6) the number of vertices $v$ such that there exists an edge $e$ both of whose vertex is $v$ is smaller than $g$.

Let $_0$ be the set of all vertices $v$ satisfying one of (8.9.1), (8.9.2), (8.9.3), (8.9.4). Then the order of $_0$ is smaller than $K + m + 2g$.

Let $v$ be a vertex which is not contained in $_0$. Then $g_v = 0$, $mark(v) = 0$, $\beta_v = 0$. Hence $sing(v) \geqslant 3$ by (8.4). Therefore, we have

$$\leqslant K + m + 2g + \#\{v \mid sing(v) \geqslant 3\}. \tag{8.10}$$

By (8.6), $\chi(T) = 1 - b_1(T) = S + 1 - g$. On the other hand,

$$\chi(T) = \frac{1}{2}\sum(2 - sing(v)) \leqslant \frac{1}{2}(\{v \mid sing(v) = 1\} - \{v \mid sing(v) \geqslant 3\}).$$

$$2S + 2 - 2g + \#\{v \mid sing(v) \geqslant 3\} \leqslant \#\{v \mid sing(v) = 1\}.$$

$$\#\{v \mid sing(v) = 1\} \leqslant \#\{v \mid g_v > 0\} + \#\{v \mid g_v = 0, m_v = 1\} + \#\{v \mid mark(v) = 1\}.$$

Hence

$$2S + 2 - 2g + \#\{v | sing(v) \geqslant 3\} \leqslant g + K + m. \tag{8.11}$$

(8.10) and (8.11) imply $\quad + 2S \leqslant 5g + 2K + 2m - 2$. Proposition 8.8 follows from this fact and the following:

LEMMA 8.12. *For each $g_0$, $c$, the set below is finite.*

$$\left\{ \beta \in H_2(M; \mathbf{Z}) \;\middle|\; \begin{array}{l} \exists h : \Sigma_g \to M \text{ pseudoholomorphic} \\ [\omega] \cdot h_*[\Sigma_g] < c \\ \Sigma_g \text{ is a Riemann surface of genus } g \leqslant g_0 \end{array} \right\}$$

This lemma will be proved in Section 11.

*Remark* 8.13. We will use a part of Proposition 8.8 in Section 11. So we remark here that we proved the following without using Lemma 8.12: Let $((\Sigma, z), h) \in C\mathcal{M}_{g,m}((M, \omega, J), \leqslant A)$. Then the number of irreducible components of $(\Sigma, z)$ is smaller than a number depending only on $M$, $\omega$, $J$, $A$.

## 9. DIFFERENTIAL GEOMETRIC DESCRIPTION OF DELIGNE – MUMFORD COMPACTIFICATION

In this section, we study the compactified moduli space $C\mathcal{M}_{g,m}$ of Riemann surfaces of genus $g$ and with $m$ marked points. Our purpose is to define a topology (and orbifold structure) on it. We also define a family of Riemann metrics on stable curves parametrized by $C\mathcal{M}_{g,m}$. (Here Riemann metric on stable curve means Riemann metric on each component.) Our metric is flat in a neighborhood of singular points.

The space $C\mathcal{M}_{g,m}$ is well studied from various points of view, and the discussion we will give in this section is not really new. However, we will give those descriptions since a similar construction is necessary to construct the charts of Kuranishi structure of our moduli space in later sections. We also want to make the exposition as elementary as possible and to avoid using deep and difficult results of algebraic geometry (though many of the arguments in principle are borrowed from algebraic geometry). To avoid using algebraic geometry may be appropriate to work in the category of symplectic manifold with compatible almost complex structure. Compare our description with [11] or [50] where various other techniques (Cohen–Macaulay scheme, stack, etc.) are used. However, we skip some part of the proof since this section is mainly of expository nature.

We first define a stratification of $C\mathcal{M}_{g,m}$. It is indexed by a set $Comb(g, m)$ which is similar to the one we used in Section 8. We consider a connected graph $T$ together with the following data.

(9.1.1) $g_v$, a nonnegative integer for each vertex $v$ of $T$.
(9.1.2) A map $o : \{1, \ldots, m\} \to \{\text{vertices of } T\}$.

Let $mark(v)$ be the number of $j$ such that $o(j) = v$, let $sing(v)$ be the number of edges containing $v$. (We count twice the edges both of whose vertices are $v$.) We put $m_v = mark(v) + sing(v)$. We assume

$$2g_v + m_v \geqslant 3. \tag{9.2}$$

We also assume

$$\textstyle\sum_v g_v + \operatorname{rank} H_1(T; \mathbf{Q}) = g. \tag{9.3}$$

Let $Comb(g, m)$ be the set of all such objects $(T, (g_v), o)$. In fact, $Comb(g, m)$ coincides to $Comb(g, m, \beta)$ if we put $\beta = 0$. Therefore, $Comb(g, m)$ is finite by Proposition 8.8. (We do not need Lemma 8.12 to prove it in this case.)

For each element $(\Sigma, z)$ of $C\mathcal{M}_{g,m}$, we associate an element of $Comb(g, m)$ as follows. We take the graph $T = T_\Sigma$ as in Section 7. Each vertex $v$ of $T$ corresponds to a component $\Sigma_v$ of $\Sigma$. Let $g_v$ be the genus of $\Sigma_v$. We put $o(j) = v$ if $z_j \in \pi_{\Sigma_v}(\Sigma_v)$. (9.2) is a consequence of the stability and (9.3) is the definition of the genus.

Let $\mathcal{M}_{g,m}(T, (g_v), o)$ be the set of all $(\Sigma, z)$ such that the associated object is $(T, (g_v), o)$.

Using the fact that the automorphism group of each element of $C\mathcal{M}_{g,m}$ is of finite order, it is rather easy to find a structure of orbifold on each stratum $\mathcal{M}_{g,m}(T, (g_v), o)$. So we omit it. We also remark that there exists a fiber bundle (in the sense of orbifold) $\mathcal{U}\mathcal{N}\mathcal{I}_{g,m}(T, (g_v), o) \to \mathcal{M}_{g,m}(T, (g_v), o)$ together with the complex structure on each fiber such that the fiber of $(\Sigma, z)$ is identified to $(\Sigma, z)$ itself.

More precisely for each $x \in \mathcal{M}_{g,m}(T, (g_v), o)$ there exists a chart $U_x = {}_x/\Gamma_x$ of it and an action of $\Gamma_x$ on $\Sigma_x$ such that the inverse image of $U_x$ in $\mathcal{U}\mathcal{N}\mathcal{I}_{g,m}(T, (g_v), o)$ is diffeomorphic to ${}_x \times \Sigma_x/\Gamma_x$. We need to require compatibility condition for these charts ${}_x \times \Sigma_x/\Gamma_x$ so that it will define a fiber bundle structure on $\mathcal{U}\mathcal{N}\mathcal{I}_{g,m}(T, (g_v), o) \to \mathcal{M}_{g,m}(T, (g_v), o)$. We omit the definition since it is almost the same as one we gave in the case of vector bundle.

It is more delicate to see how those strata are patched. We are going to describe it. Also we define a smooth family of metrics of each fiber of $\mathcal{U}\mathcal{N}\mathcal{I}_{g,m}(T, (g_v), o) \to \mathcal{M}_{g,m}(T, (g_v), o)$.

The construction is by induction on the stratum. We first define a partial order $\succ$ on the set $Comb(g, m)$.

Let $(T, (g_v), o) \in Comb(g, m)$. We consider $(T_v, (g_{v,w}), o_v) \in Comb(g_v, m_v)$ for some of the vertices $v = v_1, \ldots, v_a$ of $T$. Here $m_v$ is the sum of number of the edges containing $v$ and the number of marked points on $\Sigma_v$. (We count twice the edges both of whose vertices are $v$.)

We now replace the vertex $v$ by the graph $T_v$. We join the edge containing $v$ to the vertex $o_v(j)$ where $j \in \{1, \ldots, m_v\}$ is the suffix corresponding to this edge. We then obtain $\tilde{T}$. The number $\tilde{g}_v$ is determined from $g_v$ and $g_{v,w}$ in an obvious way. We determine $\tilde{o}$ as follows. If $o(j) \neq v_i$, $i = 1, \ldots, a$, then $\tilde{o}(j) = o(j)$. If $o(j) = v_i$ then the $j$th marked point corresponds to some of $j' \in \{1, \ldots, m_{v_i}\}$. We then put $\tilde{o}(j) = o_{v_i}(j')$. We thus obtain an element $(\tilde{T}, (\tilde{g}_v), \tilde{o}) \in Comb(g, m)$.

We write $(T, (g_v), o) \succ (\tilde{T}, (\tilde{g}_v), \tilde{o})$ if $(\tilde{T}, (\tilde{g}_v), \tilde{o})$ is obtained from $(T, (g_v), o)$ as above.

We are going to define a topology of $C\mathcal{M}_{g,m}$ so that the closure of $\mathcal{M}_{g,m}(T, (g_v), o)$ contains $\mathcal{M}_{g,m}(\tilde{T}, (\tilde{g}_v), \tilde{o})$ if and only if $(T, (g_v), o) \succ (\tilde{T}, (\tilde{g}_v), \tilde{o})$.

We construct a basis of neighborhood of each element of $\mathcal{M}_{g,m}(T, (g_v), o)$ by the induction with respect to $\prec$.

For the first step, we remark that $(T, (g_v), o)$ is minimal if the following holds.

(9.4.1) $g_v = 0$ for each $v$.
(9.4.2) $m_v = 3$ for each $v$.

In that case, the stratum $\mathcal{M}_{g,m}(T, (g_v), o)$ consists of one point $(\Sigma, z)$. We take and fix a Kähler metric on each component of this unique element (which is a sphere). We remark that the group of automorphisms of this unique element of $\mathcal{M}_{g,m}(T, (g_v), o)$ may be nontrivial. In the case when it is nontrivial, we choose our Kähler metric so that it is invariant by the action of this group.

We next construct a neighborhood of $(\Sigma, z)$ in $C\mathcal{M}_{g,m}$ together with a family of metrics on each element in this neighborhood.

Let $x \in \pi_{\Sigma_v}(x_v) = \pi_{\Sigma_w}(x_w)$ be a singular point of $\Sigma$. (In fact it may happen that $v = w$. We use the same notation in this case for simplicity.) We take the hermitian metric on $T_{x_v}\Sigma_v$ and $T_{x_w}\Sigma_w$ induced by the Kähler metric on $\Sigma_v$ and $\Sigma_w$. They induce one on the tensor product $T_{x_v}\Sigma_v \otimes T_{x_w}\Sigma_w$. For each nonzero element $\alpha \in T_{x_v}\Sigma_v \otimes T_{x_w}\Sigma_w$, we have a biholomorphic map $\Phi_\alpha : T_{x_v}\Sigma_v - \{0\} \to T_{x_w}\Sigma_w - \{0\}$ such that

$$u \otimes \Phi_\alpha(u) = \alpha.$$

Let $|\alpha| = R^{-2}$, and assume that $R$ is sufficiently large. Let $\exp_{x_v} : T_{x_v}\Sigma_v \to \Sigma_v$, $\exp_{x_w} : T_{x_w}\Sigma_w \to \Sigma_w$ be the exponential maps with respect to the Kähler metrics we have chosen. (We recall that our metric is flat in a neighborhood of singular point. Hence, $\exp_{x_v}$ is an isometry in a neighborhood of origin.)

We remove $D_{x_v}(R^{-3/2})$ from $\Sigma_v$ and $D_{x_w}(R^{-3/2})$ from $\Sigma_w$. Here $D_x(R^{-3/2})$ denote the metric ball of radius $R^{-3/2}$ centered at $x$. (We assume that $D_{x_v}(R^{-3/2})$ and $D_{x_w}(R^{-3/2})$ are both flat.)

If $R$ is sufficiently large, then $\exp_{\Sigma_w}^{-1} \circ \Phi_\alpha \circ \exp_{\Sigma_v}$ is a diffeomorphism between $D_{x_v}(R^{-1/2}) - D_{x_v}(R^{-3/2})$ and $D_{x_w}(R^{-1/2}) - D_{x_w}(R^{-3/2})$. We glue $\Sigma_v$ and $\Sigma_w$ by this diffeomorphism. In case when $\alpha = 0$, we do not make any change.

By performing this construction at each singular point, we obtain a 2-dimensional "manifold" for each element

$$(\alpha_x) \in \bigoplus_x T_{x_v}\Sigma_v \otimes T_{x_w}\Sigma_w$$

in a neighborhood of 0. (Here the sum is taken over all singular points $x$ of $\Sigma$.) It is singular when some $\alpha_x$ is 0.

We next define a metric of each "manifold" in this family and hence a complex structure on it. We do not change the metric on the complement of $D_{x_v}(R^{-1/2})$ in $\Sigma_v$.

Let us describe the metric we put on the part $D_{x_v}(R^{-1/2}) - D_{x_v}(R^{-3/2})$.

Since our Kähler metric is flat in a neighborhood of singular point, we may identify $D_{x_v}(R^{-1/2})$ with an open subset in $\mathbf{C}$ with standard metric. We then consider an isomorphism $\Phi_\alpha : z \mapsto \alpha/z$ on it. The standard metric $|dz|^2$ will become $|\alpha/z^2|^2\ |dz|^2$ by the pullback. Hence if $|\alpha| = R^{-2}$, then $\Phi_\alpha^*|dz|^2 = |dz|^2$ on the circle $|z| = R^{-1}$. We choose a function $\chi_R : (0, \infty) \to (0, \infty)$ such that $\chi_R(|z|)|dz|^2$ is invariant by $\Phi_\alpha$, and that $\chi_R(r) = 1$ if $rR > 1 + \varepsilon$. We choose such $\chi_R$ once and use it always. We perform the same construction on $D_{x_v}(R^{-1/2}) - D_{x_w}(R^{-3/2})$. Then $\Phi_\alpha^*(\chi_R(|z|)|dz|^2) = \chi_R(|z|)|dz|^2$ implies that these two metrics are compatible on the overlapping part and hence we have a metric.

origin in $\oplus_x T_{x_v}\Sigma_v \otimes T_{x_w}\Sigma_w$. (In case some component is 0, the stable curve is singular.)

We recall that our element $(\Sigma, z)$ in $C\mathcal{M}_{g,m}$ may have a nontrivial automorphism group $Aut(\Sigma, z)$. Each of them acts by isometry. Hence it acts on $\oplus_x T_{x_v}\Sigma_v \otimes T_{x_w}\Sigma_w$. Furthermore, since all the constructions are canonical (except the choice of $\chi_R$ which we may assume to use the same one for all), it follows that, for each $\gamma \in Aut(\Sigma, z)$, the semistable curve corresponding to $(\alpha_x) \in \oplus_x T_{x_v}\Sigma_v \otimes T_{x_w}\Sigma_w$ and $\gamma(\alpha_x)$ are isometric to each other, with a canonical isometry, which we write also by $\gamma$. Thus we have a family of elements of $C\mathcal{M}_{g,m}$

parametrized by a neighborhood of 0 in

$$\frac{\bigoplus_x T_{x_v}\Sigma_v \otimes T_{x_w}\Sigma_w}{Aut(\Sigma, z)}.$$

We take this neighborhood as a chart of $C\mathcal{M}_{g,m}$ as an orbifold.

Thus we have constructed a neighborhood of each point in the first (smallest) stratum. Also we have constructed a family of Riemann metrics on the element of the neighborhood.

*Remark* 9.5. In fact we need to prove that the map from a neighborhood of 0 in

$$\frac{\bigoplus_x T_{x_v}\Sigma_v \otimes T_{x_w}\Sigma_w}{Aut(\Sigma, z)}.$$

to $C\mathcal{M}_{g,m}$ is injective to show that our map gives a chart. We skip the proof of this fact. One can prove it without using algebraic geometry so much. See [30, Section 14] for the proof of it in the case of real Riemann surface of genus 0. (The proof there can be generalized to the present situation with minor change.)

Now we consider the induction step. We consider a stratum $\mathcal{M}_{g,m}(T, (g_v), o)$. By induction hypothesis we have constructed a family of metrics of each element in the neighborhood of $\mathcal{M}_{g,m}(T', (g'_v), o')$ with $(T', (g'_v), o') \prec (T, (g_v), o)$. We here use the fact that the complement of the union of such neighborhoods is compact. This is a nontrivial fact. But again we omit the proof of this fact, since it is now a part of standard theory.

Let $\mathcal{K}_{g,m}(T, (g_v), o)$ be a compact subset of $\mathcal{M}_{g,m}(T, (g_v), o)$ which contains the complement of the union of the neighborhoods of $\mathcal{M}_{g,m}(T', (g'_v), o')$ with $(T', (g'_v), o') \prec (T, (g_v), o)$.

We assume also by induction hypothesis that the metrics of the stable curves in the neighborhoods of $\mathcal{M}_{g,m}(T', (g'_v), o')$ with $(T', (g'_v), o') \prec (T, (g_v), o)$ for various $(T', (g'_v), o')$ coincides with each other on the overlapping part.

We next extend this family of metrics in any way, over all $\mathcal{M}_{g,m}(T, (g_v), o)$. We then obtain a smooth family of Kähler metrics on the fibres $\mathcal{UNI}_{g,m}(T, (g_v), o) \to \mathcal{M}_{g,m}(T, (g_v), o)$.

We next are going to use this family to construct a neighborhood of $\mathcal{K}_{g,m}(T, (g_v), o)$. We first remark that we have an orbibundle over $\mathcal{K}_{g,m}(T, (g_v), o)$ whose fiber is the direct sum $\oplus_x T_{x_v}\Sigma_v \otimes T_{x_w}\Sigma_w$ over the singular points $x$. ($\oplus_x T_{x_v}\Sigma_v \otimes T_{x_w}\Sigma_w$ consists of an orbibundle and not a vector bundle since the automorphism group $Aut(\Sigma, z)$ of elements of $\mathcal{K}_{g,m}(T, (g_v), o)$ acts on it.) The family of Kähler metrics on $\mathcal{UNI}_{g,m}(T, (g_v), o) \to \mathcal{M}_{g,m}(T, (g_v), o)$ induces a hermitian metric on our orbibundle. Therefore we can perform a parameterized version of the construction we did in the case of the first stratum, to obtain neighborhood of 0 of our orbibundle $\oplus_x T_{x_v}\Sigma_v \otimes T_{x_w}\Sigma_w$. We thus construct a neighborhood of $\mathcal{K}_{g,m}(T, (g_v), o)$. By construction, the family of metrics we obtained coincides with the one we already constructed in earlier steps at the part where they overlap. This is

Thus we have constructed an orbifold structure on $C\mathcal{M}_{g,m}$ together with a smooth family of metrics on each fiber of $\mathcal{UNI}_{g,m}(T, (g_v), o) \to \mathcal{M}_{g,m}(T, (g_v), o)$.

## 10. TOPOLOGY ON THE MODULI SPACE OF STABLE MAPS

We are going to define a topology on our moduli space $C\mathcal{M}_{g,m}(M, J, \beta)$. The definition of topology and the proof of the fact that $C\mathcal{M}_{g,m}(M, J, \beta)$ is compact is in principle due to Gromov [33]. Also there are papers by Pansu [53], Parker–Wolfson [54] and Ye [74] related to the same topic. However, we give a proof of them here since there seems to be some confusion on the way to use the word "stable map", "cusp curve", etc. Also a part of the argument of the proof will be used in later sections.

The topology is defined in a similar way to the case of $C\mathcal{M}_{g,m}$ we discussed in Section 9. One trouble however is that for a stable map $((\Sigma, z), h)$ the semistable curve $(\Sigma, z)$ may not be stable. Therefore there may be some trouble to fix a representative of $(\Sigma, z)$. (Namely there is an ambiguity controlled by a group of positive dimension (which is noncompact in many cases). In the discussion of Section 9 the ambiguity was just a finite group.)

We again start with defining a partial order of the index set $Comb(g, m, \beta)$ of the strata of $C\mathcal{M}_{g,m}(M, J, \beta)$. Let $(T, g_v, \beta_v, o) \in Comb(g, m, \beta)$. We consider $(T_v, (g_{v,w}), \beta_{v,w}, o) \in Comb(g_v, m_v, \beta_v)$ for some of the vertices $v = v_1, \ldots, v_a$ of $T$. We replace the vertex $v$ by $(T_v, (g_{v,w}), \beta_{v,w}, o_v)$ in a way similar to that in Section 9 and we obtain $(\tilde{T}, \tilde{g}_v, \tilde{\beta}_v, \tilde{o})$. We then define $(T, g_v, \beta_v, o) \succ (\tilde{T}, \tilde{g}_v, \tilde{\beta}_v, \tilde{o})$.

In later sections, we are going to define Kuranishi structure by induction of this relation $\prec$. In this section we only define a topology.

We first consider the case when $(\Sigma, z)$ is stable.

Let $((\Sigma_n, z_n), h_n) \in C\mathcal{M}_{g,m}(M, J, \beta)$ be a sequence, and assume that $(\Sigma_n, z_n) \in C\mathcal{M}_{g,m}$ namely we assume that they are stable. We assume that

$$\lim_{n\to\infty} (\Sigma_n, z_n) = (\Sigma, z)$$

in $C\mathcal{M}_{g,m}$ by the topology we defined in Section 9. Let $\Sigma = \bigcup_v \Sigma_v$ be the decomposition of $\Sigma$ into irreducible components. (We write $\Sigma_v$ in place of $\pi_{\Sigma_v}(\Sigma_v)$ for simplicity.) We *assume* that $\Sigma \in \mathcal{M}_{g,m}(T, (g_v), o)$. Then by the definition of the topology in Section 9 we find $\Sigma'_n = \bigcup_v \Sigma'_{v,n} \in \mathcal{M}_{g,m}(T, (g_v), o)$ and $(\alpha_{x_n}) \in \bigoplus_{x_n} T_{x_{v,n}}\Sigma'_{v,n} \otimes T_{x_{w,n}}\Sigma'_{w,n}$ in a neighborhood of 0 such that $(\Sigma_n, z_n) \in C\mathcal{M}_{g,m}$ corresponds to the element $(\alpha_{x_n}) \in \bigoplus_{x_n} T_{x_{v,n}}\Sigma'_{v,n} \otimes T_{x_{w,n}}\Sigma'_{w,n}$ by the chart of $C\mathcal{M}_{g,m}$ we constructed in Section 9. We also have $\lim_{n\to\infty}\alpha_{x_n} = 0$ and

$$\lim_{n\to\infty} (\Sigma'_n, z_n) = (\Sigma, z). \tag{10.1}$$

We put $R_{n,x} = |\alpha_{n,x}|^{-2}$. By the definition of Section 9, $\Sigma_n$ has a subset identified to $\Sigma'_n - \bigcup_{x_{v,n}} D_{x_{v,n}}(R_{n,x}^{-3/2})$ and that the diameter of the complement $\Sigma_n - (\Sigma'_n - \bigcup_{x_{v,n}} D_{x_{v,n}}(R_{n,x}^{-3/2}))$ converges to 0. By (10.1) we may identify $\Sigma'_n - \bigcup_{x_{v,n}} D_{x_{v,n}}(R_{n,x}^{-3/2})$ to a subset of $\Sigma$. We put

$$W_{x,n}(\mu) = (D_{x_{v,n}}(\mu) - D_{x_{v,n}}(R_{n,x}^{-1})) \cup (D_{x_{w,n}}(\mu) - D_{x_{w,n}}(R_{n,x}^{-1})).$$

Using these identifications, we define

*Definition* 10.2. We say $\lim_{n\to\infty}((\Sigma_n, z_n), h_n) = ((\Sigma, z), h)$, if the following holds.

(10.2.1) For each $\mu > 0$, the restriction of $h_n$ to $\Sigma'_n - W_{x,n}(\mu)$ converges to $h$ in $C^\infty$

(10.2.2) $\lim_{\mu\to 0}\limsup_{n\to\infty} Diam(h_n(W_{x,n}(\mu))) = 0$ for any singular point $x$.

Now we consider the general case. Because of the problem of instability we mentioned before, we use a trick to add marked points to make the semistable curve stable. Let

$$forget_\eta : C\mathcal{M}_{g,m+\eta}(M, J, \beta)_0 \to C\mathcal{M}_{g,m}(M, J, \beta)$$

be the map which forgets the marked points $z_{m+1}, \dots, z_{m+\eta}$. Here we denote by $C\mathcal{M}_{g,m+\eta}(M, J, \beta)_0$ the set of all elements of $C\mathcal{M}_{g,m+\eta}(M, J, \beta)$ which remain stable maps after forgetting marked points $z_{m+1}, \dots, z_{m+\eta}$.

*Definition* 10.3. We say $\lim_{n\to\infty}((\Sigma_n, z_n), h_n) = ((\Sigma, z), h)$, if there exist $\eta$, $((\Sigma_n, z_n), h_n)^+$, $((\Sigma, z), h)^+ \in C\mathcal{M}_{g,m+\eta}(M, J, \beta)_0$ such that

(10.3.1) $forget_\eta(((\Sigma_n, z_n), h_n)^+) = ((\Sigma_n, z_n), h_n), forget_\eta(((\Sigma, z), h)^+) = ((\Sigma, z), h)$.
(10.3.2) $(\Sigma_n, z_n)^+$ and $(\Sigma, z)^+$ are stable.
(10.3.3) $\mathrm{lims}_{n\to\infty}((\Sigma_n, z_n), h_n)^+ = ((\Sigma, z), h)^+$.

It is easy to see that this defines a topology on $C\mathcal{M}_{g,m}(M, J, \beta)$.

LEMMA 10.4. *$C\mathcal{M}_{g,m}(M, J, \beta)$ with this topology is Hausdorff.*

*Proof.* Let $\lim_{n\to\infty}((\Sigma_n, z_n), h_n) = ((\Sigma, z), h)$, $\lim_{n\to\infty}(\Sigma_n, z_n), h_n) = ((\Sigma', z'), h')$. We need to show that $((\Sigma, z), h)$ is isomorphic to $((\Sigma', z'), h')$.

Let $((\Sigma_n, z_n^+), h_n)$, $((\Sigma, z^+), h) \in C\mathcal{M}_{g,m+\eta}(M, J, \beta)_0$, and $((\Sigma_n, z_n'^+), h_n)$, $((\Sigma', z'^+), h') \in C\mathcal{M}_{g,m+\eta'}(M, J, \beta)_0$ as in Definition 10.3. By perturbing the points we add, we may assume that $z_n^+ \cap z_n'^+ = z_n$. (Namely the points we add are different from each other.) We put $z_n^+ \cup z_n'^+ = z_n''^+$. Hence we obtain a sequence $((\Sigma_n, z_n''^+), h_n) \in C\mathcal{M}_{g,m+\eta+\eta'}(M, J, \beta)_0$. By taking a subsequence and adding more marked points if necessary, we may assume that it converges to an element of $C\mathcal{M}_{g,m+\eta+\eta'+\eta''}(M, J, \beta)_0$ in the sense of $\mathrm{lims}_{n\to\infty}$. This is the most essential point of the proof. We will prove it in the next section, where we prove the compactness of $C\mathcal{M}_{g,m}(M, J, \beta)$.

Let $\mathrm{lims}_{n\to\infty}((\Sigma_n, z_n''^+), h_n) = ((\Sigma'', z''^+), h'')$. Then it is easy to see that both $((\Sigma, z), h)$ and $((\Sigma', z'), h')$ are obtained from $((\Sigma'', z''^+), h'')$ by forgetting $\eta + \eta' + \eta''$ points we add. Hence $((\Sigma, z), h)$ is ismorphic to $((\Sigma', z'), h')$ as required. □

At the end of this section, we give examples illustrating the definition of the topology.

*Example* 10.5. We consider two $\mathbf{C}P^1$'s in $\mathbf{C}P^2$. Suppose that one of them $\Sigma_1$ is of degree 2 and the other $\Sigma_2$ is of degree 1. In the generic case, they intersect each other at two different points, say $x$, $y$. We put two marked points (one for each) on these curves. It defines an element of $C\mathcal{M}_{1,2}(\mathbf{C}P^2, 3)$. We consider a sequence $(\Sigma_n, z_n, h_n)$ of them such that $\lim_{n\to\infty} d(x_n, y_n) = 0$. (Here $x_n$, $y_n$ are the points where these two curves intersect).

Geometrically the limit may look like of two $\mathbf{C}P^1$'s which intersect at one point with multiplicity 2. However such an object is not a stable map of genus 1 in our sense.

The trouble is that if we forget the map, then union of two $\mathbf{C}P^1$'s intersecting at two points and one marked point on each component has unique complex structure. Hence if we forget the map, the limit should be the same stable curve. However there is no map from this stable curve, whose image is the union of two $\mathbf{C}P^1$'s intersecting at one point.

The reason why we meet this trouble is that the limit is the union of two $\mathbf{C}P^1$'s intersecting at one point and one marked point on each component and is *unstable.*

So we proceed as follows to find the limit. We add two additional marked points $z_{3,n}$, $z_{4,n}$ one to each component so that $h_n(z_{3,n})$, $h_n(z_{4,n})$ converges in $\mathbf{C}P^2$. Let $z_n^+ = (z_{1,n}, z_{2,n}, z_{3,n}, z_{4,n})$. We consider the limit $(\Sigma_n, z_n^+)$ in the Deligne–Mumford compactification and find that the limit consists of $4\mathbf{C}P^1$'s, $\Sigma_1, \dots, \Sigma_4$ such that $\#(\Sigma_1 \cap \Sigma_3) = 1$,

$\#(\Sigma_2 \cap \Sigma_4) = 1$, $\#(\Sigma_3 \cap \Sigma_4) = 2$. We let $\Sigma_1$ be mapped to the first $\mathbf{C}P^1$ and $\Sigma_2$ to the second $\mathbf{C}P^1$ and $\Sigma_3 \cup \Sigma_4$ to the unique point $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$. We then obtain a stable map in our sense.

*Example* 10.6. Let $\mathbf{C}P^1_{i,n}$, $i = 0, \ldots, N$, be $\mathbf{C}P^1$ in, say $\mathbf{C}P^3$. We may assume that $\mathbf{C}P^1_{0,n} \cap \mathbf{C}P^1_{i,n} = \{x_{i,n}\}$ while $\mathbf{C}P^1_{i,n} \cap \mathbf{C}P^1_{j,n} = \emptyset$ if $0 < i < j$. We may assume also that $\lim_{n\to\infty} x_{i,n} = x$ and is independent of $i$. The limit in naive sense is a union of $N+1$ $\mathbf{C}P^1$'s intersecting at one point $x$.

To obtain the limit as a stable map we proceed as follows. We first add 2 marked points to each of $\mathbf{C}P^1_{i,n}$. Then $\mathbf{C}P^1_{0,n}$ has $N$ singular points and two marked points on it. The limit of $\mathbf{C}P^1$ with $N$ singular points converging to the same point $x$ and two marked points, is a tree of $\mathbf{C}P^1$'s. The type of tree depends on the speed of the convergence $\lim_{n\to\infty} x_{i,n} = x$. (This is related to the compactification of configuration space. However, our case is an easy case since we are dealing with the case of Riemann surface.) Therefore what we get as a limit is a tree of $\mathbf{C}P^1$'s which is mapped to $x$ except $N+1$ components.

*Example* 10.7. (This example is already known to Ruan–Tian [60].) Let us consider a family of tori $T^2_n$ in $\mathbf{C}P^2$ converging to a singular curve homeomorphic to $S^2$. We assume that the complex structures of $T^2_n$ remains in a compact subset of the moduli space $\mathcal{M}_{1,0}$. This gives a sequence of elements in $C\mathcal{M}_{1,0}(\mathbf{C}P^2, 3)$.

We may regard the limit as the map from a union of $T^2$ and $\mathbf{C}P^1$ where the map is degenerate on $T^2$.

## 11. COMPACTNESS OF THE MODULI SPACE OF STABLE MAPS

In this section, we are going to show the following:

THEOREM 11.1. *$C\mathcal{M}_{g,m}((M,\omega), J, \leqslant A)$ is compact.*

We remark that Lemma 8.12 is an immediate consequence of Theorem 11.1. We use the following lemma in the proof.

LEMMA 11.2. *There exists $\varepsilon_1$ independent of $L$ and depending only on $M$ with the following properties. If $h: [-L, L] \times S^1 \to (M, \omega, J)$ is a pseudoholomorphic map and if Diam $(h([-L, L] \times S^1)) < \varepsilon_1$, then*

$$\left|\frac{\partial h}{\partial \tau}(\tau, t)\right| + \left|\frac{\partial h}{\partial t}(\tau, t)\right| \leqslant C e^{-dist(\tau, \partial[-L, L])}.$$

*for $\tau \in [-L+1, L-1]$. Here $C$ is a constant independent of $L$ and $(\tau, t) \in [-L, L] \times S^1$. (We identify $S^1 = \mathbf{R}/2\pi\mathbf{Z}$.)*

This lemma probably is not new. For the convenience of the reader, we will give a proof in Section 14. We rewrite the Lemma 11.2 using polar coordinate $z = e^{\tau - L + it}$ and obtain the following Lemma 11.2′. We put

$$Annu(r, R) = \{z \in \mathbf{C} \mid r < |z| < R\}.$$

Lemma 11.2′. *There exists $\varepsilon_1$ independent of $r$ and depending only on $M$ with the following properties. If $h: Annu(r, 1) \to (M, \omega, J)$ is a pseudoholomorphic map and if $Diam(h(Annu(r, 1))) < \varepsilon_1$, then*

$$\left|\frac{\partial h}{\partial z}(z)\right| + \left|\frac{\partial h}{\partial \bar z}(z)\right| \leqslant C \max\left(1, \frac{r}{|z|^2}\right)$$

*for* $\log z \in [\log r + 1, -1]$. *Here $C$ is a constant independent of $r$ and $z$.*

*Proof of Theorem* 11.1. Let $((\Sigma_n, z_n), h_n) \in C\mathcal{M}_{g,m}((M, \omega), J, \leqslant A)$ be a sequence. We are going to find its converging subsequence.

Let $((\Sigma_n, z_n), h_n) \in \mathcal{M}_{g,m}(M, J, \beta_n), (T_n, g_{n,v}, \beta_{n,v}, o_n)$. Since we have already proved in Section 8 that there are only a finite number of possibilities of $(T_n, g_{n,v}, o_n)$, if follows that we may assume that $(T_n, g_{n,v}, o_n) = (T, g_v, o)$ is independent of $n$, by taking a subsequence if necessary. (As we remarked in Remark 8.13, we need Lemma 8.12 in the proof of Proposition 8.8 only to show the finiteness of the possibilities of $\beta_{n,v}$.) Therefore we can add a finite number of additional marked points to $(\Sigma_n, z_n)$ (whose number is independent of $n$) and obtain $(\Sigma_n, z_n^+)$ which is stable. (Hereafter we replace $(\Sigma_n, z_n^+)$ several times by adding more and more marked points.) By taking a subsequence and by using compactness of Deligne–Mumford compactification, we may assume that $(\Sigma_n, z_n^+)$ converges to $(\Sigma_\infty, z_\infty^+)$ in $C\mathcal{M}_{g,m}$. Let $\Sigma_\infty = \bigcup_v \Sigma_{\infty,v}$ be the decomposition into irreducible components. By the description of Deligne–Mumford compactification in Section 9, we have

$$\Sigma_n' = \bigcup_v \Sigma_{n,v}' \quad \text{and} \quad (\alpha_{x_n}) \in \bigoplus_{x_n} T_{x_{v,n}}\Sigma_{v,n}' \otimes T_{x_{w,n}}\Sigma_{w,n}'$$

(here $x_n$ runs over singular points of $\Sigma_n'$) such that $(\Sigma_n', z_n'^+)$ belongs to the same stratum as $(\Sigma_\infty, z_\infty^+)$, and that $(\Sigma_n, z_n^+)$ is obtained from $(\Sigma_n', z_n'^+)$ by resolving the singularity using the parameter $(\alpha_{x_n})$ as in Section 9.

Also $(\Sigma_n', z_n'^+)$ converges to $(\Sigma_\infty, z_\infty^+)$ in $C^\infty$-topology and $(\alpha_{x_n})$ converges to 0.

By the definition in Section 9, $\Sigma_n$ has a subset identified to $\Sigma_n' - \bigcup_{x_{v,n}} D_{x_{v,n}}(R_{n,x}^{-3/2})$, which we denote by the same symbol. We find that the symplectic volume of the restriction of $h_n$ is uniformly bounded. Let $\mu > 0$ be a sufficiently small number.

Proposition 11.3. *By increasing the number of marked points and by taking a subsequence if necessary, we may assume that*

$$\sup_{\Sigma_n' - \bigcup D_{x_{v,n}}(\mu)} |\nabla h_n| <$$

*Proof.* The idea of the proof is simple. If $|\nabla h_n|$ diverges then we add two marked points whose distance in $\Sigma_n$ is something like $|\nabla h_n|^{-1}$. Then we have an additional component $\mathbf{C}P^1$ where the map is nontrivial in the limit. Such a process should stop after finitely many repetitions since each pseudoholomorphic $\mathbf{C}P^1$ has mass greater than $\delta$ by Lemma 8.1. To work out this idea one needs several technicalities.

Let $\varepsilon_1$ be as in Lemma 11.2 and let $\mu > 0$ be a sufficiently small positive number. By taking a subsequence, we may assume that, for each $v$, either

$$Diam\, h_n\left(\Sigma_{n,v}' - \bigcup_{x_{v,n} \in \Sigma_v} D_{x_{v,n}}(\mu)\right) \geqslant \frac{\varepsilon_1}{1000}$$

for every $n$ or

$$Diam\, h_n\left(\Sigma'_{n,v} - \bigcup_{x_{v,n}\in\Sigma_v} D_{x_{v,n}}(\mu)\right) \leqslant \frac{\varepsilon_1}{100}$$

for every $n$. We may assume also that for each $j_1, j_2$, either $dist(h_n(z_{n,j_1}), h_n(z_{n,j_2})) \geqslant \varepsilon_1/1000$ for every $n$ or $dist(h_n(z_{n,j_1}), h_n(z_{n,j_2})) \leqslant \varepsilon_1/100$ for every $n$. (Here we recall that $z_{n,j}$ is the $j$th marked point of $(\Sigma_n, z_n^+)$.)

We put

$$\mathscr{V}_1((\Sigma_n, z_n^+), h_n) = \left\{ v \,\middle|\, Diam\, h_n\left(\Sigma'_{a,v} - \bigcup D_{x_{v,n}}(\mu)\right) \geqslant \frac{\varepsilon_1}{1000}\right\}$$

$$\mathscr{V}_2((\Sigma_n, z_n^+), h_n) = \left\{ v\in\mathscr{V}_1 \,\middle|\, \exists j_1, j_2 \text{ such that } z_{n,j_1}, z_{n,j_2}\in\Sigma'_{a,v},\ dist(h_n(z_{n,j_1}), h_n(z_{n,j_2})) > \frac{\varepsilon_1}{1000}\right\}.$$

The right-hand sides of these definitions are independent of $n$. So this set is defined from the sequence $((\Sigma_n, z_n^+), h_n)$. We furthermore define

$$\mathscr{V}_3((\Sigma_n, z_n^+), h_n) = \left\{ v\in\mathscr{V}_1 \,\middle|\, \limsup_{n\to\infty} \sup_{\Sigma'_{n,v} - \bigcup D_{x_{v,n}}(\mu)} |\nabla h_n| < \infty \right\}.$$

We remark that Lemma 8.10 implies that if $v\in\mathscr{V}_1((\Sigma_n, z_n^+), h_n)$ then

$$\int_{\Sigma'_{n,v} - \bigcup D_{x,n}} h_n^*\omega > c\varepsilon_1^2.$$

Hence the order of $\mathscr{V}_1$ is uniformly bounded.

LEMMA 11.4. *Suppose there exists* $p_n\in\Sigma'_n - \bigcup_{x_{v,n}} D_{x_{v,n}}(\mu)$ *such that* $|\nabla h_n|(p_n)$ *goes to infinity. Then we can take a subsequence and add marked points to obtain* $(\Sigma_n, z_n^{++})$ *such that*

$$\#\mathscr{V}_1((\Sigma_n, z_n^+), h_n) < \#\mathscr{V}_1((\Sigma_n, z_n^{++}), h_n)$$

*or*

$$\#\mathscr{V}_1((\Sigma_n, z_n^+), h_n) = \#\mathscr{V}_1((\Sigma_n, z_n^{++}), h_n)$$

$$\#\mathscr{V}_2((\Sigma_n, z_n^+), h_n) < \#\mathscr{V}_2((\Sigma_n, z_n^{++}), h_n)$$

*or*

$$\#\mathscr{V}_1((\Sigma_n, z_n^+), h_n) = \#\mathscr{V}_1((\Sigma_n, z_n^{++}), h_n)$$

$$\#\mathscr{V}_2((\Sigma_n, z_n^+), h_n) = \#\mathscr{V}_2((\Sigma_n, z_n^{++}), h_n)$$

$$\#\mathscr{V}_3((\Sigma_n, z_n^+), h_n) < \#\mathscr{V}_3((\Sigma_n, z_n^{++}), h_n).$$

Proposition 11.3 will follow immediately from Lemma 11.4.

For the proof of Lemma 11.4, we put

$$\sup_{- D_{x_{v,n}}(\mu)} |\nabla h_n| = C_{v,n}$$

And let $|\nabla h_n|(p_n) = C_{v,n}$ where $p_n\in\Sigma'_{n,v} - \bigcup_{x_{v,n}} D_{x_{v,n}}(\mu)$.

We first consider the case when bubble happens away from the boundary. Namely the case when $|\nabla h_n(p_n)|\, dist(p_n, \partial(\Sigma'_{n,v} - \bigcup_{x_{v,n}} D_{x_{v,n}}(\mu))) \to \infty$. We consider the composition $z \mapsto h_n \exp_{p_n}(|\nabla h_n|^{-1} z)$. (Here we identify $T_{p_n}\Sigma_n$ with $\mathbf{C}$ by an isometry.) By (11.5), the $C^1$ norm of this composition is uniformly bounded on each $D(R) \subseteq \mathbf{C}$. And also the norm of its derivative at origin is 1. Hence by using elliptic regularity, we may assume that it converges to a pseudoholomorphic map from $\mathbf{C}$, in $C^\infty$-topology on any compact subsets. It can then be extended to $\mathbf{C}P^1$ by Gromov's removable singularity theorem.

Then the diameter of this pseudoholomorphic $\mathbf{C}P^1$ (which is nontrivial), must be larger than the injectivity radius of $M$. We may assume that the injectivity radius of $M$ is larger than $\varepsilon_1$. Therefore we can find $v_0 \in \mathbf{C}$ independent of $n$ such that

$$dist(h_n \exp_{p_n}(|\nabla h_n|^{-1} v_0), h_n(p_n)) \geqslant \varepsilon_1/2.$$

We put $p'_n = \exp_{p_n}(|\nabla h_n|^{-1} v_0)$. We take $p_n$ and $p'_n$ for the new marked points. We then consider a new sequence $(\Sigma_n, z_n^{++})$ and consider its limit in Deligne–Mumford compactification. Since $|\nabla h_n|$ assumes its maximum at $p_n$, we find that a sequence of pseudoholomorphic maps on some new component (i.e. $\mathbf{C}P^1$) is nontrivial and uniformly bounded there. (If $p_n$ is away from marked points then there is only one new component. If $p_n$ goes to a marked point, then the number of new components is one or two. In each case $|\nabla h_n|$ is bounded on these new components.) It follows that we have $\#\mathscr{V}_1((\Sigma_n, z_n^+), h_n) \leqslant \#\mathscr{V}_1((\Sigma_n, z_n^{++}), h_n)$, $\#\mathscr{V}_2((\Sigma_n, z_n^+), h_n) \leqslant \#\mathscr{V}_2((\Sigma_n, z_n^{++}), h_n)$, $\#\mathscr{V}_3((\Sigma_n, z_n^+), h_n) < \#\mathscr{V}_3((\Sigma_n, z_n^{++}), h_n)$. Thus Lemma 11.4 holds in this case.

We need some more argument to study the case when $|\nabla h_n(p_n)|\, dist(p_n, \partial(\Sigma'_{n,v} - \bigcup_{x_{v,n}} D_{x_{v,n}}(\mu)))$ is bounded. Namely the case when there is a bubble near the neck region. In this case, however, we find $p_n$ such that $|\nabla h_n|(p_n) \to \infty$ and $p_n$ is away from marked points. To go further we show

SUBLEMMA 11.6. *Suppose $p_n \in \Sigma'_n - \bigcup_{x_{v,n}} D_{x_{v,n}}(\mu)$ such that $|Dh_n|(p_n)$ goes to infinity. Then, by taking a subsequence if necessary, there exists a sequence of points $p_{n,+}, p_{n,-}$ such that $\varepsilon_1/10 < dist(h_n(p_{n,+}), h_n(p_{n,-}))$ and $\lim_{n\to\infty} dist(p_{n,\pm}, p_n) = 0$.*

*Proof.* Choose $q_n$ such that $d(p_n, q_n) = \delta_n \to 0$. Let $\lambda > 0$. We first prove

$$\limsup_{n\to\infty} Diam(h_n(D_{q_n}(\lambda))) > \frac{\varepsilon_1}{2}, \tag{11.7}$$

where $\varepsilon_1$ is as in Lemma 11.2. We suppose that (11.7) is false. Then, by taking a subsequence, we may assume that $Diam(h_n(D_{q_n}(\lambda))) \leqslant \varepsilon_1$ for any $n$. We consider a map $\psi_n$: $Annu(\delta_n^2, 1) \to M$ defined by $\psi_n(z) = \exp_{q_n}(\lambda z/2)$. Here we identify $T_{q_n}\Sigma_a \cong \mathbf{C}$. We put $\psi_n(z_n) = p_n$. Then $|z_n|/\delta_n$ is bounded. We find that the image of $\psi_n$ is contained in $D_{q_n}(\lambda)$. It follows that $Diam(h_n \circ \psi_n(Annu(\delta_n^2, 1)) \leqslant \varepsilon_1$. Therefore by Lemma 11.2′ $|D(h_n \circ \psi_n)|(z_n) \leqslant C$. Hence $|Dh_n(p_n)| \leqslant C$. This contradicts our assumption. We have proved (11.7).

The rest of the proof is a standard diagonal procedure. We choose $\lambda_m \to 0$. By (11.7) we have a subsequence $n_{1,i}$ such that $Diam(h_{n_{1,j}}(D_{q_{1,j}}(\lambda_1)) \geqslant \varepsilon_1/2$ and that $\lim_{i\to\infty} n_{1,i} = \infty$. We then choose inductively subsequences $n_{m,i}$ of $n_{m-1,i}$ such that $Diam(h_{n_{m,i}}(D_{q_{m,i}}(\lambda_m))) \geqslant \varepsilon_1/2$ and that $\lim_{i\to\infty} n_{m,i} = \infty$. We take the subsequence $n_{m,m}$. So we find that $Diam(h_{n_{m,m}}(D_{q_{m,m}}(\lambda_m))) \geqslant \varepsilon_1/2$. We have two points $p_{n_{m,m},+}, p_{n_{m,m},-}$ on $D_{q_{m,m}}(\lambda_m)$ such that $dist(h_{n_{m,m},+}), h_{n_{m,m}}(p_{n_{m,m},-})) > \varepsilon_1/4$. Sublemma 11.6 follows. □

We use Sublemma 11.6 to find $p_{n,+}$, $p_{n,-}$, which we add as marked points. We get $(\Sigma_n, z_n^{++})$. Let $(\hat\Sigma_\infty, z_\infty^{++})$ be its limit in Deligne–Mumford compactification.

Since we choose $p_{n,+}$, $p_{n,-}$ so that they are uniformly away from marked points (this was possible because they are in the neighborhood of the neck region), $\hat\Sigma_\infty$ is obtained by attaching one $\mathbf{C}P^1$ to $\Sigma_\infty$. We write it $\hat\Sigma_{v_{\rm new},\infty} = \mathbf{C}P^1$.

We remark that nothing will change for components $\Sigma_{\infty,w}$ of $\Sigma_\infty$ other than $v$ and $\alpha_{\rm new}$. Namely

$$\left.\begin{matrix} w \in \mathscr{V}_j((\Sigma_n, z_n^+), h_n) \\ w \neq v \end{matrix}\right\} \Rightarrow w \in \mathscr{V}_j((\hat\Sigma_n, z_n^{++}), h_n)$$

$$\left.\begin{matrix} w \in \mathscr{V}_j((\hat\Sigma_n, z_n^{++}), h_n) \\ w \neq v, v_{\rm new} \end{matrix}\right\} \Rightarrow w \in \mathscr{V}_j((\Sigma_n, z_n^+), h_n).$$

We remark that, since $\varepsilon_1/100 < dist(h_n(p_{n,+}), h_n(p_{n,-}))$ it follows that $v_{\rm new} \in \mathscr{V}_2((\Sigma_n, z_n^{++}), h_n)$. There are three cases

*Case* 1: $v \notin \mathscr{V}_1((\Sigma_n, z_n^+), h_n)$. Since $v_{\rm new} \in \mathscr{V}_1((\hat\Sigma_n, z_n^{++}), h_n)$, we have $\#\mathscr{V}_1(((\Sigma_n, z_n^+), h_n) < \#\mathscr{V}_1((\hat\Sigma_n, z_n^{++}), h_n)$.

*Case* 2: $v \in \mathscr{V}_2(((\Sigma_n, z_n^+), h_n))$, $v \in \mathscr{V}_2((\Sigma_n, z_n^+), h_n)$ implies that $v \in \mathscr{V}_2((\hat\Sigma_a, z_a^{++}), h_a)$. Hence $\#\mathscr{V}_1(((\Sigma_a, z_a^+), h_a)) < \#\mathscr{V}_1(((\Sigma_a, z_a^{++}), h_a))$ follows from $v_{\rm new} \in \mathscr{V}_1((\Sigma_a, z_a^{++}), h_a)$.

*Case* 3: $v \in \mathscr{V}_1((\Sigma_a, z_a^+), h_a) - \mathscr{V}_2((\Sigma_a, z_a^+), h_a)$. Since $v_{\rm new} \in \mathscr{V}_2((\hat\Sigma_n, z_n^{++}), h_n)$, we have $\#\mathscr{V}_1((\Sigma_n, z_n^+), h_n) \leqslant \#\mathscr{V}_1((\hat\Sigma_n, z_n^{++}), h_n)$, $\#\mathscr{V}_2((\Sigma_n, z_n^+), h_n) < \#\mathscr{V}_2((\Sigma_n, z_n^{++}), h_n)$.

The proof of Proposition 11.3 is now complete. □

By Proposition 11.3 and diagonal procedure, we have a sequence $\mu_n$ converging to 0 such that

$$\sup_{\Sigma_n' - \bigcup D_{x_{v,n}}(\mu)} |Dh_n| < C \tag{11.8}$$

for some number $C$ independent of $n$. By (11.8) and elliptic regularity, we conclude that for each compact subset of $\Sigma_\infty - \{\text{singular points}\}$, $h_n$ converges in $C^\infty$ topology. (We remark that any compact subset of $\Sigma_\infty - \{\text{singular points}\}$ can be regarded as a subset of $\Sigma_n$ for large $n$.)

We next consider the neck region. We fix $\lambda_n \to 0$ and a small number $\lambda_0 > 0$ and put

$$W_{n,x}^+ = D_{x_{v,n}}(\lambda_0) \cup D_{x_{w,n}}(\lambda_0)$$

$$W_{n,x} = D_{x_{v,n}}(2\lambda_n) \cup D_{x_{w,n}}(2\lambda_n).$$

$W_{n,x}^+$, $W_{n,x}$ are biholomorphic to $Annu_{x,n}^+ = [-L_{x,n}^+, L_{x,n}^+] \times S^1$, $Annu_{x,n} = [-L_{x,n}, L_{x,n}] \times S^1$, respectively. Here $L_{x,n}, L_{x,n}^+ \to \infty$ and $x$ is a singular point of $\Sigma_\infty$.

We remark that the restriction of $h_n$ to the boundary of $W_{n,x}^+$ converges for each $x$. We restrict $h_n$ to $W_{n,x}^+$ and regard it as a map from $Annu_{x,n}^+ = [-L_{x,n}^+, L_{x,n}^+] \times S^1$. If the differential of $|\partial h_n/\partial\tau| + |\partial h_n/\partial t|$ on $Annu_{x,n}^+$ is not bounded, we discuss in a similar way to the proof of Proposition 11.3 to find 2 more marked points such that $\#\mathscr{V}_1((\Sigma_n, z_n^+), h_n)$ will increase for this new sequence. Therefore after finitely many steps, we may assume that $|\partial h_n/\partial\tau| + |\partial h_n/\partial t|$ is bounded on each $Annu_{x,n}^+$.

*that for each* $\varepsilon > 0$ *there exists* $L(\varepsilon)$ *and* $n(\varepsilon)$ *with* $\lim_{\varepsilon\to 0} L(\varepsilon) = \infty$ *such that if* $\tau \in [-L_{x,n} + L(\varepsilon), L_{x,n} - L(\varepsilon)]$ *and* $n > n(\varepsilon)$ *then* $|Dh_n|(\tau, t) \leqslant \varepsilon$.

*Proof.* Suppose that there is a sequence $T_n$ such that $|\partial h_n/\partial\tau|(T_n, t_n) + |\partial h_n/\partial t|(T_n, t_n)$ is bounded uniformly away from 0 and $dist(T_n \pm L_{x,n}) \to \infty$. Since the restriction of $h_n$ in a neighborhood of $(T_n, t_n)$ converges to a pseudoholmorphic map $\mathbf{R} \times S^1 \to M$, which is nontrivial, we find $(S_n, s_n)$ such that

$$dist(h_n(T_n, t_n), h_n(S_n, s_n)) \geqslant \frac{\varepsilon_1}{2}.$$

We add $(T_n, t_n)$ and $(S_n, s_n)$ as marked points. We then find that $\# \mathscr{V}_1((\Sigma_n, z_n^+), h_n)$ increases again. Therefore repeating this finitely many times, we may assume that there is no such $T_n$. The lemma follows. □

We use Lemma 11.9 to show the following

LEMMA 11.10. *$Diam(h_n([-L_{x,n} + B, L_{x,n} - B] \times S^1)) \leqslant C'e^{-B}$ for sufficiently large n. Here $C'$ is independent of n and B.*

*Proof.* We take $B_0$ such that

$$100 \int_{B_0}^{2d} Ce^{-\min\{t,d\}}\, dt \leqslant \varepsilon_1 \tag{11.11}$$

holds for each $d \geqslant B_0$. Here $C$ is as in Lemma 11.2. We next choose $\varepsilon_2$ such that $\varepsilon_1/10\varepsilon_2 \geqslant B_0 + 10$. We put $B_1 = L(\varepsilon_2)$. Here $L(\varepsilon_2)$ is as in Lemma 11.9.

Now let $\tau_0 \in [B_0 + B_1 - L_{x,n}, L_{x,n} - B_0 - B_1]$. We put $d_1 = B_0$. By Lemma 11.9, we have

$$\left|\frac{\partial h_n}{\partial \tau}(\tau, x_n)\right| + \left|\frac{\partial h_n}{\partial t}(\tau, x_n)\right| \leqslant \varepsilon_2$$

for $\tau \in [\tau_0 - d_1, \tau_0 + d_1]$. Hence

$$Diam(h_n(S^1 \times [\tau_0 - d_1, \tau_0 + d_1])) \leqslant 2\varepsilon_2(d_1 + 10) \leqslant \frac{\varepsilon_1}{5}.$$

It follows from Lemma 11.2 that if $\tau \in [\tau_0 - d_{1+1}, \tau_0 + d_1 - 1]$ then

$$\left|\frac{\partial h_n}{\partial \tau}(\tau, t)\right| + \left|\frac{\partial h_n}{\partial t}(\tau, t)\right|_{\tau_0} \leqslant Ce^{-dist(\tau, \partial[\tau_0 - d_1, \tau_0 + d_1])}.$$

Since this holds for any $\tau_0$ with $[\tau_0 - d_1, \tau_0 + d_1] \subseteq [B_1 - L_{x,n}, L_{x,n} - B_1]$, we have

$$\left|\frac{\partial h_n}{\partial \tau}(\tau, t)\right| + \left|\frac{\partial h_n}{\partial t}(\tau, t)\right| \leqslant Ce^{-dist(\tau, \partial[\tau_0 - d_1, \tau_0 + d_1])} \leqslant Ce^{-\min[dist(\tau, \partial[B_1 - L_{x,n}, L_{x,n} - B_1]), d_1)}. \tag{11.12}$$

for any $\tau \in [B_1 - L_{x,n}, L_{x,n} - B_1]$. We put $d_k = 2^k d_1$. (11.12) and (11.11) imply that

$$\begin{aligned} &Diam(h_n([\tau_0 - d_2, \tau_0 + d_2] \times S^1)) \\ &\leqslant Diam(h_n([-L_{x,n} + B_1, -L_{x,n} + B_0 + B_1] \times S^1)) \\ &\quad + Diam(h_n([L_{x,n} - B_0 - B_1, L_{x,n} - B_1] \times S^1)) \\ &\quad + Diam(h_n([\max(\tau_0 - d_2, -L_{x,n} + B_0 + B_1), \min(\tau_0 + d_2, L_{x,n} - B_0 - B_1)] \times S^1)) \\ &\leqslant \frac{\varepsilon_1}{5} + \frac{\varepsilon_1}{5} + \frac{\varepsilon_1}{10} \leqslant \varepsilon_1 \end{aligned}$$

if $[\tau_0 - d_2, \tau_0 + d_2] \subseteq [B_1 - L_{x,n}, L_{x,n} - B_1]$. Therefore if $\tau \in [\tau_0 - d_2 + 1, \tau_0 + d_2 - 1]$ then

$$\left|\frac{\partial h_n}{\partial \tau}(\tau, t)\right| + \left|\frac{\partial h_n}{\partial t}(\tau, t)\right| \leqslant Ce^{-dist\{\tau, \partial[\tau_0 - d_2, \tau_0 + d_2]\}}.$$

Since this holds for any $\tau_0$ with $[\tau_0 - d_2, \tau_0 + d_2] \subseteq [B_1 - L_{x,n}, L_{x,n} - B_1]$, we have

$$\left|\frac{\partial h_n}{\partial \tau}(\tau, t)\right| + \left|\frac{\partial h_n}{\partial t}(\tau, t)\right| \leqslant Ce^{-\min\{dist(\tau, \partial[B_1 - L_{x,n}, L_{x,n} - B_1]), d_2\}}.$$

for any $\tau \in [B_1 - L_{x,n}, L_{x,n} - B_1]$. Thus we have

$$\left|\frac{\partial h_n}{\partial \tau}(\tau, t)\right| + \left|\frac{\partial h_n}{\partial t}(\tau, t)\right| \leqslant Ce^{-\min\{dist(\tau, \partial[B_1 - L_{x,n}, L_{x,n} - B_1]), d_k\}}$$

by induction on $k$. We then conclude

$$\left|\frac{\partial h_n}{\partial \tau}(\tau, t)\right| + \left|\frac{\partial h_n}{\partial t}(\tau, t)\right| \leqslant Ce^{-dist(\tau, \partial[B_1 - L_{x,n}, L_{x,n} - B_1])}.$$

Lemma 11.10 follows immediately. □

By using Lemma 11.10, we can extent $h'_\infty$ to a pseudoholomophic map $h_\infty$ from $\Sigma_\infty$ to $X$. By definition we have

$$\lim_{n \to \infty} ((\Sigma_n, z_n^+), h_n) = ((\Sigma_\infty, z_\infty^+), h_\infty).$$

Let us next study whether $forget_\eta((\Sigma_\infty, z_\infty^+), h_\infty)$ is well defined or not. (It might be possible to show that it is well defined in the situation of the proof of Theorem 11.1 by carefully taking the way to add marked points. However, in the situation to prove Lemma 10.4, we certainly need to consider the case when $forget_\eta((\Sigma_\infty, z_\infty^+), h_\infty)$ is not well defined, and we need to remove some marked points we added.) Let $\Sigma_{\infty,v}$ be a component of $\Sigma_\infty$. We say that it is a dead component if $h_\infty$ is constant there and if it will be unstable after removing the marked points we added. We may assume $\beta \neq 0$ then the dead component must be $\mathbf{C}P^1$ and has at most two singular points. We consider the union of all dead components and take its connected component say $Y = \bigcup_{i \in I} \Sigma_{v_i}$. They are mapped to a point. We consider the union of $\bigcup_{i \in I} \Sigma_{n,v_i} - \bigcup_x D_{x_\eta,n}(\lambda_n)$ and add the necks corresponding to the singular points disjoint from $\overline{\Sigma_\infty - Y}$. We denote it by $Y_n$. It is easy to see that $Y_n$ is conformal to annuli or disk. We have $Diam(h_n(Y_n)) \to 0$.

We then remove all marked points we add which lie on the union of $Y_n$. We denote it by $((\Sigma_n, z_n^{+-}), h_n)$.

On the other hand, we remove also all the added marked points on $(\Sigma_\infty, z_\infty^+)$ in the dead component and shrink each of $Y$ to a point. We then obtain $(\Sigma_\infty, z_\infty^+)$.

It follows from $Diam(h_n(Y_n)) \to 0$ that

$$\overline{\lim s_{n \to \infty} ((\Sigma_n, z_n^{+-}), h_n)} = \overline{((\Sigma_\infty, z_\infty^{+-}), h_\infty)}.$$

# CHAPTER 3. CONSTRUCTION OF KURANISHI STRUCTURE

## 12. CONSTRUCTION OF LOCAL CHART I — APPROXIMATE SOLUTION

This chapter is devoted to the proof of Theorems 7.10 and 7.11. In Sections 12–14, we are concerned with a local construction. Namely we construct a chart around each element $\sigma = ((\Sigma, z), h)$ of $C\mathscr{M}_{g,m}(M, J, \beta)$. The construction is a combination of various ideas appeared in various other places. First it is a variant of Kuranishi's construction of local versal family of complex structures. However, we are studying a similar problem at "infinity". Also in the similar situation in Gauge theory, there are works by Taubes [67, 68] and Donaldson [12, 13] performing the gluing construction at infinity in the situation where there is an obstruction. Also for pseudoholomorphic curve, Floer [16, 19] used Taubes' type gluing argument. For the study of Gromov–Witten invariant, Ruan–Tian [60] used a gluing. Also much of the analytic part of our argument in this chapter is a copy of Appendix A of McDuff–Salamon's book [47] with some modifications.

Now we start the construction. Let $\sigma = [((\Sigma_\sigma, z_\sigma), h_\sigma)] \in \mathscr{M}_{g,m}(M, J, \beta)(T, g_v, \beta_v, o)$. We first consider the deformation complex of map $h$ as follows. Let $\Sigma = \bigcup_v \Sigma_v$ be the decomposition of $\Sigma$ into its components. We fix a Kähler metric on each component. We put

this seems to denote the moduli space (also CM(...)) so always missing the ambiguity in choice of representative

$$C^\infty(\Sigma_\sigma; h_\sigma^* TM) = \left\{ (u_v) \in \bigoplus_v C^\infty(\Sigma_{\sigma,v}; h_\sigma^* TM) \,\middle|\, u_v(p) = u_w(q) \text{ if } \pi_{\Sigma_{\sigma,v}}(p) = \pi_{\Sigma_{\sigma,w}}(q) \right\}$$

$$C^\infty(\Sigma_\sigma; h_\sigma^* TM \otimes \Lambda^{0,1}(\Sigma_\sigma)) = \bigoplus_v C^\infty(\Sigma_{\sigma,v}; h_\sigma^* TM \otimes \Lambda^{0,1}(\Sigma_\sigma)).$$

We define Sobolev spaces $L_1^p(\Sigma_\sigma; h_\sigma^* TM)$, $L^p(\Sigma_\sigma; h_\sigma^* TM \otimes \Lambda^{0,1}(\Sigma_\sigma))$ in a similar way. (Here $L_1^p$ Sobolev space consists of elements whose first derivative is of $L^p$ class.) We remark that the definition of $L_1^p(\Sigma_\sigma; h_\sigma^* TM)$ makes sense only for $p > 2$ since only in that case $L_1^p$ section is continuous.

We consider the linearization of the pseudoholomorphic curve equation. It induces an operator

$$(D_{h_\sigma} \bar\partial_{\Sigma_\sigma}) : C^\infty(\Sigma_\sigma; h_\sigma^* TM) \to C^\infty(\Sigma_\sigma; h_\sigma^* TM \otimes \Lambda^{0,1}(\Sigma_\sigma))$$

and

$$(D_{h_\sigma} \bar\partial_{\Sigma_\sigma}) : L_1^p(\Sigma_\sigma; h_\sigma^* TM) \to L^p(\Sigma_\sigma; h_\sigma^* TM \otimes \Lambda^{0,1}(\Sigma_\sigma)) \tag{12.1}$$

! The truly relevant operators will be those on local slices in $L^p_1(\Sigma_\sigma, \ldots)$ to the Aut(Σ) action

could be PSL(2, ℂ) or other Lie Group; only Aut(σ) is finite

Here we write $(D_{h_\sigma} \bar\partial_{\Sigma_\sigma})$ in order to distinguish it from the nonlinear equation $\bar\partial_{\Sigma_\sigma} h = 0$. (12.1) is a bounded operator.

LEMMA 12.2. (12.1) *is a Fredholm operator of index* $2c^1(M)\beta + 2n(1 - g)$.

*Proof.* We first consider each component separately. Namely we consider the following operator:

$$D\bar\partial : L_1^p(\Sigma_{\sigma,v}; h_\sigma^* TM) \to L^p(\Sigma_{\sigma,v}; h_\sigma^* TM \otimes \Lambda^{0,1}(\Sigma_{\sigma,v})). \tag{12.3}$$

By Riemann–Roch and Atiyah–Singer index theorem, (12.3) is a Fredholm operator of index $2c^1(M)\beta_v + 2n(1 - g_v)$. Here $g_v$ is the genus of $\Sigma_{\sigma,v}$ and $\beta_v = h_{\sigma,*}[\Sigma_{\sigma,v}]$. By definition

$$L_1^p(\Sigma_\sigma; h_\sigma^* TM) \subseteq \bigoplus_v L_1^p(\Sigma_{\sigma,v}; h_\sigma^* TM). \tag{12.4}$$

The codimension of the inclusion (12.4) is equal to $2n$ times the number of singular points. Using the graph $T_{\Sigma_\sigma}$ we introduced in Section 8, we find that the number of singular points is equal to the number of the edges of $T_{\Sigma_\sigma}$.

By Euler's formula we have

$$\#\text{edges of } T_{\Sigma_\sigma} - \# \text{ vertices of } T_{\Sigma_\sigma} = \text{rank } H_1(T_{\Sigma_\sigma}; \mathbf{Q}) - 1$$

Therefore

$$dim \frac{\bigoplus_v L_1^p(\Sigma_{\sigma,v}; h_\sigma^* TM)}{L_1^p(\Sigma_\sigma; h_\sigma^* TM)} = 2n(\text{rank } H_1(T_{\Sigma_\sigma}; \mathbf{Q}) - 1 + \# \text{ vertices of } T_{\Sigma_\sigma}). \qquad (12.5)$$

On the other hand, summing up the indices of (12.3), we find that the index of

$$D\bar\partial : \oplus L_1^p(\Sigma_{\sigma,v}; h_\sigma^* TM) \to \oplus L^p(\Sigma_{\sigma,v}; h_\sigma^* TM \otimes \Lambda^{0,1}(\Sigma_{\sigma,v})).$$

is equal to

$$\sum_v 2c^1(M)\beta_v + 2n(1 - g_v) = 2c^1(M)\beta - 2n \sum g_v + 2n\# \text{ vertices of } T_{\Sigma_\sigma}. \qquad (12.6)$$

We remark that $\sum g_v + \text{rank } H_1(T_{\Sigma_\sigma}; \mathbf{Q}) = g$ (Definition 7.2). The lemma then follows from (12.5) and (12.6). □

We next choose a subspace $E_\sigma$ of $L^p(\Sigma_\sigma; h_\sigma^* TM \otimes \Lambda^{0,1}(\Sigma_\sigma))$ with the following properties.

(12.7.1) The sum of the image of (12.1) and $E_\sigma$ is $L^p(\Sigma_\sigma; h_\sigma^* TM \otimes \Lambda^{0,1}(\Sigma_\sigma))$.

(12.7.2) $E_\sigma$ is complex linear and $Aut(\sigma)$-invariant.

(12.7.3) There exists a compact set $K_{\text{obstru}}(\sigma)$ away from the singular point such that the support of each element of $E_\sigma$ is in $K_{\text{obstru}}(\sigma)$. We assume also that $K_{\text{obstru}}(\sigma)$ is $Aut(\sigma)$-invariant.

(12.7.4) $E_\sigma$ is finite dimensional and consists of smooth sections.

To find such $E_\sigma$, we first use the unique continuation theorem ([3] see also [22]) to find $E'_\sigma$ satisfying (12.7.1), (12.7.3) and (12.7.4) and that $E'_\sigma$ is isomorphic to the cokernel of (12.1). Using the fact that the action of $Aut(\sigma)$ is complex linear, we find a finite dimensional subspace $E_\sigma$ containing $E'_\sigma$ and satisfying (12.7.1)–(12.7.4). Hence $E_\sigma$ is equal to or larger than the cokernel of (12.1).

We remark that the main point of negative multiple cover problem is that we cannot assume in general that the cokernel of (12.1) is 0 for generic almost complex structure. Hence we need to work in the situation when the bundle $E_\sigma$ is nontrivial. This is the reason we introduced the notion of Kuranishi structure.

We now consider the operator

$$\Pi_{E_\sigma} \circ (D_{h_\sigma}\bar\partial) : L_1^p(\Sigma_\sigma; h_\sigma^* TM) \to \frac{L^p(\Sigma_\sigma; h_\sigma^* TM \otimes \Lambda^{0,1}(\Sigma_\sigma))}{E_\sigma}. \qquad (12.8)$$

$$\frac{L^p(\Sigma_\sigma; h_\sigma^* TM \otimes \Lambda^{0,1}(\Sigma))}{E_\sigma}$$

is well defined as a Banach space because of (12.7.4). It is a complex vector space and $Aut(\sigma)$ acts on it because of (12.7.2).

Let ${}_{\text{map},\sigma}$ be the kernel of the operator (12.8). We next take a family of semistable curves in a neighborhood of our semistable curve $(\Sigma_\sigma, z_\sigma)$ together with a family of Riemann metrics on it. The construction is similar to one in Section 9, but there is a small difference related to the fact that $(\Sigma_\sigma, z_\sigma)$ may not be stable.

will be dealing with nondiscrete $Aut(\Sigma_6)$ here

We first consider a deformation in the same stratum (combinatorial type). We consider a stable component $\Sigma_{\sigma,v}$. We regard it as an element of $\mathcal{M}_{g_v,m_v}$. Here $g_v$ is the genus of $\Sigma_{\sigma,v}$ and $m_v$ is the number of points on $\Sigma_{\sigma,v}$ which are marked or singular. (Let $z_v$ be the collection of those points together with its order.) The assumption that $\Sigma_{\sigma,v}$ is stable means that $2g_v + m_v \geqslant 3$.

There is a family of elements of a neighborhood of $\Sigma_{\sigma,v}$ in $\mathcal{M}_{g_v,m_v}$ parameterized by the neighborhood of 0 in $\mathbf{C}^{3g_v-3+m_v}/Aut(\Sigma_v, z_v)$.

We consider the product

$$\underset{\Sigma_{v'}\text{ is stable}}{}\mathbf{C}^{3g_v-3+m_v}.$$

Let ${}_{\text{deform},\sigma}$ be a neighborhood of 0 of this space. We have a fiberwise complex structure on ${}_{\text{deform},\sigma} \times \Sigma$ which was induced by the universal family. We take the representative as follows. We consider the direct product ${}_{\text{deform},\sigma} \times \Sigma_\sigma$ and change the complex structure in a compact set $K_{\text{deform}}(\sigma) \subseteq \Sigma_\sigma$ away from the singular or marked points so that it gives a universal family. Again by unique continuation, we may assume that $K_{\text{deform}}(\sigma)$ is away from singular or marked points. We also take a family of Kähler metrics which is constant outside a compact set $K_{\text{deform}}(\sigma)$. We may also assume that ${}_{\text{deform},\sigma} \times \Sigma$ together with fiberwise complex structure and Kähler metric is equivariant by the diagonal action of $Aut(\sigma)$. (We remark that the group $Aut(\sigma)$ is contained in the group $Aut(\Sigma_\sigma)$. $Aut(\Sigma_\sigma)$ contains the direct product of $Aut(\Sigma_{\sigma,v}, z_{\sigma,v})$ but may be strictly larger than that, since there may be an automorphism interchanging the components.)

We next consider the family of vector spaces

$$\bigoplus_{x:\text{singular}} T_{x_v}\Sigma_{\sigma,v} \otimes T_{x_w}\Sigma_{\sigma,w}.$$

By exactly the same way as Section 9, we construct a family of semistable curves parametrized by a neighborhood of 0 in $\bigoplus T_{x_v}\Sigma_{\sigma,v} \otimes T_{x_w}\Sigma_{\sigma,w}$.

Let ${}_{\text{resolve},\sigma}$ be this neighborhood. Now we have a family of semistable curves together with fiberwise Kähler metric parametrized by ${}_{\text{deform},\sigma} \times {}_{\text{resolve},\sigma}$.

We remark that the metric constructed here is a direct generalization of the one in [47, Appendix A]. There is one change, which is not essential. That is we perturb the metric so that it is smooth while the metric McDuff–Salamon used to define the weighted Sobolev norm in [47, Section A.4], is singular on one circle. However this difference is not essential at all since the derivative of the metric is never used in [47].

Let $K_{\text{neck}}(\sigma)$ be the small compact neighborhood of the neck (the part we glued the spaces.) More explicitly we put

$$K_{\text{neck}}(\sigma) = \bigcup_x D_{x_v}(R_x^{-1/2}) \cup D_{x_w}(R_x^{-1/2}).$$

Here $x$ runs over the set of all singular points and $R_x = |\alpha_x|^{-1/2}$.

By construction, our semistable curve is independent of the parameter in $_{\text{deform},\sigma}\times {}_{\text{resolve},\sigma}$ outside $K_{\text{deform}}(\sigma)\cup K_{\text{neck}}(\sigma)$.

We next consider the action of automorphisms. The group $Aut(\Sigma_\sigma, z_\sigma)$ may be of positive dimension because of the presence of unstable components. However we need to require the construction equivariant only by $Aut(\sigma)$ (the set of elements $\gamma\in Aut(\Sigma_\sigma, z_\sigma)$ such that $h_\sigma\circ\gamma = h_0$), which is a finite group.

We next deal with the connected component $Aut(\Sigma_\sigma, z_\sigma)$, that is $_{v:\text{unstable}} Aut(\Sigma_{\sigma,v}, z_{\sigma,v})$. The Lie algebra of $Aut(\Sigma_{\sigma,v}, z_{\sigma,v})$ is identified to the set of all holomorphic vector fields on $\Sigma_{\sigma,v}$ which are zero at $z_{\sigma,v}$ and at singular points. Any such vector field induces an element of the kernel of $D_{h_\sigma}\bar\partial_{\Sigma_\sigma}: L^p_1(\Sigma_\sigma; h_\sigma^* TM)\to L^p(\Sigma_\sigma; h_\sigma^* TM\otimes\Lambda^{0,1}(\Sigma_\sigma))$. Hence we embed $\text{Lie}(Aut(\Sigma_\sigma, z_\sigma))\subseteq {}_{\text{map},\sigma}$. We use our metric to take its $L^2$ orthogonal component. Let $'_{\text{map},\sigma}$ be a small neighborhood of zero of it. Now let us put

$$^{+}_{\sigma} = {}_{\text{deform},\sigma}\times {}_{\text{resolve},\sigma}\times {}_{\text{map},\sigma}$$

$$'_{\sigma} = {}_{\text{deform},\sigma}\times {}_{\text{resolve},\sigma}\times {}'_{\text{map},\sigma}.$$

We remark that there is an action of $Aut(\sigma)$ on $'_\sigma$, $^+_\sigma$ and $E_\sigma$.

THEOREM 12.9. *There is an $Aut(\sigma)$-equivariant map $s_\sigma$: $^+_\sigma\to E_\sigma$ which is 0 at origin and a continuous map $\psi^+_\sigma: s_\sigma^{-1}(0)\to C\mathscr{M}_{g,m}(M, J, \beta)$. The restriction of $\psi^+_\sigma$ to $'_\sigma\cap s_\sigma^{-1}(0)/Aut(\sigma)$ gives a homeomorphism onto a neighborhood of $\sigma$.*

Let $\psi'_\sigma$ denote the restriction of $\psi^+_\sigma$ to $'_\sigma\cap s_\sigma^{-1}(0)/Aut(\sigma)$.

The proof of Theorem 12.9 (which occupies Sections 13 and 14) goes in a similar line as other gluing procedures. First we construct an approximate solution and deform it to an actual solution.

The construction of the approximate solution is the same as the one by McDuff–Salamon in [47]. We take $\zeta = ((\xi_v), (\alpha_x))\in {}_{\text{deform},\sigma}\times {}_{\text{resolve},\sigma}$. Here $(\xi_v)$ is the parameter to deform the complex structure of $\Sigma_\sigma$ in $K_{\text{deform}}(\sigma)$. We glue the component around each of the singular points and the way to glue the components is parametrized by $(\alpha_x)$. (We leave the singular point to be singular if $\alpha_x = 0$.) Let $\Sigma_\zeta$ be the semistable curve obtained in this way.

Let $u\in {}_{\text{map},\sigma}$. We define maps $h_{1,v,u}:\Sigma_{\sigma,v}\to M$ by

$$h_{1,v,u}(p) = \exp_{h(p)}(u_v(p)). \tag{12.10}$$

Since we assumed that $u_v(x_v) = u_w(x_w)$ it follows that we can glue to obtain a map $h_{1,u}$ from $\Sigma_\sigma$. We modify it and construct a map $h_{\text{approx},\zeta,u}:\Sigma_\zeta\to M$. Let $R_x = |\alpha_x|^{-1/2}$. By the description in Section 9, the semistable curve $\Sigma_\zeta$ is obtained from the disjoint union

$$\bigcup_v\left(\overline{\Sigma_{\sigma,v} -\phantom{x}}\ \overline{D_{x_v}(R_x^{-1})}\right)$$

by identifying a circle $\partial D_{x_v}(R_x^{-1})$ with another one (and deforming the complex structure on $_{\text{deform},\sigma}$). We choose a sufficiently small $\delta$ and fix it throughout. We assume $1/\delta R_x < 1/R_x^{1/2}$. We do not modify the map $h_{1,v,u}$ outside the union of the balls $D_{x_v}(2R_x^{-1}\delta^{-1})$. Namely we put

$$h_{\text{approx},\zeta,u}(p) = h_{1,v,u}(p) \tag{12.11}$$

if $p \in \Sigma_{\sigma,v} - \bigcup D_{x_v}(2R_x^{-1}\delta^{-1})$. Here $D_{x_v}(2R_x^{-1}\delta^{-1})$ is the metric ball in $\Sigma_{\sigma,v}$ centered at $x_v$ and of radius $2R_x^{-1}\delta^{-1}$. To glue them we use a partition of unity on the domain

$$D(R_x, x, v) = \bigcup D_{x_v}(2R_x^{-1}\delta^{-1}) - D_{x_v}(2R_x^{-1}). \tag{12.12}$$

as follows. We put $e_x = \exp_{h(x)} u(x_v) = \exp_{h(x)} u(x_w)$. Then the image of the set (12.10) by $h_{1,u}$ is contained in a small neighborhood of $e_x$.

Let $\chi : [0, \infty) \to [0, 1]$ be a cut off function such that

$$\chi(r) = \begin{cases} 1 & \text{if } r \geqslant 2 \\ 0 & \text{if } r \leqslant 1. \end{cases}$$

We now put

$$h_{\text{approx},\zeta,u}(\exp_x \mathbf{v}) = \begin{cases} \exp_{e_x}(\chi(R_x\delta|\mathbf{v}|)\exp_{e_x}^{-1}(h_{1,u}(\exp_x \mathbf{v}))) & \mathbf{v} \in D(2R_x^{-1}\delta^{-1}) - D(R_x^{-1}\delta^{-1}) \\ e_x & \mathbf{v} \in D(R_x^{-1}\delta^{-1}) - \operatorname{Int} D(R_x^{-1}). \end{cases} \tag{12.13}$$

Clearly they are glued to define a map $h_{\text{approx},\zeta,u} : \Sigma_\zeta \to M$. We next estimate $\| \bar\partial_{\Sigma_\zeta} h_{\text{approx},\zeta,u} \|_{L^p(\Sigma_\zeta)}$. (Here we write $\bar\partial_{\Sigma_\zeta}$ to make clear that we are using the complex structure perturbed by $(\xi_v)$ and $(\alpha_x)$.)

We remark that we have

$$|\bar\partial_{\Sigma_\zeta} h_{\text{approx},\zeta,u}(p)| \leqslant C|u(p)||\nabla u(p)|$$

for $p \notin \bigcup_{x,v}(D_{x_v}(2R_x^{-1}\delta^{-1}) - D_{x_v}(R_x^{-1}\delta^{-1})) \cup K_{\text{deform}}(\sigma)$. (We recall that $K_{\text{deform}}(\sigma)$ is the domain where we perturb the complex structure.) We have also

Lemma 12.14. $\|\bar\partial_{\Sigma_\zeta} h_{\text{approx},\zeta,u}\|_{L^p(K_{\text{deform}}(\sigma))} \leqslant C(\|(\xi_v)\| + \|u\|_{L^\infty})\|u\|_{L^p}$.

This lemma is obvious from definition. On the other hand, we have

Lemma 12.15. $\|\bar\partial_{\Sigma_\zeta} h_{\text{approx},\zeta,u}\|_{L^p(D_{x_v}(2R_x^{-1}\delta^{-1}) - D_{x_v}(R_x^{-1}\delta^{-1}))} \leqslant C(R_x\delta)^{-2/p}$.

*Proof.* The proof is the same as the proof of Lemma A4.3 of [47].

Namely the restriction of $\bar\partial_{\Sigma_\zeta} h_{\text{approx},\zeta,u}$ to $D(2R_x^{-1}\delta^{-1}) - D(R_x^{-1}\delta^{-1})$ is a sum of two terms. One involves the differential of the cut off function, the other involves the differential of $h_{1,\zeta,u}$.

The first term is estimated by $R_x\delta$ times $\|\mathbf{v}\|$ which is bounded pointwise.

The second term is estimated by the first derivative of $h_{\text{approx},\zeta,u}$ times some constant, and hence is bounded. (We remark that $\|\bar\partial h_{\text{approx},\zeta,u}\|_{L^\infty(\Sigma_\zeta)}$ is bounded.)

Since the volume of the domain $D(2R_x^{-1}\delta^{-1}) - D(R_x^{-1}\delta^{-1})$ is constant times $(R_v\delta)^{-2}$, the $L^p$ norm is estimated as asserted. □

Thus we have constructed a family of approximate solutions. To go further we need to use a right inverse to (12.1). However (12.1) has a cokernel. This is the point we need to introduce the obstruction bundle. We recall that

$$\Pi_{E_\sigma} \circ (D_{h_\sigma}\bar\partial_{\Sigma_\sigma}) : L_1^p(\Sigma_\sigma; h_\sigma^* TM) \to \frac{L^p(\Sigma_\sigma; h_\sigma^* TM \otimes \Lambda^{0,1}(\Sigma_\sigma))}{E_\sigma}$$

In place of studying the equation $\bar\partial_{\Sigma_\zeta} h = 0$, we study the following equation:

$$\text{“}\bar\partial_{\Sigma_\zeta} h \equiv 0 \mod E_\sigma.\text{”} \tag{12.17}$$

Let us make eq. (12.17) more precise. (12.17) is an equation for a pair $(\zeta, h)$ such that $\zeta = ((\xi_v)), (\alpha_x)) \in \ _{\text{deform},\sigma} \times \ _{\text{resolve},\sigma}$ and $h : \Sigma_\zeta \to M$. We remark that $K_{\text{obstru}}(\sigma)$ is away from $K_{\text{neck}}(\sigma)$, hence we may regard $K_{\text{obstru}}(\sigma) \subset \Sigma_\zeta$. We consider only $h : \Sigma_\zeta \to M$ such that

$$dist(h(p), h_\sigma(p)) \leqslant \min\left\{\frac{injrad(M)}{100}, d\right\} \tag{12.18}$$

for each $p \in K_{\text{obstr}}(\sigma)$. Here $injrad(M)$ is the injectivity radius of $M$ and $d$ is a small number depending only on $M$ and will be specified below.

Let $x, y \in M$ with $dist(x, y) < d$. We consider the parallel transform $Par_{x,y} : T_x(M) \to T_y(M)$ along the minimal geodesic joining $x$ and $y$.

We put

$$Par_{x,y} = P'_{x,y} + P''_{x,y} \tag{12.19}$$

where $P'_{x,y}$ is complex linear and $P''_{x,y}$ is anti linear. Taking $d$ enough small we may assume that $P'_{x,y}$ induces an isomorphism of the bundles if $dist(x, y) < d$. Hence by (12.18), we have an isomorphism

$$C^\infty(K_{\text{obstr}}(\sigma); h_\sigma^* TM \otimes \Lambda^{0,1}(\Sigma_\sigma)) \to C^\infty(K_{\text{obstr}}(\sigma); h^* TM \otimes \Lambda^{0,1}(\Sigma_\zeta)). \tag{12.20}$$

Here we use also the projection

$$\Lambda^{0,1}(\Sigma_\sigma)|_{K_{\text{obstr}}(\sigma)} \subseteq \Lambda^1(\Sigma_\sigma)|_{K_{\text{obstr}}(\sigma)} \cong \Lambda^1(\Sigma_\zeta)|_{K_{\text{obstr}}(\sigma)} \to \Lambda^{0,1}(\Sigma_\zeta)|_{K_{\text{obstr}}(\sigma)}$$

which can be assumed to be arbitrary close to identity by taking $\ _{\text{deform},\sigma}$ small.

We use isomorphism (12.20) together with the fact that the support of each element of $E_\sigma$ is in $K_{\text{obstr}}(\sigma)$, to regard $E_\sigma$ also as a subspace of $C^\infty(\Sigma_\zeta; h_\zeta^*(M) \otimes \Lambda^{0,1}(\Sigma_\zeta))$. Hence (12.17) makes sense.

Now we consider the approximate solution $h_{\text{approx},\zeta,u}$ and find that

$$\|\bar\partial_{\Sigma_\zeta} h_{\text{approx},\zeta,u}\|_{L^p(\Sigma_\zeta - (K_{\text{obstr}} \cup K_{\text{deform}} \cup K_{\text{neck}}))} \leqslant C\|u\|_{L^\infty}\|u\|_{L^p_1} \tag{12.21}$$

$$\|\bar\partial_{\Sigma_\zeta} h_{\text{approx},\zeta,u} - \bar\partial_{\Sigma_\sigma} u\|_{L^p(K_{\text{obstr}})} \leqslant C(\|\xi\| + \|u\|_{L^\infty})\|u\|_{L^p_1}. \tag{12.22}$$

Here we identify $\bar\partial_{\Sigma_\sigma} u \in E_\sigma$ to an element of $C^\infty(K_0; h^*(M) \otimes \Lambda^{0,1}(\Sigma_\zeta))$ by (12.20).

We prove the following proposition in the next section.

PROPOSITION 12.23. *Replacing* $\ ^+_\sigma$ *by smaller one if necessary, there exist a continuous map* $s_\sigma : \ ^+_\sigma \to E_\sigma$, *and a continuous family of smooth maps* $h_{\zeta,u} : \Sigma_\zeta \to M$ *such that*

(12.23.1) $\bar\partial_{\Sigma_\zeta} h_{\zeta,u} = s_\sigma(\zeta, u)$ *holds for every* $\zeta$.

(12.23.2) $s_\sigma : \ _\sigma \to E_\sigma$ *is* $Aut(\sigma)$ *equivariant.*

(12.23.3) *The map* $(\zeta, u) \mapsto h_{\zeta,u}$ *is* $Aut(\sigma)$ *equivariant in the following sense. We remark that we have constructed already a biholomorphic map* $\varphi_\gamma : \Sigma_\zeta \to \Sigma_{\gamma\zeta}$ *for each* $\gamma \in Aut(\sigma)$. *Then we have* $h_{\gamma(\zeta,u)} \circ \varphi_\gamma = h_{\zeta,u}$.

(12.23.4) $s_\sigma(0) = 0$.

We remark that the space $s_\sigma^{-1}(0)/Aut(\sigma)$ is mapped to $C\mathcal{M}_{g,m}(M, J, \beta)$, since if $s_\sigma(\zeta, u) = 0$ then $h_{\zeta,u}$ is pseudoholomorphic by (12.23.1). We denote that map by $\psi_\sigma^+$. Using implicit function theorem we can prove the following lemma. (The proof is in Section 13.)

LEMMA 12.24.

$$\psi'_\sigma : \frac{{}'_\sigma \cap s_\sigma^{-1}(0)}{Aut(\sigma)} \to C\mathcal{M}_{g,m}(M, J, \beta) \text{ is injective.}$$

In Section 14, we prove the following:

PROPOSITION 12.25. *$\psi_\sigma^+(s_\sigma^{-1}(0)/Aut(\sigma))$ contains a neighborhood of $\sigma$ in $C\mathcal{M}_{g,m}(M, J, \beta)$.*

We remark that

$$\psi'_\sigma\left(\frac{{}'_\sigma \cap s_\sigma^{-1}(0)}{Aut(\sigma)}\right) = \psi_\sigma^+\left(\frac{s_\sigma^{-1}(0)}{Aut(\sigma)}\right)$$

by the definition of $'_\sigma$.

Theorem 12.7 follows from Propositions 12.23 and 12.25 and Lemma 12.24.

We thus obtain a chart of Kuranishi structure around each point of $C\mathcal{M}_{g,m}(M, J, \beta)$. (Namely $'_\sigma$, $E_\sigma$ and $s_\sigma$). In section 15, we glue them to obtain a Kuranishi structure globally.

We close this section remarking that the virtual dimension of our chart is constant and is equal to $2m + 2\beta c_1 + 2(3 - n)(g - 1)$. This is necessary to verify Condition (5.3.4) of Kuranishi struture. This fact follows from Lemma 12.2 as follows:

We remark that by adding a marked point the virtual dimension increase by 2. In fact if we add a marked point on stable component then dim $_{\text{deform},\sigma}$ increase by 2 and if add a marked point to unstable component then dim $'_{\text{map},\sigma}$ increase by 2. It follows that we are only to consider the case when $\Sigma_\sigma$ is stable. Let us calculate the virtual dimension in that case.

To calculate the virtual dimension of the Kuranishi structure we need to add the index $2\beta c_1 + 2n(1 - g)$ and the dimension of the deformation of complex structure and the parameter to resolve the singularity. The latter is twice of the complex dimension of the space $C\mathcal{M}_{g,m}$ which is $m + 3g - 3$. Hence we have the dimension $2m + 2\beta c_1 + 2(3 - n)(g - 1)$. If one wants to do it more directly, the calculation of the dimension around each point can be done by counting the dimension of the deformation of each component (i.e. the dimension of $_{\text{deform},\sigma}$) and adding the dimension of the parameter resolving the singularity (i.e. the dimension of $_{\text{resolve},\sigma}$). We leave the reader to work out this elementary and standard calculation. (We are going to do a similar calculation at the end of Section 19.)

## 13. CONSTRUCTION OF LOCAL CHART II—RIGHT INVERSE TO LINEARIZED EQUATION AND CONSTRUCTION OF EXACT SOLUTION

We use the same notation as in Section 12 and are going to prove Proposition 12.23. The proof is again a copy of McDuff–Salamon's in [47] with some minor modifications to handle the existence of the obstruction and moduli parameter. We will omit some of the details of the part where we can prove in exactly the same way as in [47].

First we recall that an element $u \in {}_{\mathrm{map},\sigma}$ satisfies $(D_{h_\sigma}\bar\partial_{\Sigma_\sigma})u = v$ for some $v \in E_\sigma$. Therefore ${}_{\mathrm{map},\sigma}$ consists of smooth sections and is of finite dimension. Hence the $L^2$ orthogonal projection

$$\Pi_{V^\perp_{1,\sigma}} : s \mapsto s - \sum_i \langle s, e_i \rangle_{L^2} e_i$$

is well defined on $L^p_1(\Sigma_\sigma; h^*_\sigma TM)$. Here $e_i$ is an $(L^2)$-orthonormal basis of ${}_{\mathrm{map},\sigma}$. We let ${}^{\perp}_{\mathrm{map},\sigma}$ be its image. We have

$$L^p_1(\Sigma; h^*_\sigma TM) = {}_{\mathrm{map},\sigma} \oplus {}^{\perp}_{\mathrm{map},\sigma}.$$

It is also easy to see that

$$\Pi_{E_\sigma} \circ (D_{h_\sigma}\bar\partial_{\Sigma_\sigma}) : {}^{\perp}_{\mathrm{map},\sigma} \to \frac{L^p(\Sigma_\sigma; h^*_\sigma TM \otimes \Lambda^{0,1}(\Sigma_\sigma))}{E_\sigma}$$

is an isomorphism. (We recall that $E_\sigma$ is of finite dimension. Hence the quotient space,

$$\frac{L^p(\Sigma_\sigma; h^*_\sigma TM \otimes \Lambda^{0,1}(\Sigma_\sigma))}{E_\sigma}$$

is well defined as a Banach space.) Therefore its inverse

$$Q_\sigma : \frac{L^p(\Sigma_\sigma; h^*_\sigma TM \otimes \Lambda^{0,1}(\Sigma_\sigma))}{E_\sigma} \to {}^{\perp}_{\mathrm{map},\sigma}$$

is bounded by open mapping theorem.

We next recall that we have a map

$$I : \bigcup_v \bigcup_{x \in sing(\Sigma_{\sigma,v})} (\Sigma_{\sigma,v} - D_{X_v}(R^{-1}_{\sigma,a_x}))) \to \Sigma_\zeta \tag{13.1}$$

which is a diffeomorphism outside the boundary (union of circles). On the boundary the map is 2:1.

We recall that by our choice of $h_{\mathrm{approx},\zeta,u}$ we have, for each

$$p \in \bigcup_v \bigcup_{x \in sing(\Sigma_{\sigma,v})} (\Sigma_{\sigma,v} - D_{x_v}(R^{-1}_{\sigma,a_x}))),$$

the inequality

$$\sup dist(h_\sigma(p), h_{\mathrm{approx},\zeta,u}(Ip)) \leqslant \min\left\{\frac{injrad(M)}{100}, d\right\}. \tag{13.2}$$

We use (13.2) and obtain

$$Par_{h_\sigma(p)h_{\mathrm{approx},\zeta\,u}(Ip)} : I_{h_\sigma(p)}M \to T_{h_{\mathrm{approx},\zeta\,u}(Ip)}M. \tag{13.3}$$

Namely $Par_{h_\sigma(p)h_{\mathrm{approx},\zeta\,u}(Ip)}$ is the parallel transport along the unique geodesic in $M$ joining $h_\sigma(p)$ and $h_{\mathrm{approx},\zeta,u}(Ip)$. We write it as the sum of complex linear and antilinear parts as in (12.19) and let $P'_{h_\sigma(p)h_{\mathrm{approx},\zeta\,u}(Ip)}$ be the complex linear part. By taking $d$ small we may assume that $P'_{h_\sigma(p)h_{\mathrm{approx},\zeta\,u}(Ip)}$ is an isomorphism.

Also we have a bundle isomorphism

$$Iso_\zeta : \Lambda^{0,1}\left(\bigcup_v \bigcup_{x \in sing(\Sigma_{\sigma,v})} (\Sigma_{\sigma,v} - D_{x_v}(R^{-1}_{\sigma,a_x})))\right) \to \Lambda^{0,1}(\Sigma_\zeta), \tag{13.4}$$

which cover (13.1), which is identity outside $K_{\mathrm{deform}}(\sigma) \cup K_{\mathrm{neck}}(\sigma)$ and which satisfies

$$|Iso_\zeta - id|_{C^k} < C_k \|(\xi_v)\|. \tag{13.5}$$

on $K_{\mathrm{deform}}(\sigma)$. (We recall $(\xi_v)$ is the parameter to deform the complex structure of our semistable curve and the deformation is supported on $K_{\mathrm{neck}}(\sigma)$.) Here $D_{x_v}(R_{\sigma,a_x}^{-1})$ is a metric ball centered at $x_v$ and of radius $R_{\sigma,a_x}^{-1}$.)

Using (13.3) and (13.4), we obtain a map

$$I_{\zeta,u}: L^p(\Sigma_\zeta; h^*_{\mathrm{approx},\zeta,u} TM \otimes \Lambda^{0,1}(\Sigma_\zeta)) \to \bigoplus_v L^p(\Sigma_{\sigma,v} - D_{x_v}(R_{\sigma,a_x}^{-1}); h_\sigma^* TM \otimes \Lambda^{0,1}(\Sigma_{\sigma,v})).$$

We next use a map

$$\bigoplus_v L^p(\Sigma_v - D_{x_v}(R_{\sigma,a_x}^{-1})); h_\sigma^* TM \otimes \Lambda^{0,1}(\Sigma)) \to L^p(\Sigma; h_\sigma^* TM \otimes \Lambda^{0,1}(\Sigma))$$

by extending sections as 0. (An element of the image of this map is discontinuous. But we do not have to worry about it since we are working with $L^p$ spaces.) Let

$$I_{\zeta,u}: L^p(\Sigma; h^*_{\mathrm{approx},\zeta,u} TM \otimes \Lambda^{0,1}(\Sigma)) \to L^p(\Sigma; h_\sigma^* TM \otimes \Lambda^{0,1}(\Sigma))$$

be the composition.

We next use McDuff–Salamon's, Lemma A.1.1 in [47]. We put $\delta = e^{-2\pi/\varepsilon}$. It implies the existence of a function $\beta: [0, \infty) \to [0, 1]$ such that

$$\beta(r) = \begin{cases} 1 & \text{if } r < \delta \\ 0 & \text{if } r \geqslant 1 - o \end{cases} \tag{13.6}$$

for some $o$ and

$$\int_{|z|<1} |\nabla \beta(|z|)|^2 \leqslant 2\varepsilon. \tag{13.7}$$

We use it to define

$$Glue_{\zeta,u}: L_1^p(\Sigma_\sigma; h_\sigma^* TM) \to L_1^p(\Sigma_\zeta; h^*_{\mathrm{approx},\zeta,u} TM) \tag{13.8}$$

as follows. We first put

$$\begin{aligned} Glue_{\zeta,u}(s)(p) = P'_{h_\sigma(p) h_{\mathrm{approx},\zeta\, u}(Ip)}(s(p)) \quad &\text{if} \\ p \in \bigcup_v \bigcup_{x \in sing(\Sigma_{\sigma,v})} (\Sigma_{\sigma,v} - D_{x_v}(R_{\sigma,\alpha_x}^{-1/2}))) & \\ \text{or} & \\ p = \exp_{x_v}(\mathbf{v}),\ R_{x,\alpha_x}^{-1}\delta^{-1} \leqslant \|\mathbf{v}\| \leqslant R_{x,\alpha_x}^{-1/2}. & \end{aligned} \tag{13.9}$$

For $p = \exp_{x_v}(\mathbf{v})$ with $R_{x,\alpha_x}^{-1} \leqslant \|\mathbf{v}\| \leqslant R_{x,\alpha_x}^{-1}\delta^{-1}$, we proceed as follows. Let $x = x_v = x_w$, namely let $\Sigma_{\sigma,v}$, $\Sigma_{\sigma,w}$ be the two components of $\Sigma_\sigma$ containing the singular point $x$. We recall that, to construt $\Sigma_\zeta$, we identified $\mathbf{v}$ and $\mathbf{w} = \alpha_x/\mathbf{v}$. Let $p_v = \exp_{x_v}(\mathbf{v})$ and $p_w = \exp_{x_w}(\mathbf{w})$.

We then put

$$\begin{aligned} Glue_{\zeta,u}(s)(p_v) &= P'_{h_\sigma(p_v) h_{\mathrm{approx},\zeta\, u}(p_v)} s(p_v) \\ &\quad + (1 - \beta(1/R_{x,\alpha_x}|\mathbf{v}|))(P'_{h_\sigma(p_w) h_{\mathrm{approx},\zeta\, u}(p_v)} s(p_w) - P'_{h_\sigma(p) h_{\mathrm{approx},\zeta\, u}(p_v)} s(x)) \end{aligned} \tag{13.10}$$

if $p_v = \exp_{x_v}(\mathbf{v})$ with $R_{x,\alpha_x}^{-1} \leqslant \|\mathbf{v}\| \leqslant R_{x,\alpha_x}^{-1}\delta^{-1}$. In a neighborhooid of $\delta^{-1}R_{x,\alpha_x}^{-1} = \|\mathbf{v}\|$, the right-hand side of (13.10) is $P'_{h_\sigma(p_v)h_{2,\zeta,u}(p_v)}s(p_v)$. Hence it coincides with (13.9). In a neighborhood of $R_{x,\alpha_x}^{-1} = \|\mathbf{v}\|$, the right-hand side of (13.10) is

$$P'_{h_\sigma(p_v)h_{\mathrm{approx},\zeta\,u}(p_v)}s(p_v) + P'_{h_\sigma(p_w)h_{\mathrm{approx},\zeta\,u}(p_v)}s(p_w) - P'_{h_\sigma(x)h_{\mathrm{approx},\zeta\,u}(p_v)}s(x)$$

and is invariant by $v \leftrightarrow w$, since $h_{\mathrm{approx},\zeta,u}(p_v) = h_{\mathrm{approx},\zeta,u}(p_w)$ if $R_{x,\alpha_x}^{-1} = \|\mathbf{v}\|$. Hence $Glue_{\zeta,u}(s)$ is continuous there.

Now we define

$$Q'_{\zeta,u}\colon \frac{L^p(\Sigma_\zeta; h^*_{\mathrm{approx},\zeta,u}TM \otimes \Lambda^{0,1}(\Sigma_\zeta))}{E_\sigma} \to L^p_1(\Sigma_\zeta; h^*_{\mathrm{approx},\zeta,u}TM)$$

by

$$Q'_{\zeta,u} = Glue_{\zeta,u} \circ Q_\sigma \circ I_{\zeta,u}.$$

Here we regard $Q_\sigma$ as a map

$$Q_\sigma\colon L^p(\Sigma_\sigma; h^*_\sigma TM \otimes \Lambda^{0,1}(\Sigma_\sigma)) \to L^p_1(\Sigma_\sigma; h^*_\sigma TM)$$

which is 0 on $\Sigma_\sigma$. By the choice of the way we embed $E_\sigma$ in $L^p(\Sigma_\zeta; h^*_{\mathrm{approx},\zeta,u}TM \otimes \Lambda^{0,1}(\Sigma_\zeta))$, we find that $I_{\zeta,u}(E_\sigma) \subseteq E_\sigma$. It follows that $Glue_{\zeta,u} \circ Q_\sigma \circ I_{\zeta,u}$ is 0 on $E_\sigma$ and defines a map from the quotient space.

Lemma 13.11. *$\|\Pi_{E_\sigma} \circ (D_{h_{\mathrm{approx},\zeta\,u}}\bar\partial_{\Sigma_\zeta}) \circ Q'_{\mathrm{approx},\zeta,u}(s) - s\|_{L^p/E_p} \leqslant \frac{1}{2}\|s\|_{L^p}$, if $\delta$ and $\|\zeta\|$ are sufficiently small.*

*Proof.* The argument we need to control the error term coming from gluing is exactly the same as the proof of Lemma A.4.2 of [47]. We can control the term coming from deformation of complex structure since it is parameterized by $\zeta$. □

Using Lemma 13.11, it is an exercise of functional analysis to find

$$Q_{\zeta,u}\colon \frac{L^p(\Sigma_\zeta; h^*_{\mathrm{approx},\zeta,u}TM \otimes \Lambda^{0,1}(\Sigma_\zeta))}{E_\sigma} \to L^p_1(\Sigma_\zeta; h^*_{\mathrm{approx},\zeta,u}TM)$$

such that

$$(\Pi_{E_\sigma} \circ D_{h_{2,\zeta,u}}\bar\partial_{\Sigma_\zeta}) \circ Q_{\zeta,u}(s) = s. \tag{13.12}$$

Lemma 13.13.

$$Q_{\zeta,u}\colon \frac{L^p(\Sigma_\zeta; h^*_{\mathrm{approx},\zeta,u}TM \otimes \Lambda^{0,1}(\Sigma_\zeta))}{E_\sigma} \to L^p_1(\Sigma_\zeta; h^*_{\mathrm{approx},\zeta,u}TM)$$

*is bounded uniformly of $\zeta, u$.*

The proof is the same as Lemma A.4.2 of [47].

Now we can use Newton's method to construct exact solutions of eq. (12.17) parametrized by $\ {}^{+}_{\sigma}$ as follows.

We first put

$$h^2_{\mathrm{approx},\zeta,u}(p) = \exp_{h_{\mathrm{approx},\zeta\,u}(p)}(-(Q_{\zeta,u}(\Pi_{E_\sigma}\bar\partial h_{\mathrm{approx},\zeta,u}))(p)).$$

By Lemmata 12.14 and 12.15, Formulae (12.21), (12.22), (13.12) and Lemma 13.13, we find

$$d_{L_1^p}(h^2_{\text{approx},\zeta,u}, h_{\text{approx},\zeta,u}) \leqslant C(\delta^{-2/p}|\alpha|^{4/p} + (\|\xi\| + \|u\|_{L^\infty})\|u\|_{L_1^p}), \tag{13.14}$$

and

$$\begin{aligned}\|\Pi_{E_\sigma}\bar\partial h^2_{\text{approx},\zeta,u}\|_{L^p/E_\sigma} &\leqslant C\|Q_{\zeta,u}(\Pi_{E_\sigma}\bar\partial h_{\text{approx},\zeta,u})\|_{L^\infty}\|\Pi_{E_\sigma}\bar\partial h_{\text{approx},\zeta,u}\|_{L^p/E_\sigma}\\ &\leqslant C(\delta^{-2/p}|\alpha|^{4/p} + (\|\xi\| + \|u\|_{L^\infty})\|u\|_{L_1^p})\|\Pi_{E_\sigma}\bar\partial h_{\text{approx},\zeta,u}\|_{L^p/E_\sigma}. \end{aligned}\tag{13.15}$$

Here $\Pi_{E_\sigma}$ is a projection to the quotient space by $E_\sigma$. Using (13.14) and (13.15), we can repeat the same procedure if $u$, $\delta$, $(\alpha_x)$ and $\xi$ are sufficiently small and obtain $h^3_{\text{approx},\zeta,u}$ as follows:

$$h^3_{\text{approx},\zeta,u}(p) = \exp_{h^2_{\text{approx},\zeta,u}(p)}(-(Q_{\zeta,u}(\Pi_{E_\sigma}\bar\partial h^2_{\text{approx},\zeta,u}))(p)).$$

We have

$$\begin{aligned}\|\Pi_{E_\sigma}\bar\partial h^3_{\text{approx},\zeta,u}\|_{L^p/E_\sigma} &\leqslant C\|Q_{\zeta,u}(\Pi_{E_\sigma}\bar\partial h^2_{\text{approx},\zeta,u})\|_{L^\infty}\|\Pi_{E_\sigma}\bar\partial h^2_{\text{approx},\zeta,u}\|_{L^p/E_\sigma}.\\ &\leqslant C\|\Pi_{E_\sigma}\bar\partial h^2_{\text{approx},\zeta,u}\|^2_{L^p/E_\sigma}.\end{aligned}$$

We define $h^i_{\text{approx},\zeta,u}$, $i = 4, \ldots,$ in a similar way. We then have

$$h_{\text{exact},\zeta,u} = \lim_{m\to\infty} h^m_{\text{approx},\zeta,u}$$

such that

$$\bar\partial h_{\text{exact},\zeta,u} \equiv 0 \mod E_\sigma.$$

Thus we obtain solutions of (12.17). It is immediate from construction that it satisfies Conditions (12.22.1)–(12.22.4).

*Remark* 13.16. It might be possible to show that $s_\sigma$ is smooth. However we do not need it since we can use Lemma 3.12 instead. (Roughly speaking, continuous section is enough for our purpose since Euler class is an invariant of the $C^0$-structure of the bundle.) It seems cumbersome to prove smoothness at the point where $\alpha_x = 0$ for some $x$.

We next prove Lemma 12.24. First we recall that $Aut(\Sigma_\sigma, z_\sigma)$ is the group of automorphisms of the semistable curve fixing marked points, and $Aut(\sigma)$ is a subgroup of $Aut(\Sigma_\sigma, z_\sigma)$ consisting of elements $\vartheta$ such that $h_\sigma \circ \vartheta = h_\sigma$. Let $LieAut(\Sigma_\sigma, z_\sigma)_0$ be the neighborhood of identity of the Lie algebra of $Aut(\Sigma_\sigma, z_\sigma)$. We will construct an "action" of $LieAut(\Sigma_\sigma, z_\sigma)_0$ on ${}_{\text{deform},\sigma} \times {}_{\text{resolve},\sigma}$. (It is not an action in the usual sense. Namely if we identify $LieAut(\Sigma_\sigma, z_\sigma)_0$ with the neighborhood of the identify in $Aut(\Sigma_\sigma, z_\sigma)$, then $(\gamma_1\gamma_2)(x) \neq \gamma_1(\gamma_2(x))$. This is the reason we write $LieAut(\Sigma_\sigma, z_\sigma)_0$ in place of $Aut(\Sigma_\sigma, z_\sigma)_0$. We mention this point again later.) The finite group $Aut(\sigma)$ acts on $LieAut(\Sigma_\sigma, z_\sigma)_0$.

We next consider universal family $\pi: Uni \to {}_{\text{deform},\sigma} \times {}_{\text{resolve},\sigma}$. Namely for each $\zeta \in {}_{\text{deform},\sigma} \times {}_{\text{resolve},\sigma}$ the fibre $\pi^{-1}(\zeta)$ is identified with $\Sigma_\zeta$. We remark that $Aut(\sigma)$ acts on $Uni$ and ${}_{\text{deform},\sigma} \times {}_{\text{resolve},\sigma}$ and $\pi$ is $Aut(\sigma)$-equivariant.

We also remark that $Uni$ is a smooth manifold together with fiberwise complex structure. In fact, it is obvious that we have such a smooth structure outside the singular points of $\Sigma_\zeta$. It is true at singular points also since $Uni$ looks like $\{(x, y, \alpha) \in \mathbf{C}^3 \mid xy = \alpha\} \times \mathbf{C}^N$ in its neighborhood and $\pi(x, y, \alpha, Z) = (\alpha, Z)$.

We then are going to find an open neighborhood ${}'_{\text{deform},\sigma} \times {}'_{\text{resolve},\sigma}$ of origin and maps

$$act : LieAut(\Sigma_\sigma, z_\sigma)_0 \times {}'_{\text{deform},\sigma} \times {}'_{\text{resolve},\sigma} \to {}_{\text{deform},\sigma} \times {}_{\text{resolve},\sigma}$$

$$\widehat{act} : LieAut(\Sigma_\sigma, z_\sigma)_0 \times Uni' \to Uni \text{ (where } Uni' = \pi^{-1}({}'_{\text{deform},\sigma} \times {}'_{\text{resolve},\sigma}))$$

such that the diagram

$$\begin{array}{ccc} LieAut(\Sigma_\sigma, z_\sigma)_0 \times Uni' & \xrightarrow{\widehat{act}} & Uni \\ \downarrow & & \downarrow \\ LieAut(\Sigma_\sigma, z_\sigma)_0 \times {}'_{\text{deform},\sigma} \times {}'_{\text{resolve},\sigma} & \xrightarrow{act} & {}_{\text{deform},\sigma} \times {}_{\text{resolve},\sigma} \end{array}$$

Diagram 13.17.

commutes and the following holds.

LEMMA 13.18. *Let $\zeta, \zeta' \in {}'_{\text{deform},\sigma} \times {}'_{\text{resolve},\sigma}$. If $\Sigma_\zeta$ and $\Sigma_{\zeta'}$ together with their marked points are biholomorphic to each other and let $\varphi : \Sigma_\zeta \to \Sigma_{\zeta'}$ be a biholomorphic map. Suppose that $h_\sigma \circ \varphi$ is sufficiently close to $h_\sigma$ on $\Sigma_\zeta - K_{\text{neck}}$. (We remark that $\Sigma_{\zeta'} - K_{\text{neck}}$ is identified to $\Sigma_\sigma - K_{\text{neck}}$. Hence $h_\sigma \circ \varphi$ does make sense.) Then there exist $\gamma_0 \in LieAut(\Sigma_\sigma, z_\sigma)_0$ and $\gamma_1 \in Aut(\sigma)$ such that $\zeta' = \gamma_1\gamma_0\zeta$. Moreover $\varphi = \gamma_1\gamma_0$. Here $\gamma_0 : \Sigma_\zeta \to \Sigma_{\gamma_0\zeta}$ is the map induced by $\widehat{act}$ and $\gamma_1 : \Sigma_{\gamma_0\zeta} \to \Sigma_{\gamma_1\gamma_0\zeta} = \Sigma_{\zeta'}$ is a map induced by the action of $\gamma_1 \in Aut(\sigma)$ on Uni. Here we write $\gamma\zeta = act(\gamma, \zeta)$.*

Now let us assume that $(\zeta, u), (\zeta', u') \in {}'_\sigma$ and that $(\Sigma_\zeta, h_{\zeta,u})$ is equivalent to $(\Sigma_{\zeta'}, h_{\zeta',u'})$. Namely we assume that there exists a biholomorphic map $\vartheta : \Sigma_{\zeta'} \to \Sigma_\zeta$ such that $h_{\zeta,u}\vartheta = h_{\zeta',u'}$. It suffices to find $\mu \in Aut(\sigma)$ such that $\mu(\zeta, u) = (\zeta', u')$.

We have $\gamma_0 \in LieAut(\Sigma_\sigma, z_\sigma)_0$, $\gamma_1 \in Aut(\sigma)$ satisying $\gamma_1\gamma_0\zeta = \zeta'$.

We remark that ${}'_\sigma = {}_{\text{deform},\sigma} \times {}_{\text{resolve},\sigma} \times {}'_{\text{map},\sigma}$ and ${}'_{\text{map},\sigma}$ is perpendicular to the Lie algebra $Aut(\Sigma_\sigma, z_\sigma)_0$, (which are regarded as holomorphic vector fields.) Therefore the by construction, we have $\gamma_0 = 1$. (We remark that the support of elements of ${}'_{\text{map},\sigma}$ is away from singular points. Hence we can apply implicite theorem.) The proof of Lemma 12.24 is complete. □

Finally, we construct $\widehat{act} : Aut(\Sigma_\sigma, z_\sigma)_0 \times Uni' \to Uni$ and prove Lemma 13.18.

We put additional marked points $z'_1, \ldots, z'_{m'}$ to $\Sigma_\sigma$ so that each unstable component of it will become stable. The position of $z'_i$ is arbitrary but we require that the number of the additional marked points is as small as possible. Namely we put one more marked point for each unstable $S^2$ with two special points and two additional marked points for each unstable $S^2$ with one special point. We put $z'_\sigma = (z'_1, \ldots, z'_{m'})$.

We call $(\Sigma_\sigma, (z_\sigma, z'_\sigma)) \in C\mathcal{M}_{g,m+m'}$ as $\Sigma_{\sigma'}$. We remark that ${}_{\text{deform},\sigma} = {}_{\text{deform},\sigma'}$, ${}'_{\text{resolve},\sigma} = {}'_{\text{resolve},\sigma'}$ since the number of additional marked points is as small as possible. Hence we obtain an open embedding:

$$Addmark : \frac{{}_{\text{deform},\sigma} \times {}_{\text{resolve},\sigma}}{Aut(\Sigma_\sigma, (z_\sigma, z'_\sigma))} = \frac{{}_{\text{deform},\sigma'} \times {}_{\text{resolve},\sigma'}}{Aut(\Sigma_\sigma, (z_\sigma, z'_\sigma))} \to C\mathcal{M}_{g,m+m'}. \tag{13.19}$$

We remark that $Aut(\Sigma_\sigma, (z_\sigma, z'_\sigma))$ is a finite subgroup of $Aut(\Sigma_\sigma, z_\sigma)$. We next consider the product

$$LieAut(\Sigma_\sigma, z_\sigma)_0 \times {}_{\text{deform},\sigma} \times \Sigma_\sigma \to LieAut(\Sigma_\sigma, z_\sigma)_0 \times {}_{\text{deform},\sigma}. \tag{13.20}$$

We construct a family of complex structures on the fibers of this map by deforming $\Sigma_\sigma$ at $K_{\mathrm{deform},\sigma}$.

On each fiber of (13.20), we put $m + m'$ marked points as follows. For the first $m$ marked points we take marked points $z_\sigma$ and do not move it. For additional $m'$ marked points, we move them by using $LieAut(\Sigma_\sigma, z_\sigma)_0$ components. (Here we identify $LieAut(\Sigma_\sigma, z_\sigma)_0$ with a small neighborhood of identity in $Aut(\Sigma_\sigma, z_\sigma)_0$).

We next resolve singularities of each fiber of (13.20) in the same way as in Section 9 by taking additional factor $_{\mathrm{resolve},\sigma}$. We get

$$\pi^+ : Uni^+ \to LieAut(\Sigma_\sigma, z_\sigma)_0 \times {}_{\mathrm{deform},\sigma} \times {}_{\mathrm{resolve},\sigma}. \tag{13.21}$$

We remark that $Aut(\sigma)$ and $Aut(\Sigma_\sigma, (z_\sigma, z'_\sigma))$ generate a finite group $G$. We use a $G$ invariant metric to use the method of Section 6 to construct the map (13.21).

We now construct the maps $act$, $\hat{act}$. Let $\zeta \in {}_{\mathrm{deform},\sigma} \times {}_{\mathrm{resolve},\sigma}$ be in a small neighborhood of 0 (which we call ${}'_{\mathrm{deform},\sigma} \times {}'_{\mathrm{resolve},\sigma}$) and let $\gamma \in LieAut(\Sigma_\sigma, z_\sigma)_0$.

The fibre $(\pi^+)^{-1}(\gamma, \zeta)$ of the map (13.21) (together with $m + m'$ marked points) is regarded as an element of $C\mathcal{M}_{g,m+m'}$. Hence we find $\zeta' \in {}_{\mathrm{deform},\sigma} \times {}_{\mathrm{resolve},\sigma}$ such that $(\pi^+)^{-1}(\gamma, \zeta)$ is biholomorphic to $(\pi^+)^{-1}(0, \zeta') = \Sigma_{\zeta'}$. Let

$$\varphi_{\gamma,\zeta} : (\pi^+)^{-1}(\gamma, \zeta) \to (\pi^+)^{-1}(0, \zeta')$$

be the biholomorphic map. The pair $(\zeta', \varphi_{\gamma,\zeta})$ is unique modulo the action of finite group $Aut(\Sigma_\sigma, (z_\sigma, z'_\sigma))$. By requiring $dist(x, \varphi_{\gamma,\zeta}(x))$ to be small, we can choose $(\zeta', \varphi_{\gamma,\zeta})$ uniquely. (Here we use a metric on $Uni^+$ to define $dist(x, \varphi_{\gamma,\zeta}(x))$.)

By construction, $(\pi^+)^{-1}(\gamma, \zeta)$ and $(\pi^+)^{-1}(0, \zeta)$ together with its first $m$-marked points are biholomorphic and there is a canonical biholomorphic map. This is because our construction is trivial on $Aut(\Sigma_\sigma, z_\sigma)_0$ factor except $m'$ additional marked points.

We put $\zeta' = act(\gamma, \zeta)$ and let $\varphi_{\gamma,\zeta} : (\pi^+)^{-1}(\gamma, \zeta) = \Sigma_\zeta \to (\pi^+)^{-1}(0, \zeta') = \Sigma_{\zeta'}$ be the restriction of $\hat{act}$ to $\pi^{-1}\zeta = \Sigma_\zeta$.

We need to remark however that this map is not so natural. In fact, we have

$$act(\gamma'\gamma, \zeta) \neq act(\gamma', act(\gamma, \zeta)),$$

in general. (Here $\gamma'\gamma$ is the multiplication in $Aut(\Sigma_\sigma, z_\sigma)_0$.) However the "orbit" of this "action" is well defined. Namely we have the following:

LEMMA 13.22. *There exists* $Aut(\Sigma_\sigma, z_\sigma)'_0 \subseteq Aut(\Sigma_\sigma, z_\sigma)_0$ *such that if* $\gamma', \gamma \in Aut(\Sigma_\sigma, z_\sigma)'_0$ *and if* $\zeta \in {}'_{\mathrm{deform},\sigma} \times {}'_{\mathrm{resolve},\sigma}$ *then there exists* $\gamma'' \in Aut(\Sigma_\sigma, z_\sigma)_0$ *such that*

$$act(\gamma'', \zeta) = act(\gamma', act(\gamma, \zeta)).$$

*Moreover* $\varphi_{\zeta,\gamma''} = \varphi_{\gamma\zeta,\gamma'} \circ \varphi_{\zeta,\gamma}$.

*Proof.* Let us first describe $\varphi_{\gamma\zeta,\gamma'} \circ \varphi_{\zeta,\gamma}$. Let us consider $z' \in \Sigma_{\gamma\zeta}$. We pull it back to $\varphi_{\zeta,\gamma}^{-1}(z') \in \Sigma_\zeta - K_{\mathrm{neck}} = \Sigma_\sigma - K_{\mathrm{neck}}$. We deform the complex structure and resolve the singularity using $\zeta$. We then get a $m + m'$ pointed Riemann surface. This Riemann surface is biholomorphic to $\Sigma_{act(\gamma', act(\gamma,\zeta))}$. It then is isomorphic to some $(\pi^+)^{-1}(\gamma'', \zeta)$ since $\Sigma_{act(\gamma', act(\gamma,\zeta))}$ is isomorphic to $\Sigma_\zeta$ after forgetting $m'$ marked points. Therefore we have $act(\gamma'', \zeta) = act(\gamma', act(\gamma, \zeta))$. The equality $\varphi_{\zeta,\gamma''} = \varphi_{\gamma\zeta,\gamma'} \circ \varphi_{\zeta,\gamma}$ follows from the above mentioned uniqueness of $\varphi_{\zeta,\gamma''}$. □

We now prove Lemma 13.18. Let $\zeta, \zeta' \in {}'_{\mathrm{deform},\sigma} \times {}'_{\mathrm{resolve},\sigma}$, $\varphi : \Sigma_\zeta \to \Sigma_{\zeta'}$ be as in Lemma 13.18. Since $h_\sigma \circ \varphi$ is sufficiently close to $h_\sigma$, there exists $\gamma_1 \in Aut(\sigma)$ such that $\gamma_1^{-1}\varphi(z_i')$ is close to $z_i'$ for $i = 1, \dots, m'$. (Since $z_i' \in \Sigma_\zeta - K_{\mathrm{neck}}$, we may regard $\gamma_1^{-1}\varphi(z_i') \in \Sigma_\sigma - K_{\mathrm{neck}}$, $z_1' \in \Sigma_\sigma - K_{\mathrm{neck}}$.) Therefore $Addmark(\gamma_1^{-1}\zeta')$ is close to $Addmark(\zeta)$. (We use here the fact that the metric we use to construct $Addmark$ is $G$ invariant.)

We furthermore remark that $Addmark(\gamma_1^{-1}\zeta') = (\pi^+)^{-1}(0, \gamma_1^{-1}\zeta')$ is biholomorphic to $Addmark(\zeta) = (\pi^+)^{-1}(0, \zeta)$ after removing additional $m'$ marked points. Therefore, there exists $\gamma_0 \in LieAut(\Sigma_\sigma, \mathbf{z}_\sigma)_0$ such that $(\pi^+)^{-1}(0, \gamma_1^{-1}\zeta')$ is biholomorphic to $(\pi^+)^{-1}(\gamma_0, \zeta)$. Therefore $\gamma_1^{-1}\zeta' = \gamma_0\zeta$. Here $\zeta' = \gamma_1\gamma_0\zeta$.

We now prove that the map $\gamma_1 \circ \varphi_{\gamma_0,\zeta}$ coincides with $\varphi : \Sigma_\zeta \to \Sigma_{\zeta'}$. By construction, we have

$$\gamma_1\varphi_{\gamma_0',\zeta}(z_i') = \varphi(z_i'). \tag{13.23}$$

Here we regards

$$z_i' \in (\pi^+)^{-1}(0, \zeta') - K_{\mathrm{neck}} = \Sigma_\sigma - K_{\mathrm{neck}}.$$

Therefore $\gamma_1\varphi_{\gamma_0',\zeta}$ coincides with $\varphi$ on unstable components of $\Sigma_\sigma$. Let $G'$ be the subgroup of $Aut(\Sigma_\sigma)$ consisting of elements which is identity on unstable components. $G'$ is a finite group. We find that $\varphi^{-1}\gamma_1\varphi_{\gamma_0',\zeta}$ is in $G'$. We also find that both $h_\sigma \circ \gamma_1 \circ \varphi_{\gamma_0,\zeta}$ and $h_\sigma \circ \varphi$ is close to $h_\sigma$. Therefore, using the finiteness of $G'$, we have $\gamma_1\varphi_{\gamma_0',\zeta}\varphi^{-1} \in Aut(\sigma)$. By changing $\gamma_1$ we may assume that $\varphi^{-1}\gamma_1\varphi_{\gamma_0',\zeta} = 1$. The proof of Lemma 13.18 is now complete. □

## 14. CONSTRUCTION OF LOCAL CHART III—SURJECTIVITY

In this section, we are going to prove Proposition 12.25. We first need to prove an a priori estimate for pseudoholomorphic curve. That is we need to prove Lemma 11.2.

*Proof of Lemma* 11.2. Choose $p_0$ such that $h(t, \tau) \in D_{2\varepsilon_1}(p_0)$. Let $\ell_\tau(t) = h(t, \tau)$. We may assume that $\varepsilon_1$ is smaller than the injectivity radius. Hence we write

$$\ell_\tau(t) = \exp_{p(\tau)}\left(\sum_k a_k(\tau)e^{kti}\right). \tag{14.1}$$

SUBLEMMA 14.2. *There is a unique $p(\tau)$ such that $a_0(\tau) = 0$.*

$$\overline{x \mapsto} \overline{\int_0^{2\pi} \exp_x^{-1}(\ell_\tau(t))\,\mathrm{d}t.}$$

a function $U \to \mathbf{C}^n$. Here $U$ is a small neighborhood of $\mathrm{Im}\,\ell_\tau$. It is easy to see that the differential of it is invertible. Hence it hits zero at unique point $p(\tau)$, as required. □

Now we rewrite the equation $\bar\partial h = 0$ using $a_k(\tau)$ and $p(\tau)$ and obtain the following. We remark that $a_k(\tau)$ may be regarded as a vector field along the curve $p(\tau)$ hence its covariant derivative $Da_k/d\tau$ makes sense.

SUBLEMMA 14.3. *We have*

$$\frac{dp}{d\tau} = O_{1,2}\left(a_k, \frac{dp(\tau)}{d\tau}\right)$$

$$\sum_k \left(\frac{Da_k}{d\tau} - ka_k\right)\mathrm{e}^{kti} = O_{2,2}\left(a_k, \frac{dp(\tau)}{d\tau}\right). \tag{14.4}$$

*Here $O_{1,2}(a_k, dp(\tau)/d\tau)$, $O_{2,2}(a_k, dp(\tau)/d\tau)$ are terms estimated by*

$$\left\|O_{1,2}\left(a_k, \frac{dp(\tau)}{d\tau}\right)\right\| \leqslant C_m\left(\left\|\sum_k a_k(\tau)\mathrm{e}^{kti}\right\|_{L^2_m(S^1\times[r-1/2,\tau+1/2])} + \left\|\frac{dp(\tau)}{d\tau}\right\|_{L^2_m([r-1/2,\tau+1/2])}\right)^2$$

$$\left\|O_{2,2}\left(a_k, \frac{dp(\tau)}{d\tau}\right)\right\|_{L^2_{m-1}(S^1\times[\tau])} \leqslant C_m\left(\left\|\sum_k a_k(\tau)\mathrm{e}^{kti}\right\|_{L^2_m(S^1\times[\tau-1/2,\tau+1/2])} + \left\|\frac{dp(\tau)}{d\tau}\right\|_{L^2_m([\tau-1/2,\tau+1/2])}\right)^2$$

*where $\|\ \|_{L^2_m}$ is a Sobolev $L^2_m$ norm (the sum of the $L^2$ norms of the derivatives up to order $m$) and $m$ is a large but fixed number.*

*Proof.* We put

$$a_\tau(t) = \sum a_k(\tau)\mathrm{e}^{kti}.$$

Since the problem is local on $\tau$, we consider at $\tau = \tau_0$. We take a normal coordinate at $p_0 = p(\tau_0)$ and identify its neighborhood with Euclidean space. We consider

$$\exp: D_\varepsilon(p_0)\times L^2_m(S^1; D(\varepsilon)) \to L^2_m(S^1; D_{2\varepsilon}(p_0))$$

such that

$$\exp(p, a(t)) = \exp_p a(t).$$

Here $D(\varepsilon)$ is the metric ball in $\mathbf{R}^{2n}$ of radius $\varepsilon$ centered at 0. exp is a smooth map between Banach manifolds. We identify $D(\varepsilon)$ with $D_\varepsilon(p_0)$ by an exponential map. Let $\tau \mapsto (q(\tau), b_\tau(t))$ be a curve in $D_\varepsilon(p_0)\times L^k_2(S^1; D(\varepsilon))$ such that $(q(0), b_0(t)) = (p_0, \mathbf{0})$. We then have

$$\frac{d}{d\tau}\exp(q(\tau), b_\tau(t)) = \frac{dq(\tau)}{d\tau} + \frac{db_\tau(t)}{d\tau}.$$

Therefore we have

$$\left\|\frac{d}{d\tau}\exp_{p(\tau)}(a_\tau(t)) - \frac{dp(\tau)}{d\tau} - \frac{d}{d\tau}(a_\tau(t))\right\|_{L^2_{m-1}(S^2\times\{\tau_0\})} \leqslant C_m\left(\|a_\tau(t)\|_{L^2_m(S^1\times[\tau-1/2,\tau+1/2])} + \left\|\frac{dp(\tau)}{d\tau}\right\|_{L^2_m([\tau-1/2,\tau+1/2])}\right)^2.$$

$$\left\|\frac{d}{dt}\exp_{p_0}a_{\tau_0}(t) - \frac{da_{\tau_0}(t)}{dt}\right\|_{L^2_{m-1}(S^1)} \leqslant C\|a_\tau(t)\|^2_{L^2_m(S^1\times[\tau_0-1/2,\tau_0+1/2])}.$$

We put $g_+(\tau) = \|\sum_{k>0} a_k e^{kti}\|_{L^2}$, $g_-(\tau) = \|\sum_{k<0} a_k e^{kti}\|_{L^2}$, $g_0(\tau) = |dp(\tau)/d\tau|$. By using elliptic regularity, we have

$$\left\| \sum_k a_k(\tau) e^{kti} \right\|_{L^2_m(S^1 \times [\tau - 1/2, \tau + 1/2])} \leqslant C_m \int_{\tau-1}^{\tau+1} (g_+(x) + g_-(x) + g_0(x))\, dx.$$

Hence using Sublemma 14.3, we have

$$\begin{aligned} \frac{dg_+}{d\tau} &\geqslant +g_+ - C\left( \int_{\tau-1}^{\tau+1} (g_+(x) + g_-(x) + g_0(x))\, dx \right)^2 \\ \frac{dg_+}{d\tau} &\leqslant -g_- + C\left( \int_{\tau-1}^{\tau+1} (g_+(x) + g_-(x) + g_0(x))\, dx \right)^2 \qquad (14.4) \\ |g_0| &\leqslant C\left( \int_{\tau-1}^{\tau+1} (g_+(x) + g_-(x) + g_0(x))\, dx \right)^2. \end{aligned}$$

We remark that in case $(M, J) = \mathbf{C}^n$ with standard complex structure (or in case when $J$ is integrable) we can take $C = 0$ in (14.4). Then Lemma 11.2 follows immediately. The main part of the proof is an estimate of the contribution of the nonlinear term in (14.4).

Next we estimate

SUBLEMMA 14.5. *For each $\varepsilon$ there exists $\varepsilon_1$ such that if Diam$(h(S^1 \times [-L, L])) \leqslant \varepsilon_1$ then $g_\pm(\tau), g_0(\tau) \leqslant \varepsilon$ for $\tau \in [-L+1, L-1]$.*

*Proof.* It is obvious that $g_\pm(\tau) \leqslant \varepsilon$. By elliptic regularity we have

$$\|p(\tau)\|_{C^2} < C. \qquad (14.6)$$

for $\tau \in [-L+1/2, L-1/2]$. Suppose that $|dp(\tau_0)/d\tau| > \delta$ for $\tau_0 \in [-L+1, L-1]$. Then we take a coordinate and find $i$ such that

$$dp_i(\tau)/d\tau > c\delta > 0.$$

Then by (14.6) we have

$$dp_i(\tau)/d\tau > c\delta/2$$

for $\tau \in [\tau_0 - c\delta/2C, \tau_0 + c\delta/2C]$. We may assume $c\delta/2C < \frac{1}{2}$. Hence we have

$$p_i(\tau_0 + c\delta/2C) > p_i(\tau_0) + c^2\delta^2/4C.$$

Therefore by assumption we have $c^2\delta^2/4C < C'\varepsilon_1$. Sublemma 14.5 holds. □

We use (14.4) and Sublemma 14.5 together with the following Lemma 14.7. Lemma 11.2 then will follow from elliptic regularity.

SUBLEMMA 14.7. *For each $C$ there exists $\varepsilon$ independent of $L$, such that if $g_+, g_-, g_0 : [-L, L] \to \mathbf{R}$ satisfies* (14.4) *and if $|g_\pm|, |g_0| < \varepsilon$ then we have*

$$\begin{aligned} |g_+(\tau)| &\leqslant C' e^{-\min(|\tau - L|, |\tau + L|)} \\ |g_-(\tau)| &\leqslant C' e^{-\min(|\tau - L|, |\tau + L|)} \\ |g_0(\tau)| &\leqslant C' e^{-\min(|\tau - L|, |\tau + L|)}. \end{aligned}$$

*Proof of Sublemma* 14.7. The proof is an analogue of [24, Sublemma 9.8; 28, Sublemma 7.20]. We choose $C'$ later. We prove

$$\begin{aligned}|g_+(\tau)| &\leqslant C'(\varepsilon e^{-\min(|\tau-L|,|\tau+L|)} + \varepsilon^{k/2})\\ |g_-(\tau)| &\leqslant C'(\varepsilon e^{-\min(|\tau-L|,|\tau+L|)} + \varepsilon^{k/2}) \qquad (14.8.k)\\ |g_0(\tau)| &\leqslant C'(\varepsilon e^{-\min(|\tau-L|,|\tau+L|)} + \varepsilon^{k/2}).\end{aligned}$$

by induction on $k$. ($C'$ is independent of $k$.) When $k = 1$ the inequality follows from assumption $|g_\pm|, |g_0| < \varepsilon$. The case $k = \infty$ is the conclusion. Suppose that (14.8.k) holds for $k$, then it follows that

$$\int_{\tau-1}^{\tau+1} (g_+(x) + g_-(x) + g_0(x))dx \leqslant 100C'(\varepsilon e^{-\min(|\tau-L|,|\tau+L|)} + \varepsilon^{k/2}). \tag{14.9}$$

We put $\hat{g}_+(\tau) = e^{-(\tau-\tau_0)}g_+(\tau)$. Then by (14.4) we have

$$\begin{aligned}\frac{d\hat{g}_+}{d\tau} &> C\left(\int_{\tau-1}^{\tau+1} (g_+(x) + g_-(x) + g_0(x))dx\right)^2 e^{-(\tau-\tau_0)}\\ &\geqslant -10000CC'^2(\varepsilon e^{-\min(|\tau-L|,|\tau+L|)} + \varepsilon^{k/2})^2 e^{-(\tau-\tau_0)}.\end{aligned}$$

We may choose $\varepsilon$ and $C'$ such that the

$$\begin{aligned}&C\int_{\tau_0}^{\infty} 10\,000C'^2(\varepsilon e^{-\min(|\tau-L|,|\tau+L|)} + \varepsilon^{k/2})^2 e^{-(\tau-\tau_0)}d\tau\\ &\quad < C'\varepsilon^{(k+1)/2} + \frac{C'}{100}\varepsilon^{3/2}e^{-\min\{|\tau_0-L|,|\tau_0+L|\}}.\end{aligned}$$

Hence

$$\begin{aligned}g_+(\tau_0) = \hat{g}_+(\tau_0) &\leqslant C'\varepsilon^{(k+1)/2} + \frac{C'}{100}\varepsilon^{3/2}e^{-\min\{|\tau_0-L|,|\tau_0+L|\}} + \hat{g}_+(L)\\ &\leqslant C'\varepsilon^{(k+1)/2} + C'e^{-\min\{|\tau_0-L|,|\tau_0+L|\}}\varepsilon.\end{aligned}$$

The first inequality of (14.8.k + 1) holds. The proof of the second inequality is similar. The third inequality of (14.8.k + 1) is then obvious from (14.4) and the first and the second inequalities of (14.8.k + 1). The proof of Sublemma 14.7 is complete. □

The proof we gave above is an analog of the argument by Uhlenbeck used in the proof of removable singularity theorem of Yang–Mills conncection.

A different proof (based on local Hölder estimate of integral operators $P$, $T$ which we define later) is due to referee. We give an outline of it below. The proof is similar to [4, pp. 166–170], by Sikorav (which proves a similar estimate in case of $h: D^2(1) \to (M, J)$).

It is easy to see that Lemma 11.2 is equivalent to Lemma 11.2′. So we are going to prove Lemma 11.2′. Let $h: Annu(r, 1) \to (M, \omega, J)$ is as in Lemma 11.2′. We put $p_0 = h(z_0)$ where $z_0 \in Annu(r, 1)$ and

$$\hat{h}(z) = (\exp_{p_0})^{-1}(h(z)). \tag{14.10}$$

We regards $T_{p_0}M = \mathbf{C}^n$ and $\hat{h}: Annu(r, 1) \to \mathbf{C}^n$. Let $U$ be a small neighborhood of 0 in $\mathbf{C}^n$. There exists $q \in \Gamma(U, Hom_{\mathbf{R}}(\mathbf{C}^n, \mathbf{C}^n))$ such that when we identify $U$ and a neighborhood of $p_0$ in $M$, we have

$$\bar{\partial} h = \bar{\partial}\hat{h} + q(\hat{h})\partial\hat{h}. \tag{14.11}$$

(Here $\bar{\partial} h$ is defined by using the almost complex structure $J$ of $M$ and $\bar{\partial}\hat{h}$ is defined by using the standard complex structure of $U \subseteq \mathbf{C}^n$.) Also we have

$$q(0) = Dq(0) = 0. \tag{14.12}$$

(14.11) and pseudoholomorphicity of $h$ implies

$$\bar{\partial}\hat{h} + q(\hat{h})\partial\hat{h} = 0. \tag{14.13}$$

Let $\lambda: (0, \infty) \to [0, 1]$ be a cut-off function such that

$$\lambda(t) = \begin{cases} 0 & t \geqslant 1 \\ 1 & t < \mathrm{e}^{-1/2}. \end{cases}$$

We put

$$h_1(z) = \lambda(z)\lambda(r/|z|)\hat{h}(z) \tag{14.14.1}$$

$$g_1 = \bar{\partial} h_1 + q(\hat{h})\partial h_1. \tag{14.14.2}$$

It is easy to see that

$$|g_1(z)| \leqslant C(1 + r/|z|^2). \tag{14.15.1}$$

$$\sup \|q(z)\| \leqslant o(\varepsilon_1). \tag{14.15.2}$$

Here $o(\varepsilon_1) \to 0$ as $\varepsilon_1 \to 0$. We put

$$(H_{1/2}h_1)(z) = \sup\left\{ \frac{(h_1(z') - h_1(z))}{|z - z'|^{1/2}} \,\middle|\, z' \in D(1) \right\}.$$

We use it to prove

$$(H_{1/2}h_1)(z) \leqslant C\left(1 + \frac{r}{|z|^2}\right). \tag{14.16}$$

$$Pg = \frac{1}{2\pi \mathrm{i}} \iint \frac{g(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}$$

$$Tg(z) = \lim_{\varepsilon \to 0} \iint_{\{\zeta \mid |\zeta - z| \geqslant \varepsilon, |\zeta| \leqslant 1\}} \frac{g(\zeta)}{(\zeta - z)^2}\, d\zeta \wedge d\bar{\zeta}. \tag{14.17.2}$$

We use the fact $\bar{\partial} \circ P = \mathrm{Id}$, $\partial \circ P = \mathrm{T}$ (see [4, p. 166]) to find

$$h_1 = P(Id + q(\hat{h})T)^{-1} g_1. \tag{14.18}$$

(14.16) then follows from (14.15) and the boundedness of $P$, $T$ with respect respect to an appropriate local Hölder norm. The proof of boundedness of $P$, $T$ with respect to a local Hölder norm is similar to [71]. We omit the detail. (See [4, pp. 166–170].)

We next put

$$h_2(z) = \lambda(e^{1/2}z)\lambda(e^{1/2}r/|z|)\hat{h}(z)$$

$$g_2 = \bar{\partial}h_2 + q(h_1)\partial h_2.$$

It is easy to see that

$$|g_2(z)| \leqslant C\left(1 + \frac{r}{|z|^2}\right). \tag{14.19}$$

Using (14.16)–(14.19) in a way similar to [4, pp. 166–170] using [71] (and in a way similar to the proof of (14.16)), we obtain the conclusion of Lemma 11.2′. (We omit the detail.) □

We now prove Proposition 12.25 by contradiction. Our proof is similar to Donaldson's argument in [12]. (See also [23].) Let $\sigma_i = (\Sigma_{\sigma_i}, h_{\sigma_i}) \in C\mathcal{M}_{g,m}(M, J, \beta)$ be a sequence. Suppose that $\sigma_i$ converges to $\sigma \in C\mathcal{M}_{g,m}(M, J, \beta)$ but is not equivalent to any of the element of $s_\sigma^{-1}(0)/Aut(\sigma)$. We then can take a representative of $\sigma_i = (\Sigma_{\sigma_i}, h_{\sigma_i})$ such that $\Sigma_{\sigma_i} = \Sigma_{\zeta_i}$ for $\zeta_i \in {}_{\text{deform},\sigma} \times {}_{\text{resolve},\sigma}$ converging to zero. By the definition of the topology on the space of stable maps, after adding marked points, $\Sigma_{\sigma_i}$ converges to $\Sigma_\sigma$. In particular, outside of the neck region $K_{\text{neck}}(\Sigma_\sigma)$, our semistable curve $\Sigma_{\sigma_i}$ is canonically diffeomorphic to $\Sigma_\sigma$. Hence there exists $u_i' \in C^\infty(\Sigma_\sigma - K_{\text{neck}}(\sigma); h_\sigma^* TM)$ such that

$$h_i(p) = \exp_{h(p)} u_i'(p) \tag{14.20}$$

on $\Sigma_{\zeta_i} - K_{\text{neck}} = \Sigma_\sigma - K_{\text{neck}}(\sigma)$. One the neck region, we can use estimate Lemma 11.2′ as follows. We choose $\varepsilon$ later. Then by assumption, we find $\sigma$ such that if $i$ is enough large then

$$Diam(h_{\sigma_i}(D_{x_v}(\mu) - D_{x_v}(R^{-1}_{x_v,\alpha_{i,v}}))) < \varepsilon. \tag{14.21}$$

Let $\Sigma_{\sigma,v}$ and $\Sigma_{\sigma,w}$ be two components of $\Sigma_\sigma$ containing the singular point $x$. We define a conformal isomorphism

$$\psi_{i,x} : (D_{x_v}(\mu) - D_{x_v}(R^{-1}_{x_v,\alpha_{i,v}})) \cup (D_{x_w}(\mu) - D_{x_w}(R^{-1}_{x_w,\alpha_{i,w}})) \to Annu(\mu^{-2}R^{-2}_{x_v,\alpha_{i,v}}, 1)$$

(note $R_{x_v,\alpha_{i,v}} = R_{x_w,\alpha_{i,v}}$) by

$$\psi_{i,x}(p) = \begin{cases} \mu^{-1}\exp^{-1}_{x_v}(z) & z \in D_{x_v}(\mu) - D_{x_v}(R^{-1}_{x_v,\alpha_{i,v}}) \\ \mu^{-1}\alpha_{i,v}/\exp^{-1}_{x_w}(z) & z \in D_{x_w}(\mu) - D_{x_w}(R^{-1}_{x_w,\alpha_{i,v}}). \end{cases} \tag{14.22}$$

Now by Lemma 11.2′ we have

$$\|h_{\sigma_i} \circ \psi^{-1}_{i,x}\|_{C^1(Annu(\mu^{-1}R^{-1}_{x_v,\alpha_{i,v}}, 1))} < C. \tag{14.23}$$

$\|h_i\|_{C^1} < C$

on $h_i(D_{x_v}(\mu) - D_{x_v}(R^{-1}_{x_v,\alpha_{i,v}}))$. We also obtain from (14.23) that

$$Diam(h_i(D_{x_v}(R^{-1}) - D_{x_v}(R^{-1}_{x_v,\alpha_{i,v}}))) < CR^{-1}. \tag{14.25}$$

Hence

$$\|u_i'(z)\| < C\|z\| \tag{14.26}$$

on $h_{\sigma_i}(D_{x_v}(\mu) - D_{x_v}(R^{-1}_{x_v,\alpha_{i,v}}))$. We use these two estimates to show the following:

SUBLEMMA 14.27. *There exists* $u_i'' \in \Gamma(\Sigma_\sigma; h_\sigma^* TM)$ *and* $\lambda_i \to 0$ *such that*

(14.28.1) $u_i''$ *coincides with* $u_i'$ *on* $\bigcup_v \bigcup_{x \in sing(\Sigma_{\sigma,v})} (\Sigma_{\sigma,v} - D_{x_v}(\delta^{-1} R^{-1}_{x_v,\alpha_i})) \subset \Sigma_\sigma - K_{\mathrm{neck}}(\sigma) = \Sigma_{\zeta_i} - K_{\mathrm{neck}}$.

(14.28.2) $\|(D_{h_\sigma}\bar\partial_{\Sigma_\sigma})u_i''\|_{L^p(K_{\mathrm{neck}}(\sigma))} < \lambda_i$.

*Proof.* Choose $\delta_i \to 0$ such that $\delta_i R_{x_v,\alpha_i} \to \infty$ and take a cut-off function $\chi_i$ such that

$$\chi_i(r) = \begin{cases} 0 & r < \delta_i^{-1} R^{-1}_{x_v,\alpha_i} \\ 1 & r > 2\delta_i^{-1} R^{-1}_{x_v,\alpha_i}. \end{cases}$$

We have

$$\sup|d\chi_i/dr| < C\delta_i R_{x_v,\alpha_i} \tag{14.29}$$

and $d\chi_i/dr$ is supported on $[\delta_i^{-1} R^{-1}_{x_v,\alpha_i}, 2\delta_i^{-1} R^{-1}_{x_v,\alpha_i}]$. On $D_{x_v}(\mu) - D_{x_v}(R^{-1}_{x_v,\alpha_{i,v}})$ we put

$$u_i''(x) = \chi_i(|x|) u_i'(x).$$

Then by (14.29) and (14.26) we have

$$\|(D_{h_\sigma}\bar\partial_{\Sigma_\sigma})u_i''(z)\| < \left|\frac{\partial \chi_i}{dr}(|z|)\right| \|u_i'(z)\| + \|(D_{h_\sigma}\bar\partial_{\Sigma_\sigma})u_i'(z)\| \leqslant C \tag{14.30}$$

if $\|z\| \in [\delta_i^{-1} R^{-1}_{x_v,\alpha_i}, 2\delta_i^{-1} R^{-1}_{x_v,\alpha_i}]$. If $\|z\| \notin [\delta_i^{-1} R^{-1}_{x_v,\alpha_i}, 2\delta_i^{-1} R^{-1}_{x_v,\alpha_i}]$ we have

$$(D_{h_\sigma}\bar\partial_{\Sigma_\sigma})u_i''(z) = (D_{h_\sigma}\bar\partial_{\Sigma_\sigma})u_i'(z).$$

On the other hand by (14.19) and the fact that $h_{\sigma_i}$ and $h$ are both pseudoholomorphic on $K_{\mathrm{neck}}(\sigma)$, we have

$$\|(D_{h_\sigma}\bar\partial_{\Sigma_\sigma})u_i'\|_{C^0} \leqslant C(\|h_i\|_{C^1} + \|h\|_{C^1})\|u_i'\|_{C^0}.$$

$$\sup_{\|z\| \notin [\delta_i^{-1} R^{-1}_{x_v,\alpha_i}, 2\delta_i^{-1} R^{-1}_{x_v,\alpha_i}]} \|(D_{h_\sigma}\bar\partial_{\Sigma_\sigma})u_i''(z)\| \to 0. \tag{14.31}$$

Using (14.30) and (14.31) and the fact that the volume of the domain of $z$ satisfying $\|z\| \in [\delta_i^{-1} R^{-1}_{x_v,\alpha_i}, 2\delta_i^{-1} R^{-1}_{x_v,\alpha_i}]$ converges to 0, we obtain the required estimate (14.28.2). The proof of Sublemma 14.27 is now complete. □

We put $u_i = u_i'' - Q_\sigma \circ (D_{h_\sigma}\bar\partial_{\Sigma_\sigma})(u_i'')$. Here

$$Q_\sigma : \frac{L^p(\Sigma_\sigma; h_\sigma^* TM \otimes \Lambda^{0,1}(\Sigma_\sigma))}{E_\sigma} \to {}^{\perp}_{1,\sigma} \quad \text{and}$$

$$\Pi : L^p(\Sigma_\sigma; h_\sigma^* TM \otimes \Lambda^{0,1}(\Sigma_\sigma)) \to \frac{L^p(\Sigma_\sigma; h_\sigma^* TM \otimes \Lambda^{0,1}(\Sigma_\sigma))}{E_\sigma}$$

is as in Section 13. It then follows that $(D_{h_\sigma}\bar\partial_{\Sigma_\sigma})u_i \equiv 0 \bmod E_\sigma$. Namely $u_i \in {}_{\mathrm{map},\sigma}$. We use Sublemma 14.27 and (14.25) to obtain an estimate

$$\sup dist(h_{\zeta_i,u_i}(p), h_i(p)) \leqslant C\lambda_i \to 0. \tag{14.32}$$

We now put

$$\hat u_i(p) = \exp^{-1}_{h_i(p)}(h_{\zeta_i,u_i}(p)) \tag{14.33}$$

$$h_i(s,p) = \exp_{h_i(p)}(s\hat u_i(p)).$$

We show

Sublemma 14.34. $\lim_{i\to\infty} \|\Pi_{E_\sigma}\bar\partial_{\Sigma_{\sigma i}} h_i(s,\cdot)\|_{L^p} = 0.$

*Proof.* It is easy to see that

$$s\exp^{-1}_{h_i(p)}(h_{\zeta_i,u_i}(p)) - su_i(p)$$

converges to 0 as $i\to\infty$ in $C^\infty$ topology outside the neck region. Hence it suffices to estimate $\bar\partial_{\Sigma_{\sigma_i}} h_i(s,\cdot)$ on $D_{x_v}(\mu) - D_{x_v}(R^{-1}_{x_v,\alpha_{i,v}})$. There we use the fact that $h_{\zeta_i,u_i}$ is holomorphic and use Lemma 11.2 in the same way as we did to prove (14.24). We then obtain

$$\|h_{\zeta_i,u_i}\|_{C^1} < C. \tag{14.35}$$

Applying (14.35), (14.24) and (14.30) in the same way as we did to prove (14.31) we have

$$\sup_{\|z\|\notin[\sigma,\delta_i R^{-1}_{x_v,\alpha_i}]} \|(D_{h_{\zeta,u_i}}\bar\partial_{\Sigma_{\sigma_i}})\hat u_i(z)\| \leqslant C(\|h_i\|_{C^1} + \|h_{\zeta_i,u_i}\|_{C^1})\|\hat u_i\|_{C^0} \to 0.$$

Sublemma 14.34 follows. □

Thus, for each $s\in[0,1]$, we obtain an approximate solution $h_i(s,\cdot):\Sigma_{\zeta_i}\to M$ of (12.17). For sufficiently large $i$, we can make an exact solution of (12.17) from it by using the argument of Section 13. Namely we obtain maps $h_i'(s,\cdot):\Sigma_{\zeta_i}\to M$ for $s\in[0,1]$ such that

$$h_i'(0,p) = h_i(p),\ h_i'(1,p) = h_{\zeta_i,u_i}(p) \tag{14.35.1}$$

$$\bar\partial_{\Sigma_{\sigma_i}} h_i'(s,\cdot) \equiv 0 \bmod E_\sigma$$

$$\left|\frac{\partial h_i'}{\partial s}\right| \to 0 \quad \text{as} \quad i\to\infty. \tag{14.35.3}$$

We remark that linearized equation for (12.17) is of maximal rank. It follows from implicit function theorem that the family of solutions we constructed in Section 13 is one of the maximal dimension. We then can use the fact that $h_i'(1, p) = h_{\zeta_i, u_i}(p)$ to show that family $h_i'(s, \cdot)$ for $s \in [0, 1]$ is contained in the family of solutions we constructed in Section 13. Therefore $h_i'(0, p) = h_i(p)$ is also contained in the family of solutions we constructed in Section 13. The proof of Proposition 12.25 is now complete. □

## 15. GLUING

We are going to glue the charts constructed in Sections 12–14 to obtain a Kuranishi structure. To glue the charts, one trouble is that the moduli space $C\mathcal{M}_{g,m}(M, J, \beta)$ can be quite pathological, because it is in general the zero set of a continuous function which can be an arbitrary closed set.

The second trouble is to find an appropriate way to fix representative of elements of $C\mathcal{M}_{g,m}(M, J, \beta)$. We remark that the representative is well defined modulo the group $\mathit{Diff}(\sigma, z)$, the group of the diffeomorphisms fixing marked points. Namely $\mathcal{M}_{g,m}(M, J, \beta)$ is regarded as a subspace of $(J(\Sigma) \times \mathrm{Map}(\Sigma, M))/\mathit{Diff}(\sigma, z)$, here $J(\Sigma)$ is the space of all complex structures on $\Sigma$ and $\mathit{Map}(\Sigma, M)$ is the space of all maps from $\Sigma$ to $M$. So if we try to embed $\mathcal{M}_{g,m}(M, J, \beta)$ to a single function space, we need to make precise the definition of $(J(\Sigma) \times \mathit{Map}(\Sigma, M))/\mathit{Diff}(\sigma, z)$ by fixing function space and prove some kind of slice theorem, etc. Then we immediately meet a trouble directly related to the stability of complex structure, etc. The difficulty is that $\mathit{Diff}(\sigma, z)$ is very far from being compact. In a similar problem of Gauge theory, slice theorem (see [23, Section 3]) is proved. In that case, the proof depends on the fact that the image of the group of gauge transformations of $L^2_{k+1}$ class into the group of gauge transformations of $L^2_k$ class is compact. This is because the Gauge group ($SU(2)$ for example) is compact and we can then use Rellich's theorem. The corresponding fact in our case is not true. Moreover the isotropy group of the action of $\mathit{Diff}(\sigma, z)$ for some element at "infinity" of $J(\Sigma) \times \mathit{Map}(\Sigma, M)$ may be noncompact. This causes a trouble in studying the space $(J(\Sigma) \times \mathit{Map}(\Sigma, M))/\mathit{Diff}(\sigma, z)$. However by using the fact that isotropy group of element of $C\mathcal{M}_{g,m}(M, J, \beta)$ is finite, one may probably be able to prove a slice theorem in a neighborhood of $\mathcal{M}_{g,m}(M, J, \beta)$. Namely the quotient space $(J(\Sigma) \times \mathit{Map}(\Sigma, M))/\mathit{Diff}(\sigma, z)$ is Hausdorff there. However, because of the trouble we mentioned above, we do not use this infinite dimensional space and work more directly without using infinite dimensional manifold.

Let us point out the third trouble to make our charts compatible. The trouble is that our charts are constructed by solving eq. (12.17) which *depends* on the choice of the subspace $E_\sigma$. Unfortunately, it seems that there is no canonical choice of this subspace. Usually to work out Kuranishi theory, one takes $L^2$ orthogonal complement to the image of the linearized operator to find a representative of the obstruction bundle. This is the way, for example, taken by Furuta [32] to study the case of monopole equation. However, in our situation, we cannot use $L^2$ orthogonal complement to the image of the operator (12.1). The reason is as follows. First of all we cannot put $p = 2$ in (12.1) since $L^2_1$ function on 2 manifold is not continuous and we cannot make sense the condition $\pi_v(p) = \pi_w(q) \Rightarrow u_v(p) = u_w(q)$ we put at the beginning of Section 12. To clarify the situation, let us consider the dual operator

$$(D_{h_\sigma}\bar{\partial}_{\Sigma_\sigma})^* : L^q(\Sigma_\sigma; h_\sigma^* TM \otimes \Lambda^{0,1}(\Sigma_\sigma)) \to (L^p_1(\Sigma_\sigma : h_\sigma^* TM))^*$$

to (12.1). Here $1/p + 1/q = 1$. The dual space $(L_1^p(\Sigma_\sigma : h_\sigma^* TM))^*$ contains a delta function supported on a finite number of (singular) points. Hence there exists a form like $f\, d\bar{z}/\bar{z}$ in the kernel of dual operator. This form is not of $L^2$-class. In algebraic geometry, corresponding phenomenon is observed. Namely one needs to study logarithmic forms to consider Dolbeault cohomology of singular variety. Thus it seems difficult to find a canonical choice of $E_\sigma$. Because of this problem, we take an arbitrary choices locally and "glue" them to patch the charts. Then the Kuranishi structure itself will depend on the choice of such subspaces. However the cobordism class and hence the fundamental cycle of it is well defined, as we will prove in Section 17.

Now we start the gluing construction. The construction of the obstruction bundle $E_\sigma$ will be done inductively on neighborhoods of the strata, beginning with the ones that are minimal (and nonempty) with respect to the partial order $\prec$.

Let $\mathscr{M}_{g,m}(M, J, \beta)(T, g_v, \beta_v, o)$ be such stratum. Namely we assume that $\mathscr{M}_{g,m}(M, J, \beta)(T', g'_v, \beta'_v, o')$ is empty if $(T', g'_v, \beta'_v, o') \prec (T, g_v, \beta_v, o)$. It follows that the stratum $\mathscr{M}_{g,m}(M, J, \beta)(T, g_v, \beta_v, o)$ is compact. By Proposition 8.7, there exists such a stratum.

so M=CM are always quotiented out by Aut

First we consider the problem to fix a representative of elements of $\mathscr{M}_{g,m}(M, J, \beta)(T, g_v, \beta_v, o)$. It might seem that an appropriate way is to construct the universal family of semistable curves over $\mathscr{M}_{g,m}(M, J, \beta)(T, g_v, \beta_v, o)$ as a "fiber bundle" and use its trivialization. (This is the way we took to study Deligne–Mumford compactification in Section 9.) The universal family, however, is not a "fiber bundle" but is an "orbibundle" because of the presence of nontrivial automorphism. The trouble is that our space $\mathscr{M}_{g,m}(M, J, \beta)(T, g_v, \beta_v, o)$ is not in general an orbifold so it does not make sense to say that universal family is an orbibundle on it. So we take more direct way, that is to specify the choice of representatives locally.

Let us denote by $\psi'_\sigma$ the homeomorphism $s_\sigma^{-1}(0) \cap {}'_\sigma \to \mathscr{M}_{g,m}(M, J, \beta)$ to an open set constructed in Theorem 12.9. We choose finitely many elements $\tau_i \in \mathscr{M}_{g,m}(M, J, \beta)(T, g_v, \beta_v, o)$ such that the images of the homeomorphism $\psi'_{\tau_i}$ in Theorem 12.9 cover $\mathscr{M}_{g,m}(M, J, \beta)(T, g_v, \beta_v, o)$. We put $\hat{\Omega}_{\tau_i} = \operatorname{Im} \psi'_{\tau_i} \cap \mathscr{M}_{g,m}(M, J, \beta)(T, g_v, \beta_v, o)$. For each $i$ we *fix* a representative $(\Sigma_{\tau_i}, h_{\tau_i})$ of $\tau_i$. Here $h_{\tau_i} : \Sigma_{\tau_i} \to M$. We remark that we have already chosen $E_{\tau_i} \subseteq \Gamma(\Sigma_{\tau_i}, h_{\tau_i}^*(TM) \otimes \Lambda^{0,1}(\Sigma_{\tau_i}, J_{\tau_i}))$. Let $\Omega_{\tau_i}$ be a *closed* subset of $\hat{\Omega}_{\tau_i}$ such that their interiors cover $\mathscr{M}_{g,m}(M, J, \beta)(T, g_v, \beta_v, o)$.

This is a nontrivial transversality requirement on the choices of $E_{\tau_i}$

Roughly speaking we take $E_\sigma = \bigoplus_{\sigma \in \Omega_{\tau_i}} E_{\tau_i}$. To be precise one needs to identify $E_{\tau_i} \subseteq \Gamma(\Sigma_i, h_{\tau_i}^*(TM) \otimes \Lambda^{0,1}(\Sigma_i, J_{\tau_i}))$ as a subspace of $\Gamma(\Sigma, h^*(TM) \otimes \Lambda^{0,1}(\Sigma))$ in a way as canonical as possible. We need to find a "canonical" map from $\Sigma$ to $\Sigma_{\tau_i}$. In order to overcome this trouble, we modify eq. (12.7) a bit in the way we will explain below. and smoothly although Aut(Σ)×Γ→Γ isn't differentiable

differentiability troubles coming in here

Let $\zeta \in {}_{\mathrm{deform},\tau_i} \times {}_{\mathrm{resolve},\tau_i}$. We consider $(\zeta, 0) \in {}'_{\tau_i}$. We obtain $(\Sigma_{\tau_i,\zeta}, h_{\zeta,0})$ by Theorem 12.9. (Hereafter we write $\Sigma_{\tau_i,\zeta}$ in place of $\Sigma_\zeta$ to clarify that it is constructed out of $\Sigma_{\tau_i}$.) We recall that for each $(\zeta, 0) \in {}'_{\tau_i}$ we embed $E_{\tau_i} \subseteq C^\infty(\Sigma_{\tau_i,\zeta}, h_{\mathrm{exact},\zeta,0}^*(TM) \otimes \Lambda^{0,1}(\Sigma_{\tau_i,\zeta}))$ as follows. We remark that we have fixed a representative of $(\Sigma_{\tau_i,\zeta}, h_{\mathrm{exact},\zeta,0})$ not only its equivalence class. We may regard $K_{\mathrm{obstr}}(\tau_i) \subseteq \Sigma_{\tau_i,\zeta}$ since $K_{\mathrm{obstr}}(\tau_i) \subseteq \Sigma_{\tau_i}$ is disjoint to $K_{\mathrm{neck}}(\tau_i)$. For each $p \in K_{\mathrm{obstr}}(\tau_i) \subseteq \Sigma_{\tau_i,\zeta}$ we consider the parallel transport $Par_{h_{\tau_i}(p), h_{\zeta,0}(p)}$. Its complex linear part induces an isomorphism

$$\Lambda^{0,1}(K_{\mathrm{obstr}}(\tau_i)) \otimes h_{\tau_i}^* TM \cong \Lambda^{0,1}(K_{\mathrm{obstr}}(\tau_i)) \otimes h_{\mathrm{exact},\zeta,0}^* TM.$$

We use this isomorphism to regard $E_{\tau_i} \subseteq C^\infty(\Sigma_{\tau_i,\zeta}, h_{\mathrm{exact},\zeta,0}^*(TM) \otimes \Lambda^{0,1}(\Sigma_{\tau_i,\zeta}))$.

Now let $(\Sigma, h)$ be a pair of a semistable curve and a map $\Sigma \to M$. We assume that it is equivalent to an element close to $(\Sigma_\sigma, h_\sigma)$ with $\sigma \in \Omega_{\tau_i}$ in the following sense.

There exists $\zeta \in {}_{\mathrm{deform},\sigma} \times {}_{\mathrm{resolve},\sigma}$ and a biholomorphic map $\eta : \Sigma_{\sigma,\zeta} \to \Sigma_0$ such that

$$\sup_p dist(h_{\mathrm{exact},\zeta,0}(p), h\eta(p)) < \min\left\{\frac{injrad(M)}{100}, d\right\}. \tag{15.1}$$

Then by taking $\Omega_{\tau_i}$ enough small, we have $\zeta \in {}_{\mathrm{deform},\tau_i} \times {}_{\mathrm{resolve},\tau_i}$ and a biholomorphic map $\vartheta : \Sigma \to \Sigma_{\tau_i,\zeta}$ such that

$$\sup_p dist(h_{\mathrm{exact},\zeta,0}\,\vartheta(p), h(p)) < \min\left\{\frac{injrad(M)}{100} + d, 2d\right\}. \tag{15.2}$$

We use $\vartheta : \Sigma \to \Sigma_{\tau_i,\zeta}$ and parallel transport $Par_{h_{\mathrm{exact},\zeta,0}\vartheta(p),h_{(p)}}$ to embed

$$Emb_{(\zeta,\vartheta),\tau_i} : E_{\tau_i} \to C^\infty(\Sigma, h^*(TM) \otimes \Lambda^{0,1}(\Sigma)). \tag{15.3}$$

Now we modify eq. (12.7) as follows. *why "⊕" subspaces ∀(ζ,ϑ)*

*should be e' bundle over {h∈W^{1,p}} despite differentiability failure of Aut(Σ) action and pregluing constructions. Also needs to be Aut(σ) invariant?*

*⊛*

$$\bar\partial_\Sigma h \equiv 0 \mod \bigoplus_{\sigma \in \Omega_{\tau_i}} Emb_{(\zeta,\vartheta),\tau_i}(E_{\tau_i}). \tag{15.4}$$

*(15.8)*

We need however to handle with one more trouble. Namely the pair $(\zeta, \vartheta)$ is *not* unique. In case when there is no unstable component, $\Sigma_{\tau_i,\zeta} \cong \Sigma_{\tau_i,\zeta'}$ implies that $\zeta' = \gamma\zeta$ for some $\gamma \in Aut(\Sigma_{\tau_i})$, and $Aut(\Sigma_{\tau_i})$ is a finite group in this case. Let $\gamma : \Sigma_{\tau_i,\zeta} \cong \Sigma_{\tau_i,\zeta'}$ denote this biholomorphic map. Requiring that $(\zeta', \vartheta \circ \gamma)$ also satisfies (15.2), we have $\gamma \in Aut(\tau_i)$. Since the space $E_{\tau_i}$ is invariant of the action of $Aut(\tau_i)$, eq. (15.4) is independent of the choice of $(\zeta, \vartheta)$ in this case.

However, if there are unstable components, $\Sigma_{\tau_i,\zeta} \cong \Sigma_{\tau_i,\zeta'}$ does not imply $\zeta' = \gamma\zeta$ for some $\gamma \in Aut(\tau_i)$. The extra symmetry is parametrized by a neighborhood $Lie(Aut(\Sigma_{\tau_i}))_0$ of 0 of the Lie algebra of the group of automorphism of $\Sigma_{\tau_i}$ (Lemma 13.18). So we have $\gamma_0 \in Lie(Aut(\Sigma_{\tau_i}))_0$, $\gamma \in Aut(\Sigma_{\tau_i})$ such that $\zeta' = \gamma\gamma_0\zeta$ and there exists a biholomorphic map $\gamma\gamma_0 : \Sigma_{\tau_i,\zeta} \cong \Sigma_{\tau_i,\zeta'}$. Requiring $(\zeta', \vartheta \circ \gamma\gamma_0)$ also satisfies (15.2) we have $\gamma \in Aut(\tau_i)$. The trouble here is that $E_{\tau_i}$ is *not* invariant by the "action" of $Lie(Aut(\Sigma_{\tau_i}))_0$ (in other words by the isomorphism $\gamma_0 : \Sigma_\zeta \to \Sigma_{\gamma_0\zeta}$). So eq. (15.4) has an extra parameter described by ${}_i\, Lie(Aut(\Sigma_{\tau_i}))_0$.

This is an important point since the heart of "negative multiple covered problem" is presence of unstable component for the stable map.

We need to kill this extra parameter to obtain a moduli space we need. The way to do so must be canonical. To be more precise it should be independent of $\sigma$ but may depend on the data related to $\tau_i$.

We can do it in the following way. (This argument is a kind of center of mass technique developed by [34]. Another argument is discussed in Appendix.) Let $\Sigma_{\mathrm{uns},\tau_i} \subseteq \Sigma_{\tau_i}$ be the union of unstable components minus a neighborhood of singular points. We assume that it is invariant of $Aut(\tau_i)$. Since we remove neighborhoods of singular points, we may regard $\Sigma_{\mathrm{uns},\tau_i} \subseteq \Sigma_{\tau_i,\zeta}$.

Next let $dist^{2\prime} : M \times M \to [0, \infty)$ be a smooth function which is $C^2$-close to the square of Riemannian distance in a neighborhood of diagonal and that $(dist^{2\prime})^{-1}(0) =$ diagonal.

*The approach I looked at (using hypersurface intersections to stabilize) runs into differentiability issues at this point. I wonder whether/how this is avoided here.*

For $\vartheta : \Sigma_0 \to \Sigma_{\tau_i,\zeta}$ we consider

$$meandist_{\tau_i}((\vartheta, \zeta), h) = \int_{x \in \Sigma_{\mathrm{uns},\tau_i}} dist^{2\prime}(h_{\mathrm{exact},\zeta,0}(x), h\vartheta^{-1}(x))dx. \tag{15.5}$$

We use Riemann metric on $\Sigma_{\tau_i}$ to obtain the volume element $dx$.

Since $h_{\tau_i}$ is nontrivial on unstable component by assumption (7.4.1), it follows that $h_{\tau_i}$ is not $LieAut(\Sigma_{\tau_i})_0$ invariant. We can then use uniform convexity of distance function and $|\nabla dist^{2\prime}(h_{\mathrm{exact},\zeta,0}(x), h\vartheta^{-1}(x)| \ll 1$ to obtain

$$\left(\frac{\partial^2}{\partial\gamma_{0,a}\partial\gamma_{0,b}}\, meandist_{\tau_i}(\gamma_0(\vartheta,\zeta),h) : a, b\right) > c > 0. \tag{15.6}$$

Namely the symmetric matrix in the left-hand side is uniformly positive. Here we put $\gamma_0 = (\gamma_{0,a}) \in Lie(Aut(\Sigma_{\tau_i}))_0$, by taking a coordinate of $Lie(Aut(\Sigma_{\tau_i}))_0$.

On the other hand we can find $\vartheta$ such that

$$\inf_{\gamma_0 \in \partial Lie(Aut(\Sigma_{\tau_i}))_0} meandist_{\tau_i}(\gamma_0(\vartheta,\zeta),h) - meandist_{\tau_i}(\vartheta,h) > c. \tag{15.7}$$

It follows from (15.6) and (15.7) that we can choose $\Omega_{\tau_i}$ small enough and can prove that there exists unique $\gamma_0 \in Lie(Aut(\Sigma_{\tau_i}))_0$ such that $meandist_{\tau_i}(\gamma_0(\vartheta,\zeta),h_0)$ is locally minimal. It implies that we can kill this extra parameter $LieAut(\Sigma_{\tau_i})_0$ by requiring

$$meandist_{\tau_i}(\gamma_0(\vartheta,\zeta),h_0) \geqslant meandist_{\tau_i}((\vartheta,\zeta),h_0) \quad \text{for any } \gamma_0 \in Lie(Aut(\Sigma_{\tau_i}))_0. \tag{15.8}$$

We remark that Lemma 13.22 implies the consistency of the condition (15.8). Assuming this additional condition on biholomorphic map $\vartheta : \Sigma_0 \to \Sigma_{\tau_i,\zeta}$, eq. (15.4) has moduli space of correct dimension.

Note that, this way can be arbitrary close to the way we did in Section 12, by choosing $\Omega_{\tau_i}$ and $\;'_\sigma$ small. In fact we require in Section 12 that $\exp^{-1}_{h_{\tau_i}(x)}(h_{\sigma,\zeta}\mu^{-1}(x))$ is perpendicular to $LieAut(\Sigma_\sigma)_0$. This condition is asymptotically equal to (15.8). (We do not make it precise since we do not need it.)

We next remark that in a way similar to Sections 12–14, we can construct the family of solutions (15.4) with condition (15.8) as follows.

We start with a family of approximate solutions parametrized by $\;'_\sigma$ in exactly the same way as in Section 12. We next use the product $\;_i\, Lie(Aut(\Sigma_{\tau_i}))_0$ to parametrize the maps $\vartheta_i : \Sigma_{\sigma,\zeta} \to \Sigma_{\tau_i,\zeta_i}$. We then apply the implicit function theorem in the same way as in Sections 12 and 13, to find a solution of (15.4) parametrized by $\;'_\sigma \times \;_i\, Lie(Aut(\Sigma_{\tau_i}))_0$. Let us denote it by $h_{\zeta,u,(\vartheta_i,\zeta_i)}$. Now on this family we consider the condition (15.8). By (15.6) we have

$$\left(\frac{\partial^2}{\partial\gamma_{0,a}\partial\gamma_{0,b}} meandist_{\tau_i}(\gamma_0(\vartheta_i,\zeta_i),h_{\zeta,u,(\vartheta_i,\zeta_i)}) : a, b\right) > c > 0. \tag{15.9}$$

Furthermore, by (15.7) we have

$$\inf_{\gamma_0 \in \partial Lie(Aut(\Sigma_{\tau_i}))_0} meandist_{\tau_i}(\gamma_0(\vartheta_i,\zeta_i),h_{\zeta,u,(\gamma_0(\vartheta_i,\zeta_i))}) - meandist_{\tau_i}((\vartheta_i,\zeta_i),h_{\zeta,u,(\vartheta_i,\zeta_i)}) > c. \tag{15.10}$$

On the other hand by taking $\;'_\sigma$ small we may assume that

$$\left|\frac{\partial}{\partial\gamma_{0,a}} meandist_{\tau_i}(\vartheta_i, h_{\zeta,u,(\gamma_0(\vartheta_i,\zeta_i))})\right| <$$

since the approximate solution we start with is independent of $(\vartheta_i,\zeta_i)$. Here $\varepsilon$ is a number sufficiently small compared to $c$ in (15.9) and (15.10). It follows from (15.9), (15.10) and (15.11) that we can find submanifold $\;_\sigma \subseteq \;'_\sigma \times \;_i\, Lie(Aut(\Sigma_{\tau_i}))_0$ parametrizing the solution satisfying (15.4) with condition (15.8) and that $\;_\sigma \to \;'_\sigma$ is a diffeomorphism, if we replace

$'_{\sigma}$ by a smaller one if necessary. (We remark that (15.8) is written as

$$\frac{\partial}{\partial \gamma_{0,a}} meandist_{\tau_i}(\gamma_0(\vartheta_i, \zeta_i), h_{\zeta,u,(\vartheta_i)}) = 0.)$$

Thus we obtain $\ _{\sigma}$, and $h_{\zeta,u}: \Sigma_\zeta \to M$ for $(\zeta, u) \in \ _{\sigma}$ such that $h_{\zeta,u}$ solves (15.4) with condition (15.8).

Lemma 15.12. *There exists $\varepsilon > 0$, with the following properties. If $(h, \Sigma)$ solves* (15.4) *with condition* (15.8) *and if there exist $\zeta \in \ _{\sigma,\mathrm{deform}} \times \ _{\sigma,\mathrm{resolve}}$ and a holomorphic map $\mu: \Sigma \to \Sigma_\zeta$ such that*

$$\sup_p dist(h_{\mathrm{exact},\zeta,0}(p), h\mu(p)) < \varepsilon.$$

*Then there exists $(\zeta', u') \in \ _{\sigma}$ and $\mu: \Sigma \to \Sigma_{\zeta'}$ such that $h = h_{\zeta,u} \circ \mu$. The pair $(\zeta', u') \in \ _{\sigma}$ is unique up to the action of $Aut(\sigma)$.*

The proof is the same as the argument of Section 14.

We next construct the coordinate change. The key observation is that neither eq. (15.4) nor the condition (15.8) depends on $\sigma$.

We require that $U_\sigma = \ _{\sigma}/Aut(\sigma)$ satisfies the following:

(15.13) If $\rho \in \operatorname{Im} \psi'_\sigma$ and if $\rho \in \Omega_{\tau_i}$ then $\sigma \in \Omega_{\tau_i}$.

We can assume (15.13), since $\Omega_{\tau_i}$ is closed.

We consider $\rho \in \operatorname{Im} \psi_\sigma \cap \mathscr{M}_{g,m}(M, J, \beta)(T, g_v, \beta_v, o)$. (15.13) implies $E_p \subseteq E_\sigma$. We then conclude that if $(\Sigma, h)$ solves (15.4) for $\rho$ with condition (15.8), then it solves (15.4) for $\sigma$ with condition (15.8).

Therefore by using Lemma 15.12, we find the required embeddings $\varphi_{\sigma\rho}: \ _{\rho} \to \ _{\sigma}$ and $\hat{\varphi}_{\sigma\rho}$ in Definition 5.1. Properties (5.1.4)–(5.1.7) are immediate from construction. Thus, we have constructed a Kuranishi structure on a neighborhood of the stratum $\mathscr{M}_{g,m}(M, J, \beta)(T, g_v, \beta_v, o)$.

Now we are going to construct a Kuranishi structure by an induction on the partial order $\prec$. Namely we assume that we have glued the charts and constructed a Kuranishi structure on a union of neighborhoods of $\mathscr{M}_{g,m}(M, J, \beta)(T', g'_v, \beta'_v, o')$ with $(T', g'_v, \beta'_v, o') \prec (T, g_v, \beta_v, o)$ and are going to construct a Kuranishi structure on a neighborhood of $\mathscr{M}_{g,m}(M, J, \beta)(T, g_v, \beta_v, o)$.

By induction hypothesis, Theorem 11.1 and by Proposition 12.25, we have finitely many $\tau_i$ contained in some $\mathscr{M}_{g,m}(M, J, \beta)(T', g'_v, \beta'_v, o')$ with $(T', g'_v, \beta'_v, o') \prec (T, g_v, \beta_v, o)$ and maps $\psi'_{\tau_i}: s_{\tau_i}^{-1}(0) \to C\mathscr{M}_{g,m}(M, J, \beta)$ such that $\mathscr{M}_{g,m}(M, J, \beta)(T, g_v, \beta_v, o)$ minus the union of images of $\psi'_{\tau_i}$ is compact. We then choose finitely many $\tau'_i$ on $\mathscr{M}_{g,m}(M, J, \beta)(T, g_v, \beta_v, o)$ such that

$$\bigcup_i \operatorname{Im} \psi'_{\tau_i} \cup \bigcup_i \operatorname{Im} \psi'_{\tau'_i} \supseteq \mathscr{M}_{g,m}(M, J, \beta)(T, g_v, \beta_v, o).$$

Here $\psi'_{\tau'_i}: s_{\tau'_i}^{-1}(0) \to C\mathscr{M}_{g,m}(M, J, \beta)$ is the map constructed by Theorem 12.9. Now we repeat the argument of this section.

Choose closed subset $\Omega_{\tau_i} \subseteq \operatorname{Im} \psi'_{\tau_i}$, $\Omega_{\tau'_i} \subseteq \operatorname{Im} \psi'_{\tau'_i}$ such that its interior cover $\mathscr{M}_{g,m}(M, J, \beta)(T, g_v, \beta_v, o)$. For each $\sigma \in \mathscr{M}_{g,m}(M, J, \beta)(T, g_v, \beta_v, o)$ we put

$$E_\sigma = \bigoplus_{\sigma \in \Omega_{\tau_i}} E_{\tau_i} \oplus \bigoplus_{\sigma \in \Omega_{\tau'_i}} E_{\tau'_i}.$$

Here to identify $E_{\tau_i} \subset L^q(\Sigma_\sigma; h_\sigma^* TM \otimes \Lambda^{0,1}(\Sigma_\sigma))$ we use parallel transport in a similar way. (We remark that we can identify $K_0(\sigma) \cong K_0(\tau_i)$ by a biholomorphic map.) We use this subspace to define an equation similar to (15.4). Condition (15.8) is defined in the same way. Thus by the same argument we used to study the first stratum, we can extend the Kuranishi structure to a neighborhood of $\mathscr{M}_{g,m}(M, J, \beta)(T, g_v, \beta_v, o)$.

Thus the proof of Theorem 7.10 except the construction of stably almost complex structure is complete.

missing: Cocycle condition

Let us turn to the proof of Theorem 7.11. (7.11.1)–(7.11.3) are immediate from construction. Also the differential $D_\sigma \pi_\sigma$ of the projection to the Deligne–Mumford compactification, is surjective by construction. Let us consider $D_\sigma\, ev|_{D_\sigma \pi_\sigma}$, the restriction of the differential of evaluation map to the kernel of $D_\sigma \pi_\sigma$.

We remark that $\ker D_\sigma \pi_\sigma \subseteq T \quad _\sigma$ is identified with the kernel of operator (12.16). If $u \in L_1^p(\Sigma_\sigma; h_\sigma^* TM)$ is an element of $\ker D_\sigma \pi_\sigma$, then

$$D_\sigma\, ev(u) = (u(z_i) : i = 1, \dots, m) \in \quad T_{h_\sigma(z_i)} M.$$

Here $z_1, \dots, z_m$ are marked points. Hence the surjectivity of $D_\sigma\, ev|_{D_\sigma \pi_\sigma}$ is equivalent to the surjectivity of the restriction of the operator $\Pi_{E_\sigma} \circ (D_{h_\sigma} \bar\partial_{\Sigma_\sigma})$ (in (12.6)) to

$$\{u \in L_1^p(\Sigma_\sigma, h_\sigma^* TM) \mid u(z_1) = \cdots = u(z_m) = 0\}.$$

The surjectiviy of it holds if we take $E_\sigma$ is enough large [3]. The proof of Theorem 7.11 is now completed. □

## 16. ORIENTATION

In this section we show that the Kuranishi structure we constructed in Section 15 is stably almost complex. It then follows that it is stably oriented. Hence by Lemma 5.17 it is oriented.

We first prove the following.

Proposition 16.1. *The Kuranishi structure of $C\mathscr{M}_{g,m}(M, J, \beta)$ we constructed has a tangent bundle.*

*Proof.* Let $\sigma, \rho \in C\mathscr{M}_{g,m}(M, J, \beta)$. We assume that $\rho \in \operatorname{Im} \psi_\sigma$. Here $\psi_\sigma : s_\sigma^{-1}(0) \to C\mathscr{M}_{g,m}(M, J, \beta)$. We consider $\tau_i$ as in Section 15 such that

$$E_\sigma = \bigoplus_{\sigma \in \Omega_{\tau_i}} E_{\tau_i}, \qquad E_\rho = \bigoplus_{\rho \in \Omega_{\tau_i}} E_{\tau_i}.$$

We may assume that $\rho \in \Omega_{\tau_i} \;\Rightarrow\; \sigma \in \Omega_{\tau_i}$. Therefore we have $E_\rho \subseteq E_\sigma$. We recall that the Kuranishi neighborhood $U_\sigma = \quad _\sigma/\Gamma_\sigma$ of $\sigma$ is by definition the set of $(\zeta, h)$ such that

$$\bar\partial_\zeta h \equiv 0 \mod E_\sigma$$

divided by $Aut(\sigma)$ and the Kuranishi neighborhood $U_\rho = \quad _\rho/\Gamma_\rho$ of $\rho$ is the set of $(\zeta, h)$ such that

$$\bar\partial_\zeta h \equiv 0 \qquad \mod E_\rho$$

divided by $Aut(\rho)$. Since $E_\rho \subseteq E_\sigma$ we have ${}_\rho \subseteq {}_\sigma$. This is the map $\varphi_{\rho\sigma}$ in the definition of Kuranishi structure. Now we take a point $(\zeta, h) \in {}_\rho$. By construction the map

$$\Pi_{E_\rho} \circ (D_h \bar{\partial}_{\Sigma_\zeta}) : L_1^p(\Sigma_\zeta : h^*TM) \to \frac{L^p(\Sigma_\zeta : h^*TM \otimes \Lambda^{0,1}(\Sigma_\zeta))}{E_\rho}$$

is surjective and the direct sum

$$\frac{\ker(\Pi_{E_\rho} \circ (D_h \bar{\partial}_{\Sigma_\zeta}))}{Lie(Aut(\Sigma_\rho))} \oplus T_{(\zeta,h)}( {}_{\rho,\mathrm{deform}} \times {}_{\rho,\mathrm{resolve}})$$

is the tangent space $T_{(\zeta,h)} {}_\rho$. Similarly

$$T_{(\zeta,h)} {}_\sigma = \frac{\ker(\Pi_{E_\sigma} \circ (D_h \bar{\partial}_{\Sigma_\zeta}))}{Lie(Aut(\Sigma_\sigma))} \oplus T_{(\zeta,h)}( {}_{\sigma,\mathrm{deform}} \times {}_{\sigma,\mathrm{resolve}}).$$

We remark that

$$T_{\pi[\Sigma_\sigma]} C\mathcal{M}_{g,m} = \frac{T_{(\zeta,h)}( {}_{\sigma,\mathrm{deform}} \times {}_{\sigma,\mathrm{resolve}})}{Lie(Aut(\Sigma_\sigma))} \quad \text{if } 2g + m \geqslant 3$$

$$\frac{Lie(Aut(\Sigma_\sigma))}{\bigoplus_{v:\text{stable component of } \Sigma_\sigma} T_{\pi[\Sigma_{\sigma,v}]} C\mathcal{M}_{g_v,m_v} \oplus \bigoplus_{x:\text{ singular point of } \Sigma_\sigma} T_{x_v}\Sigma_{\sigma,v} \otimes T_{x_w}\Sigma_{\sigma,w}}$$

$$\cong \frac{Lie(Aut(\Sigma_\sigma))}{T_{(\zeta,h)}( {}_{\sigma,\mathrm{deform}} \times {}_{\sigma,\mathrm{resolve}})} \quad \text{if } 2g + m < 3.$$

Hence the restriction of $\Pi_{E_\rho} \circ (D_h \bar{\partial}_{\Sigma_\zeta})$ to $\ker \Pi_{E_\sigma} \circ (D_h \bar{\partial}_{\Sigma_\zeta})$ induces a surjective map

$$\ker \Pi_{E_\sigma} \circ (D_h \bar{\partial}_{\Sigma_\zeta}) \to E_\sigma / E_\rho. \tag{16.4}$$

The kernel of the map (16.4) is $\ker \Pi_{E_\rho} \circ (D_h \bar{\partial}_{\Sigma_\zeta})$. Thus we obtain a required isomorphism

$$\Phi_{\sigma\rho} : \frac{T_{(\zeta,h)} {}_\sigma}{T_{(\zeta,h)} {}_\rho} \to \frac{E_\sigma}{E_\rho}.$$

The commutativity of Diagram 5.7 is immediate from construction. The proof of Proposition 16.1 is complete. □

PROPOSITION 16.5. *The Kuranishi structure of $C\mathcal{M}_{g,m}(M, J, \beta)$ we constructed is stably almost complex.*

*Proof.* The proof uses family of indices and is based on the fact that symbol of our elliptic complex is complex linear.

We first cover $C\mathcal{M}_{g,m}(M, J, \beta)$ by finitely many charts $\psi_{\tau_i} : s_{\tau_i}^{-1}(0) \to C\mathcal{M}_{g,m}(M, J, \beta)$. For each $(\Sigma_\zeta, h) \in {}_{\tau_i}$, we have an operator

$$(D_h \bar{\partial}_{\Sigma_\zeta}) : L_1^p(\Sigma_\zeta : h^*TM) \to L^p(\Sigma_\zeta : h^*TM \otimes \Lambda^{0,1}(\Sigma_\zeta)).$$

The operator $(D_h \bar{\partial}_{\Sigma_\zeta})$ is not necessary complex linear since the complex structure on $M$ is not necessary integrable. However the symbol of $(D_h \bar{\partial}_{\Sigma_\zeta})$ is complex linear. We divide $(D_h \bar{\partial}_{\Sigma_\zeta})$ to complex linear part and anti linear part, then the complex linear part is again a Fredholm operator. We denote it by $(D_h \bar{\partial}_{\Sigma_\zeta})'$.

By construction $(D_h\bar{\partial}_{\Sigma_\zeta})'$ is glued on the part where charts are glued by $\varphi_{\sigma\rho}$. We construct a family of operators parametrized by $_{\tau}\times[-1,1]$ by

$$P_{\zeta,h,t} = t(D_h\bar{\partial}_{\Sigma_\zeta}) + (1-t)(D_h\bar{\partial}_{\Sigma_\zeta})'.$$

For each $\tau$ we find $F_\tau \subseteq L^p(\Sigma_\tau : h_\tau^* TM \otimes \Lambda^{0,1}(\Sigma_\tau))$ such that

$$\Pi_{F_\tau} \circ P_{\zeta,h,\tau} : L_1^p(\Sigma_\zeta : h^*TM) \to \frac{L^p(\Sigma_\zeta : h^*TM \otimes \Lambda^{0,1}(\Sigma_\zeta))}{F_\tau}$$

is surjective for every $(\zeta, h)$ and $t$. We assume also that $F_\tau$ is a complex linear subspace of finite dimension, that is invariant by $Aut(\tau)$, and that its element is a smooth section supported in $K_0(\tau)$. Here we embed $F_\tau \subseteq L^p(\Sigma_\zeta : h^*TM \otimes \Lambda^{0,1}(\Sigma_\zeta))$ in a way similar to Sections 12, 13 and 15. We may assume that $E_r \subseteq F_\tau$. Using $F_\tau$ we define

$$F_\sigma = \bigoplus_{\sigma\in\Omega_\tau} F_\tau$$

in a similar way to Section 15.

Now we construct a bundle system on the space $C\mathcal{M}_{g,m}(M, J, \beta) \times [0,1]$ as follows. On the charts $_{\sigma}\times[-1,1]$ we take two orbibundles

$$F_{1,\sigma}(\zeta, h, t) = \ker\left(\Pi P_{\zeta,h,\tau} : L_1^p(\Sigma_\zeta : h^*TM) \to \frac{L^p(\Sigma_\zeta : h^*TM \otimes \Lambda^{0,1}(\Sigma_\zeta))}{F_\sigma}\right) \subseteq L_1^p(\Sigma_\zeta : h^*TM)$$
$$F_{2,\sigma}(\zeta, h, t) = F_\sigma.$$

We can construct the isomorphisms in Definition 5.6 (which defines the notion that Kuranishi structure has a tangent bundle) in the same way as the proof of Proposition 16.1. Hence we obtain a bundle system, which we write $Index\ P_{\zeta,h,t}$. Since the index is of constant rank when one moves $t$, one can use homotopy lifting property of usual vector bundle to show that the restriction of $Index\ P_{\zeta,h,t}$ to $C\mathcal{M}_{g,m}(M, J, \beta) \times \{0\}$ is isomorphic to the restriction of $Index\ P_{\zeta,h,t}$ to $C\mathcal{M}_{g,m}(M, J, \beta) \times \{1\}$.

On the other hand, since $(D_h\bar{\partial}_{\Sigma_\zeta})'$ is complex linear it follows from definition that the restriction of $Index\ P_{\zeta,h,t}$ to $C\mathcal{M}_{g,m}(M, J, \beta) \times \{0\}$ is a complex bundle system. Hence using the fact that $TC\mathcal{M}_{g,m}$ is a complex orbibundle and the following Lemma 16.6 we can prove Proposition 16.5.

LEMMA 16.6.

$$[Index\ P_{\zeta,h,\tau}]|_{C\mathcal{M}_{g,m}(M,J,\beta)\times\{0\}} + \pi^*[TC\mathcal{M}_{g,m}] \cong [TC\mathcal{M}_{g,m}(M, J, \beta)]$$

*in* $KO(C\mathcal{M}_{g,m}(M, J, \beta))$ *if* $2g + m \geqslant 3$ *and*

$$[Index\ P_{\zeta,h,t}]|_{C\mathcal{M}_{g,m}(M,J,\beta)\times\{0\}} \cong [TC\mathcal{M}_{g,m}(M, J, \beta)]$$

*if* $2g + m < 3$.

$$F_{1,\sigma}(\zeta, h, 0) = \ker\left(\Pi_{F_\sigma} \circ \bar{\partial}_{\zeta,h} : L_1^p(\Sigma_\zeta : h^*TM) \to \frac{L^p(\Sigma_\zeta : h^*TM \otimes \Lambda^{0,1}(\Sigma_\zeta))}{F_\sigma}\right),$$

we can construct an isomorphism

$$\frac{F_{1,\sigma}(\zeta,h,0)}{\ker(\Pi_{E_\sigma}\circ(D_{h_\sigma}\bar\partial_{\Sigma_\zeta}))}\cong\frac{F_\sigma}{E_\sigma},\quad T_{(\zeta,h)}\ \ _\sigma\cong\ker(\Pi_{E_\sigma}\circ(D_{h_\sigma}\bar\partial_{\Sigma_\zeta}))\oplus\pi^*[TC\mathscr{M}_{g,m}]$$

in the same way as the proof of Proposition 16.1. It follows that we have an isomorphism

$$\pi^*TC\mathscr{M}_{g,m}\oplus \mathit{Index}\ P_{\zeta,h,t}|_{C\mathscr{M}_{g,m}(M,J,\beta)\times\{0\}}\cong TC\mathscr{M}_{g,m}(M,J,\beta)\oplus\bigoplus_i\left(\frac{F_{\tau_i}}{E_{\tau_i}},\frac{F_{\tau_i}}{E_{\tau_i}}\right).$$

Here

$$\bigoplus_i\left(\frac{F_{\tau_i}}{E_{\tau_i}},\frac{F_{\tau_i}}{E_{\tau_i}}\right)$$

is a bundle system $(G_{1,\sigma},G_{2,\sigma})$ such that if $\{i_1,\dots,i_N\}=\{i\mid\sigma\in\Omega_{\tau_i}\}$ then

$$(G_{1,\sigma},G_{2,\sigma})=\bigoplus_{j=1}^{N}\left(\frac{F_{\tau_{i(j)}}}{E_{\tau_{i(j)}}},\frac{F_{\tau_{i(j)}}}{E_{\tau_{i(j)}}}\right).$$

Since

$$\bigoplus_i\left(\frac{F_{\tau_i}}{E_{\tau_i}},\frac{F_{\tau_i}}{E_{\tau_i}}\right)$$

is a trivial bundle system by Definition 5.9 it follows that

$$[\mathit{Index}\ P_{\zeta,h,\tau}]|_{C\mathscr{M}_{g,m}(M,J,\beta)\times\{0\}}+\pi^*[TC\mathscr{M}_{g,m}]\cong[TC\mathscr{M}_{g,m}(M,J,\beta)]$$

in $KO(C\mathscr{M}_{g,m}(M,J,\beta))$, as required.

The case $2g+m<3$ is similar. □

## CHAPTER 4: APPLICATIONS

### 17. GROMOV–WITTEN INVARIANT

In this section, we use results of Chapters 1–3 to construct Gromov–Witten invariant for arbitrary symplectic manifold. We first extend our definition of Kuranishi structure to Kuranishi structure with boundary.

*Definition* 17.1. *A local model of n-dimensional orbifold with boundary* is a pair ( , $\Gamma$) where $\Gamma$ is a finite group which has a linear representation to $\mathbf{R}^{n-1}$ or $\mathbf{R}^n$, and  is a $\Gamma$ invariant open neighborhood of $\mathbf{0}$ in $[0,\infty)\times\mathbf{R}^{n-1}$ or $\mathbf{R}^n$. We assume that the action of $\Gamma$ on  is effective.

*Definition* 17.2. Let $X$ be a compact metric space. An *n-dimensional orbifold structure with boundary* on $X$ is an open covering $X=\bigcup_i U_i$, local models ( $_i,\Gamma_i$) of $n$-dimensional orbifold with boundary for each $i$, and homeomorphisms $\phi_i$:  $_i/\Gamma_i\to U_i$ which satisfy the same properties as Definition 2.2.

We can define an orbibundle on orbifold with boundary in the same way. The definition of orientation of orbifold with boundary is similar. Also we can define a notion of multisection and prove a transversality theorem in the same way.

*Definition* 17.3. A *Kuranishi structure with boundary* of dimension $n$ on $X$ is a collection $(U_p, E_p, s_p, \psi_p, \varphi_{pq}, \hat{\varphi}_{pq})$ for each $p \in X$ such that

(17.3.1) $U_p = \;\; _p/\Gamma_p$ is a germ of orbifold with boundary and $E_p$ is an orbibundle on it.
(17.3.2) $s_p$ is a germ of (single valued) continuous section of $E_p$.
(17.3.3) $\psi_p$ is a germ of homeomorphism from $s_p^{-1}(0)$ to a neighborhood of $p$ in $Y$.
(17.3.4) Let $q \in \psi_p(s_p^{-1}(0))$. Then there exists a germ of an embedding $\varphi_{pq}: U_q \to U_p$ in the category of orbifolds, which is covered by a germ of an embedding of orbibundle $\hat{\varphi}_{pq}: E_q \to E_p$.
(17.3.5) $s_p \circ \varphi_{pq} = \hat{\varphi}_{pq} \circ s_q$, $\psi_p \circ \varphi_{pq} = \psi_q$.
(17.3.6) If $r \in \psi_q(s_q^{-1}(0))$, then $\varphi_{pq} \circ \varphi_{qr} = \varphi_{pr}$, $\hat{\varphi}_{pq} \circ \hat{\varphi}_{qr} = \hat{\varphi}_{pr}$.
(17.3.7) $\dim U_p - \operatorname{rank} E_p = n$ is independent of $p$.

We define orientation and stably almost complex structure for Kuranishi structure with boundary in the same way as in Section 5.

The following relative version of Theorem 6.4 can also be proved in the same way. Let $X = (X, (\;\; _q, \Gamma_q, E_q, s_q, \psi_q))$ be a space with Kuranishi structure with boundary. Let $K \subseteq X$ be a compact space and $K^+$ be its neighborhood. Let $(P, ((U_p, \psi_q, s_q): p \in P), \varphi_{pq}, \hat{\varphi}_{pq})$ be a sufficiently fine good coordinate system of $X$. (Good coordinate system on Kuranishi structure with boundary can be defined in the same way as Definition 6.1.) Put $P(K) = P \cap K^+$.

Lemma 17.4. *Suppose that there exists a sequence of smooth multisections $s_{0,q,n}$ for each $p \in P(K)$ such that*

(17.5.1) $s_{0,p,n} \circ \varphi_{pq} = \hat{\varphi}_{pq} \circ s_{0,q,n}$,
(17.5.2) $\lim_{n\to\infty} s_{0,q,n} = s_q$ *in* $C^0$ *-topology*
(17.5.3) $s_{0,q,n}$ *is transversal to* 0.
(17.5.4) *Let $\varphi_{pq}(x) \in U_p$. Then the restriction of the differential of the composition of any branch of $s_{0,q,n}$ and the projection $E_p \to E_p/E_q$ coincides with the isomorphism $\Phi_{pq}: N_{U_p} U_q \cong E_p/E_q$.*

*Then there exists a sequence of smooth multisections $s_{q,n}$ for each $q \in P$ such that*

(17.6.1) $s_{p,n} \circ \varphi_{pq} = \hat{\varphi}_{pq} \circ s_{q,n}$,
(17.6.2) $\lim_{n\to\infty} s_{q,n} = s_q$ *in* $C^0$*-topology*
(17.6.3) $s_{q,n}$ *is transversal to* 0.
(17.6.4) *Let $\varphi_{pq}(x) \in U_p$. Then the restriction of the differential of the composition of any branch of $s_{q,n}$ and the projection $E_p \to E_p/E_q$ coincides with the isomorphism $\Phi_{pq}: N_{U_p} U_q \cong E_p/E_q$.*
(17.6.5) $s_{q,n} = s_{0,q,n}$ *on $K$ if $q \in P(K)$.*

We can also prove an existence of good coordinate system on $X$ extending a given one on a neighborhood $K$. The proof of it is similar to the proof of Lemma 6.3

Let $X$ be an orbifold with boundary. The boundary of $X$ is by definition the set of all points whose neighborhood is identified to an open subset of $[0, \infty) \times \mathbf{R}^{n-1}/\Gamma$. Let $\partial X$ be the boundary of $X$. If $X$ is an $n$ dimensional orbifold with boundary then $\partial X$ is an $n-1$ dimensional orbifold.

Let $X = (X, (\;\; _q, \Gamma_q, E_q, s_q, \psi_q))$ be a space with Kuranishi structure with boundary.

*Definition* 17.7. Let $p \in X$ and $U_p = {}_p/\Gamma_p$ be its chart where $p$ corresponds to $p' \in U_p$. We say $p \in \partial X$ if $p' \in \partial U_p$.

The following lemma is obvious from definition.

Lemma 17.8. *Let $X = (X, ({}_q, \Gamma_q, E_q, s_q, \psi_q))$ be a space with $n$ dimensional Kuranishi structure with boundary. Then the space $\partial X$ has an $n - 1$ dimensional Kuranishi structure (without boundary) such that $\partial X \times [0,1)$, together with its Kuranishi structure with boundary is diffeomorphic to an open neighborhood $\partial X$ in $X$. If $X$ is oriented (resp stably almost complex) then so is $\partial X$ and the diffeomorphism between $\partial X \times [0,1)$ and open neighborhood of $X$ is orientation preserving (resp. preserving the stably almost complex structures).*

Now we can use these machineries to perform standard cobordism argument and prove the class defined in Section 6 is cobordism invariant. More precisely we consider the following situation. Let $X = (X, ({}_q, \Gamma_q, E_q, s_q, \psi_q))$ be a space with oriented $n$-dimensional Kuranishi structure with boundary. Suppose

$$\partial X = X_1 \cup - X_2.$$

Suppose that $f : X \to Y$ is a strongly continuous map in the sense of Definition 6.6. We write it $f : X \to Y$ for simplicity. It induces $f|_{X_i} : X_i \to Y$.

Lemma 17.9.

$$(f|_{X_1})_*([X_1]) = (f|_{X_2})_*([X_2]) \in H_{n-1}(Y; \mathbf{Q}).$$

*Here $(f|_{X_i})_*([X_i])$ is defined in Section 6.*

*Proof.* We take multisections $s_i$ on $X_i$ such that $s_i^{-1}(0)$ is the fundamental class of $X_i$. We extend it to $X_i \times [0,1)$ such that it is constant in the second factor. We identify them to a neighborhood of $X_i$ in $X$ by using Lemma 17.8. We then apply Lemma 17.4 and extend it to a multisection $s$ on $X$. We use $s^{-1}(0)$ to define a singular $\mathbf{Q}$-chain $f[s^{-1}(0)]$ on $Z$. By the same way as in the proofs of Lemmas 6.11 and 4.7, we find that

$$\partial f[s^{-1}(0)] = f[s_1^{-1}(0)] - f[s_2^{-1}(0)].$$

Now we start the definition of Gromov–Witten invariant. Let $(M, \omega)$ be a compact symplectic manifold and let $J$ be a compatible almost complex structure. For $\beta \in H_2(M; \mathbf{Z})$, let $C\mathscr{M}_{g,m}(M, J, \beta)$ be the moduli space of stable maps of genus $g$, $m$ marked points and of homology class $\beta$ defined in Chapter 2. We constructed a stably almost complex Kuranishi

$$\Pi : C\mathscr{M}_{g,m}(M, J, \beta) \to C\mathscr{M}_{g,m} \times M^m, \qquad (17.10)$$

by

$$\Pi([(\Sigma, z), h]) = ([\Sigma, z], (h(z_1), \ldots, h(z_m))).$$

We remark that the map $C\mathcal{M}_{g,m}(M, J, \beta) \to C\mathcal{M}_{g,m}$ is of maximal rank.

However the Kuranishi structure we constructed may depend on various choices we made. Especially it may depend on the choice of the subspace $E_\sigma$ of $L^p(\Sigma_\sigma; h_\sigma^* TM \otimes \Lambda^{0,1}(\Sigma_\sigma))$, we took in (12.7). Also we fixed and use various partition of unity. We use Lemma 17.9 to show

THEOREM 17.11. *$\Pi_*(C\mathcal{M}_{g,m}(M, J, \beta)) \in H_\mu(C\mathcal{M}_{g,m} \times M^m; \mathbf{Q})$ depends only on $(M, \omega)$, $g, m, \beta$ and is independent of compatible almost complex structure $J$ and various choices we made to define a Kuranishi structure.*

*Proof.* Let $J$, $J'$ be two almost complex structures compatible to the symplectic structure $\omega$. By [33], they are homotopic. So we have a family $J_s$ of compatible almost complex structures such that $J_s = J$ for $s \in [0, \varepsilon]$ and $J_s = J'$ for $s \in [1-\varepsilon, 1]$. We put

$$C\mathcal{M}_{g,m}(M, J_{\mathrm{para}}, \beta) = \bigcup_{s \in [0,1]} \{s\} \times C\mathcal{M}_{g,m}(M, J_s, \beta).$$

In the same way as in Sections 10 and 11, we can define a topology on $C\mathcal{M}_{g,m}(M, J_{\mathrm{para}}, \beta)$ and prove that it is compact and Hausdorff.

We take two choices of $E_\sigma$ and partition of unity, etc. for $s = 0, 1$ and extend it to $[0, \varepsilon] \cup [1-\varepsilon, 1]$ so that it is constant in $s$. It determines Kuranishi structure on $\bigcup_{s \in [0,\varepsilon] \cup [1-\varepsilon, 1]} C\mathcal{M}_{g,m}(M, J_s, \beta)$ such that its restriction to $C\mathcal{M}_{g,m}(M, J_s, \beta)$, $s = 0, 1$, coincides with one which we used to define $\Pi_*(C\mathcal{M}_{g,m}(M, J, \beta))$ for each two choices. Therefore in view of Lemma 17.9 it suffices to show that we can define a Kuranishi structure on $C\mathcal{M}_{g,m}(M, J_{\mathrm{para}}, \beta)$ extending one on $\bigcup_{s \in [0,\varepsilon/2] \cup [1-\varepsilon/2, 1]} C\mathcal{M}_{g,m}(M, J_s, \beta)$.

For the choices other than $E_\sigma$, we can extend it to $[0, 1]$ so that it depends smoothly on $s$. For the choice of $E_\sigma$, we can use a similar procedure as the gluing argument in Section 15 as follows. For each $(\tau, s)$ with $\tau \in C\mathcal{M}_{g,m}(M, J_s, \beta)$ we take $E_{\tau,s}$ satisfying (12.7). Then the same condition ((12.7.1) especially) holds for $(\tau', s')$ if they are sufficiently close to $(\tau, s)$. Thus we can use it to find a chart in each neighborhood. Using compactness we cover $\bigcup_{s \in [2\varepsilon/3, 1-2\varepsilon/3]} C\mathcal{M}_{g,m}(M, J_s, \beta)$ by finitely many such charts.

Then for each $(\sigma, s)$, $\sigma \in C\mathcal{M}_{g,m}(M, J_s, \beta)$ we consider all $(\tau_i, s_i)$ such that $(\sigma, s)$ is contained in the charts centered at $(\tau_i, s_i)$. Then we use the same identification (using exponential map) to regard $E_{\tau_i, s_i}$ as a section on $\Sigma_\sigma$ of $\Lambda^{0,1}\Sigma_\sigma \otimes h_\sigma T^*M$. We take sum of all of them. In case when $s \in [2\varepsilon/3, \varepsilon]$ or $s \in [1-\varepsilon, 1-2\varepsilon/3]$, we add also $E_\sigma$ (the choice we fixed at the beginning). We thus defined $E_{\sigma,s}$. (For $s \in [0, \varepsilon/2]$ or $s \in [1-\varepsilon/2, 1]$ we take $E_\sigma$.)

Now using these choices we can repeat the construction of Chapter 3 to find a Kuranishi structure on $C\mathcal{M}_{g,m}(M, J_{\mathrm{para}}, \beta)$. The proof that it is stably almost complex is the same as that given in Section 16. The proof of Theorem 17.11 is now complete. □

By Theorem 17.11 we have a class $\Pi_*(C\mathcal{M}_{g,m}(M, J, \beta)) \in H_{\dim C\mathcal{M}_{g,m} \times M^m - \mu}(C\mathcal{M}_{g,m} \times M^m; \mathbf{Q})$. It induces a map

$$I^M_{g,m,\beta} : H^*(M, \mathbf{Q})^{\otimes m} \to H^{*+\mu}(C\mathcal{M}_{g,m}, \mathbf{Q})$$

by

$$I^M_{g,m,\beta}(\gamma) = PD(\gamma\backslash\Pi_*(C\mathscr{M}_{g,m}(M,J,\beta)). \tag{17.12}$$

where $PD$ is the Poincaré duality and $\backslash$ is the slant product. Here

$$\mu = 2n(g-1) - 2\beta c_1.$$

*Definition* 17.13. We call the map (17.12) the *Gromov–Witten invariant.*

We study the properties of Gromov–Witten invariant in Section 23.

So far in this section, we assumed $m + 2g - 3 \geqslant 0$. But we can define a Kuranishi structure in the case $m + 2g - 3 < 0$ as well. In fact proof of Chapter 3 works without change. We however need to remark one point.

The main difference between the case $m + 2g - 3 < 0$ and $m + 2g - 3 \geqslant 0$ is that in case $m + 2g - 3 < 0$ our curve $\Sigma$ is not stable for an element of $(\Sigma, h) \in \mathscr{M}_{g,m}(M,J,\beta)$ (that is an element in the main stratum). Moreover, the generic element in the main stratum may have a nontrivial automorphism.

We remark that we assumed that the action of $\Gamma$ is effective in the definition of orbifold. So this assumption may not be satisfied even in the case when the moduli space $\mathscr{M}_{g,m}(M,J,\beta)$ is transversal.

This phenomenon actually happens in the following way. Let us put $M = S^2 \times T^2$ and $\beta = [1 \times T^2]$. By perturbing almost complex structure we may assume that $\mathscr{M}_{1,0}(M,J,\beta)$ consists of two elements which are transversal. We then find that $\mathscr{M}_{1,0}(M,J,k\beta)$ consists of elements of $k$-fold covering of the elements of $\mathscr{M}_{1,0}(M,J,\beta)$ and that $\mathscr{M}_{1,0}(M,J,k\beta)$ is transversal. Then $\mathscr{M}_{1,0}(M,J,k\beta)$ consists of finitely many points. However each point has a nontrivial symmetry. So to find an invariant we need to put the weight $1/\#Aut$.

But we can proceed as follows and perform the constructions of Chapter 3 without change. Suppose that we have a neighborhood ${}_\tau/\Gamma_\tau$ such that the action of $\Gamma_\tau$ is not effective on ${}_\tau$. In that case we change $E_\tau$ as follows. We consider an action of $\Gamma_\tau$ on $C^\infty(\Sigma_\tau;\Lambda^{0,1}(\Sigma_\tau)\otimes h_\tau^* TM)$. This action is effective and hence we can find a finite dimensional subspace $E'_\tau$ of $C^\infty(\Sigma_\tau;\Lambda^{0,1}(\Sigma_\tau)\otimes h_\tau^* T_*M)$ on which the action of $\Gamma_\tau$ is effective. We change our $E_\tau$ by adding $E'_\tau$. Then in the new Kuranishi structure the action of $\Gamma_\tau$ is effective.

So when we consider again the example discussed above we have an obstruction bundle even if $\mathscr{M}_{1,0}(M,J,k\beta)$ is transversal. We do not need to change other part of the argument.

Using this Kuranishi structure and evaluation map

$$ev: C\mathscr{M}_{0,m}(M,J,\beta) \to M^m$$

for $m = 1,2$, we find an element

$$ev_*([C\mathscr{M}_{0,m}(M,J,\beta)]) \in H_{2m+\beta c_1+2(n-3)}(M^m;\mathbf{Q}) \tag{17.14}$$

which depends only on $(M,\omega)$ and $\beta$ and is independent of $J$ and various other choices involved. If $g = 0$ and $m = 0$, there is no evaluation map either. So the invariant counting the order only makes sense. Hence in the case $g = 0$ and

$$2\beta c_1 - 2(3-n) = 2\beta c_1 + 2(3-n)(g-1) = 0, \tag{17.15}$$

we have a rational number

$$[\mathcal{CM}_{0,0}(M,J,\beta)] \in \mathbf{Q} \tag{17.16}$$

as an invariant of $(M,\omega)$ and $\beta$. One important case where (17.15) holds is $c_1 = 0$ and $n = 3$. This the counting problem of rational curves in Calabi–Yau 3 fold.

We finally consider the case when $g = 1$, $m = 0$. In this case the space $\mathcal{CM}_{1,0}$ is a (real) 2 dimensional orbifold, which is homeomorphic to $S^2$. Hence we have an invariant in the case when

$$2\beta c_1 = 2\beta c_1 + 2(3-n)(g-1) = 0 \text{ or } 2.$$

In the case $2\beta c_1 = 2$ the invariant is

$$[\mathcal{CM}_{1,0}(M,J,\beta)] \cap \Pi^*[\mathcal{CM}_{1,0}] \in \mathbf{Q}.$$

This is (in the case everything is transversal) the number of pseudoholomorphic $T^2$ with fixed complex structure.

In the case, $2\beta c_1 = 0$. Invariant is

$$[\mathcal{CM}_{1,0}(M,J,\beta)] \in \mathbf{Q}.$$

This is (in the case everything is transversal) the number of pseudoholomorphic $T^2$ with arbitrary complex struture.

The case $\beta c_1 = 0$, $g = 1$, $m = 0$ appeared in Taubes' work on the relation between Seiberg–Witten and Gromov–Witten invariants. Taubes [69] gave an argument how to handle this case (when $n = 2$). There is one difference between the number he defined and ours. That is we counted only connected curves in a given homology class, while Taubes counts disconnected one also. It seems that in case we count only connected one, rational number (which is not an integer) appeared. Taubes' counting always gives an integer. It seems interesting for us to know the reason why after summing up various contributions from various components (which corresponds to the number of connected components) finally gives an integer. It seems likely that our invariant (17.16) coincides with Taubes' if we take into account the difference we mentioned above appropriately. However we have not checked it yet.

## 18. REVIEW OF FLOER HOMOLOGY

In this section, we summarize the theory of Floer homology for periodic hamiltonian system [19, 35, 52, 62]. The result stated in this section is not new, we refer to [19, 35, 45, 52, 62] for their proofs.

Let $(M,\omega)$ be a $2n$-dimensional closed symplectic manifold and $H: M \times S^1 \to \mathbf{R}$ a smooth Hamiltonian. We identify $S^1$ with $\mathbf{R}/\mathbf{Z}$. We write $H_t(x) =$
vector fields $X_{H_t}$ of $H_t$ by

$$dH_t = i(X_{H_t})\,\omega.$$

We call a family of vector field $X_{H_t}$ the *hamiltonian system.*

*Definition* 18.1. We call a map $\ell: S^1 \to M$ to be a *1-periodic solution* of the hamiltonian system $X_{H_t}$ if $\ell$ satisfies the following equation.

$$\frac{d\ell}{dt} + X_{H_t}(\ell(t)) = 0. \tag{18.1}$$

Let $\phi_t : M \to M$ be the one parameter group of transformations associated with $-X_{H_t}$. We put $\phi = \phi_1$ and call it the *time-one map*. There is a one to one correspondence between 1-periodic solution of the hamiltonian system and fixed point set of the time-one map. (Namely we associate $\ell(0)$ to $\ell$.)

*Definition* 18.2. A 1-periodic solution is called *non-degenerate* if 1 is not an eigenvalue of $d\phi$ on $T_{\ell(0)}M$.

Arnold [1, 2] conjectured that the number of 1-periodic solutions of a periodic hamiltonian system is at least the smallest number of critical points of smooth functions on $M$. In case when all 1-periodic solutions of the system are nondegenerate, the conjecture also states that the number of 1 periodic solutions of the system is at least the smallest number of critical points of Morse functions on $M$.

Morse theory tells us that the number of critical points of a Morse function is at least the sum of Betti numbers and torsion numbers. In this paper, we study the estimate by the Betti numbers (i.e. the rank of homology group over $\mathbf{Q}$).

From now on, we call 1 periodic solutions as periodic solutions, and denote by $\mathscr{P}(H)$ the set of all contractible periodic solutions of (18.1). Equation (18.1) can be considered as the Euler–Lagrange equation of a functional $\mathscr{A}_H$ on a covering space of the space $LM$ of contractible loops in $M$, which we shall explain below. Write

$$\tilde{L}(M) = \left\{ (x,u) \,\middle|\, \begin{array}{l} x \in L(M), \\ u : D^2 \to M \\ x = u|_{\partial D^2} \end{array} \right\} \Big/ \sim$$

$$(x,u) \sim (y,v) \Leftrightarrow \begin{cases} x = y \\ \int_D u_2^* \omega = \int_D v_2^* \omega \\ \int_{D^2} u^* c_1 = \int_{D^2} \gamma^* c_1 \end{cases}$$

The covering transformation group of $\tilde{L}(M) \to L(M)$ is

$$\Gamma = \frac{\pi_2(M)}{\ker \phi_{c_1} \cap \ker \phi_\omega},$$

where $\phi_{c_1} : \pi_2(M) \to \mathbf{R}$ and $\phi_\omega : \pi_2(M) \to \mathbf{R}$ are evaluation maps of $c_1$ and $\omega$. The action functional is defined by

$$\mathscr{A}_H(x,u) = -\int_{D^2} u^* \omega + \int_0^1 H(x(t),t)\, dt.$$

Floer initiated an analog of Morse theory for the action functional $\mathscr{A}_H$.

First of all, we formally compute the equation for gradient flow lines. Let $J$ be an almost complex structure compatible with $\omega$. Then the Riemann metric $g_J$ defined by $g_J(\mathbf{v},\mathbf{w}) = \omega(\mathbf{v}, J\mathbf{w})$ induces $L^2$ inner product on the tangent space of $LM$, so does on $\tilde{L}M$. A curve $\gamma : \mathbf{R} \to LM$ is identified with a map $h : \mathbf{R} \times S^1 \to M$. Under this identification, the

$$\frac{\partial h}{\partial \tau} + J(h)\frac{\partial h}{\partial t} + \nabla H_t(h) = 0, \tag{18.3}$$

where $\nabla H_t$ is the gradient vector field of $H_t$ with respect to $g_J$. Here and hereafter we use $\tau$ for the coordinate of $\mathbf{R}$, and $t$ for the coordinate of $S^1$.

*Definition* 18.4. A map $h:\mathbf{R}\times S^1 \to M$ solving (18.3) is called the *connecting orbits.*

*Definition* 18.5. The *energy* $E_H(h)$ of $h:\mathbf{R}\times S^1 \to M$ is defined by

$$E_H(h) = \frac{1}{2}\int_{-\infty}^{\infty}\int_0^1 \left\{\left|\frac{\partial h}{\partial \tau}\right|^2 + \left|\frac{\partial h}{\partial t} + X_{H_t}\right|^2\right\} dt\, d\tau$$

Let $h_i$ be a sequence of connecting orbits with $E_H(h_i) < C$ for some constant $C$. An argument similar to the proof of Gromov's compactness for pseudoholomorphic curves implies that there is a subsequence, also denoted by $\{h_i\}$, which coverges, outside of a finite set of points in $\mathbf{R}\times S^1$, to a connecting orbit $h_\infty$ locally uniformally. The bubbling-off is analyzed in the same way as the case of pseudoholomorphic curves and we possibly get bubble tree. (The phenomenon of splitting into several connecting orbits will be discussed later in this chapter.)

Using the above argument, one can prove the following:

THEOREM 18.6. *The following two conditions on connecting orbit h are equivalent.*

(18.6.1) $E_H(h) < \infty$.

(18.6.2) *There are* $(\ell^\pm, u^\pm)\in\tilde{L}M$ *with* $\ell^\pm$ *being solutions of* (18.1) *such that*

$$\lim_{\tau\to\infty} h(\tau,t) = \ell^\pm(t)$$

*and* $(\ell^+, u^+) \sim (\ell^+, u^- \# h) \bmod \ker\phi_{c_1} \cap \ker\phi_\omega$. *Here* $u^-\#h: D^2 \to M$ *is a map obtained by gluing* $u^-: D^2 \to M$ *and* $h:\mathbf{R}\times S^1 \to M$ *along* $\ell^- = u|_{\partial D^2} = h|_{\{-\infty\}\times S^1}$.

We denote by $\tilde{\mathscr{P}}(H) \subseteq \tilde{L}M$ the inverse image of $\mathscr{P}(H)$, the set of all contractible periodic solutions of (18.1). $\tilde{\mathscr{P}}(H)$ is the set of critical points of the functional $\mathscr{A}_H$. We have an action of $\Gamma$ on $\tilde{\mathscr{P}}(H)$ by $(A, \tilde{\ell}) \mapsto u\#\tilde{\ell}$, where $[u] = A$ and $\#$ is as in Theorem 18.6.

*Definition* 18.7. Let $\tilde{\ell}^\pm = (\tilde{\ell}^\pm, u^\pm)\in\tilde{\mathscr{P}})(H)$. We denote by $\tilde{\mathscr{M}}(\tilde{\ell}^-, \tilde{\ell}^+)$ the set of all connecting orbits $h$ satisfying

$$\lim_{\tau\to\pm\infty} h(\tau,t) = \ell^\pm(t) \tag{18.7.1}$$

$$h\#\tilde{\ell}^- \sim \tilde{\ell}^+ \bmod \ker\phi_{c_1} \cap \ker\phi_\omega. \tag{18.7.2}$$

Since connecting orbits are solutions of (18.3), which is an equation of gradient line for $\mathscr{A}_H$, we have

$$E_H(h) = \mathscr{A}_H(\tilde{\ell}^-) - \mathscr{A}_H(\tilde{\ell}^+) \tag{18.8}$$

if $h\in\tilde{\mathscr{M}}(\tilde{\ell}^-, \tilde{\ell}^+)$. We remark that the right-hand side depends only on $\tilde{\ell}^-, \tilde{\ell}^+$ and is

We remark also that, since eq. (18.3) is invariant by the translation symmetry in $\tau$ variable, it follows that $\mathbf{R}$ acts on $\tilde{\mathscr{M}}(\tilde{\ell}^-, \tilde{\ell}^+)$. This action is free if $\tilde{\ell}^- \neq \tilde{\ell}^+$.

We are going to review the definition of Floer chain complex. For this purpose we introduce a completion of the group ring of $\Gamma$, with respect to the homomorphism $\phi_\omega : \pi_2(M) \to \mathbf{R}$. The ring we obtain by completion is called the *Novikov ring*, since it is first introduced by Novikov [51].

*Definition* 18.9. The Novikov ring $\Lambda$ is the set of all formal sums $\sum_{A\in\Gamma} \lambda_A \delta_A$ with $\lambda_A \in \mathbf{Z}$, which satisfies the following conditions

For any $c \in \mathbf{R}$ the set

$$\{A \in \Gamma \mid \lambda_A \neq 0, \quad \phi_\omega(A) < c\}$$

is finite. The sum in the coefficients of

$$\lambda_1 \lambda_2 = \sum_{A\in\Gamma} \left( \sum_{B\in\Gamma} \lambda_{1,B} \lambda_{2,A-B} \right) \delta_A \tag{18.10}$$

is a finite sum and is well defined. One can also check that $\lambda_1\lambda_2 \in \Lambda$. Thus $\Lambda$ has a structure of ring. The Floer chain complex we are going to define is a chain complex with coefficient in $\Lambda$.

The Floer chain complex is analogous of Morse's (or Witten's [72]) complex associated to our functional $\mathscr{A}_H$. Hence the set of generators of it (as an abelian group) is the set of critical points of $\mathscr{A}_H$, which is identified with $\tilde{\mathscr{P}}(H)$. The definition of degree of each critical point is different from finite dimensional case. Since both the numbers of positive eigenvalues and negative eigenvalues of the Hessian operator at a critical point of $\mathscr{A}_H$ are infinite, the definition in the finite dimensional case, which uses Morse index does not make sense in the case of $\mathscr{A}_H$. However one can make sense the difference of Morse index of two critical points $\tilde{\ell}, \tilde{\ell}' \in \tilde{\mathscr{P}}(H)$, by taking the index of spectral flow of a certain 1-parameter family of ordinary differential opeators on $S^1$. In fact, there is a map $\mu : \tilde{\mathscr{P}}(H) \to \mathbf{Z}$, so-called Conley–Zehnder index. (See [10, 19, 62].)

Now we are ready to introduce Floer's chain complex.

*Definition* 18.11. $C_k = C_k(H, J)$ is the set of all formal sums

$$\sum_{\substack{\tilde{\ell}\in\tilde{\mathscr{P}}(\Lambda)\\ \mu(\tilde{\ell})=k}} \xi(\tilde{\ell}) \delta_{\tilde{\ell}}$$

satisfying the following conditions.

For each $c \in \mathbf{R}$, the set

$$\{\tilde{\ell} \in \tilde{\mathscr{P}}(H) \mid \xi(\tilde{\ell}) \neq 0, \mathscr{A}_H(\tilde{\ell}) > c\}$$

is finite.

Let $\lambda = \sum_{A\in\Gamma} \lambda_A \delta_A \in \Lambda$ and

$$\xi = \sum_{\substack{\tilde{\ell}\in\tilde{\mathscr{P}}(\Lambda)\\ \mu(\tilde{\ell})=k}} \xi(\tilde{\ell}) \delta_{\tilde{\ell}} \in C_k(H, J),$$

$$\lambda\xi = \sum_{\tilde{\ell}\in\tilde{\mathscr{P}}(H)} \left( \sum_{A\in\Gamma} \lambda_A \xi_{[-A]\#\tilde{\ell}} \right) \delta_{\tilde{\ell}}$$

is well defined and is an element of $C_k(H,J)$. (We can prove it by using (18.8).) Hence $C_k(H,J)$ is a module over $\Lambda$.

The rough idea to define a boundary operator $\partial : C_k \to C_{k-1}$ is to count the order of the quotient space $\mathcal{M}(\tilde{\ell}^-, \tilde{\ell}^+) = \tilde{\mathcal{M}}(\tilde{\ell}^-, \tilde{\ell}^+)/\mathbf{R}$ and put

$$\delta_{\tilde{\ell}} = \sum_{\tilde{\ell}} \# \mathcal{M}(\tilde{\ell}, \tilde{\ell}') \cdot \delta_{\tilde{\ell}'}.$$

We will discuss it in more detail later.

In the case when $(M, \omega)$ is semi-positive (or weakly monotone), the strategy explained above was carried out rigorously in [19, 35]. There it was also shown that $\partial \circ \partial = 0$.

Thus we obtain a chain complex $(C_*(H,J), \partial)$. The homology of this chain complex is called *Floer homology* and is denoted by $HF_*(H,J)$.

It is proved also in [9, 35] that the homology $HF_*(H,J)$ is invariant under the deformation of $(H,J)$. The idea of the proof was as follows. We study the $\tau$-dependent analogue of eq. (18.3). Namely, let $(H_\tau, J_\tau)$ be a smooth 1 parameter family of pairs $H_\tau : M \times S^1 \to \mathbf{R}$ and almost complex structures $J_\tau$ compatible with $\omega$. We assume that there exists $R > 0$ such that

$$\begin{aligned} (H_\tau, J_\tau) &= (H_\alpha, J_\alpha) \quad \text{for } \tau < -R \\ (H_\tau, J_\tau) &= (H_\beta, J_\beta) \quad \text{for } \tau > +R \end{aligned} \tag{18.12}$$

where $\alpha, \beta$ are independent of $\tau$. We then define a map

$$\Phi^{\alpha;\beta}_{(H_\tau, J_\tau)} : C_*(H_\alpha, J_\alpha) \to C_*(H_\beta, J_\beta)$$

by counting, with sign, the number of solutions of the equation

$$\frac{\partial h}{\partial \tau} + J_\tau(h)\frac{\partial h}{\partial t} + \nabla(H_\tau)(h) = 0. \tag{18.13}$$

One can then prove that $\Phi^{\alpha;\beta}_{(H_\tau, J_\tau)}$ is a $\Lambda$-module map and is a chain map. It is also proved that it is independent of the choice of $(H_\tau, J_\tau)$, up to chain homotopy. So we obtain a map

$$\Phi^{\alpha,\beta} : HF_*(H_\alpha, J_\alpha) \to HF_*(H_\beta, J_\beta). \tag{18.14}$$

We then have $\Phi^{\alpha,\gamma} = \Phi^{\beta,\gamma} \circ \Phi^{\alpha,\beta}$. One can then prove that (18.14) is an isomorphism

To obtain an estimate of number of periodic orbits we need also to calculate the Floer homology as follows:

$$HF_*(H,J) = H_*(M) \otimes \Lambda. \tag{18.15}$$

(18.5) is proved by Floer [19] in the case when $M$ is monotone, and by Hofer–Salamon [35]: in the case either $\phi_{c_1} = 0$ or the minimal Chern number $N$ is at least $n = 1/2 \cdot \dim M$. The second named author [52] proved a similar isomorphism for weakly monotone symplectic manifolds after modifying the definition of Floer homology group. Later, Piunikhin *et al.* [55] and Ruan–Tian [59] announced the isomorphism with multiplicative structure.

Definition and invariance of Floer homology and its calculation (18.15) imply the estimate of the number of periodic solutions in terms of the Betti numbers, in the case all periodic solutions are nondegenerate.

These are the summary of Floer homology theory for periodic hamiltonian systems on weakly monotone symplectic manifolds. To apply this strategy to an arbitrary symplectic manifold, we need to deal with "negative multiple cover problem" for the bubbling-off of pseudoholomorphic spheres from solutions of eqs (18.3) and (18.13). This is the reason why Theorem 1.1 was proved only in the weakly monotone case.

We are going to apply our method developed in Chapters 1–3 to prove Theorem 1.1.

## 19. KURANISHI STRUCTURE ON THE SPACE OF CONNECTING ORBITS

The aim of this section is to define a Kuranishi structure on the space of connecting orbits, which enables us to define Floer homology (with rational coefficient) for periodic hamiltonian systems.

We pick and fix $p > 2$. For $\tilde{\ell}^{\pm} = (\ell^{\pm}, u^{\pm}) \in \tilde{\mathscr{P}}(H)$, we first define spaces $\mathscr{B}(\tilde{\ell}^-, \tilde{\ell}^+)$ and $\mathscr{E}(\tilde{\ell}^-, \tilde{\ell}^+)$. We take a map $h_0 : \mathbf{R} \times S^1 \to M$ such that

$$h_0(\tau, t) = \begin{cases} \tilde{\ell}^+(t) & \text{if } \tau > R \\ \tilde{\ell}^-(t) & \text{if } \tau < -R, \end{cases} \tag{19.1}$$

Let $L^p_1(\mathbf{R} \times S^1; h_0^* TM)$ be the Banach space of $L^p_1$-sections of $h_0^* TM$. (Here $L^p_1$-section stands for the $L^p$-section whose first derivative is of $L^p$ class.) We remark that, since $p > 2$, elements of $L^p_1(\mathbf{R} \times S^1; h_0^* TM)$ are continuous.

*Definition* 19.2. $\mathscr{B}(\tilde{\ell}^-, \tilde{\ell}^+)$ is the set of all locally $L^p_1$-maps $h : \mathbf{R} \times S^1 \to M$ with the following properties. There exists $\xi \in L^p_1(\mathbf{R} \times S^1; h_0^* TM)$ such that

$$h(\tau, t) = \exp_{h_0(\tau,t)} \xi(\tau, t)$$

on $S^1 \times ((-\infty, -R'] \cup [R', \infty))$ for some $R' \geqslant R$ and that $(\tilde{\ell}^+, u^+ \# h) \sim (\ell^-, u^-) \bmod \ker \phi_{c_1} \cap \ker \phi_{\omega}$.

Here exp is the exponential map with respect to the metric $g_J$. Since $h$ is continuous the last condition makes sense. It is easy to see that $\mathscr{B}(\tilde{\ell}^-, \tilde{\ell}^+)$ is independent of the choice of $h_0 : \mathbf{R} \times S^1 \to M$.

It is well known that $\mathscr{B}(\tilde{\ell}^-, \tilde{\ell}^+)$ is a Banach manifold.

*Definition* 19.3. $\mathscr{E}(\tilde{\ell}^-, \tilde{\ell}^+)$ is the Banach bundle over $\mathscr{B}(\tilde{\ell}^-, \tilde{\ell}^+)$, whose fibre at $h \in \mathscr{B}(\tilde{\ell}^-, \tilde{\ell}^+)$ is $L^p(\mathbf{R} \times S^1; h^* TM)$.

For $h \in \mathscr{B}(\tilde{\ell}^-, \tilde{\ell}^+)$, we put

$$\bar{\partial}_{J,H} h = \frac{\partial h}{\partial \tau} + J(h(t, \tau)) \frac{\partial h}{\partial t} + \nabla H_t(h(t,\tau)). \tag{19.4}$$

It is easy to see that $\bar{\partial}_{J,H} h \in L^p(\mathbf{R} \times S^1;\, h^* TM)$. The following lemma is also easy to see

LEMMA 19.4. *$h \to \bar{\partial}_{J,H} h$ is a Fredholm section of the Banach bundle $\mathscr{E}(\tilde{\ell}^-, \tilde{\ell}^+)$.*

Hence its differential defines a Fredholm operator:

$$D_h \bar{\partial}_{J,H} : L^p_1(\mathbf{R} \times S^1 : h^* TM) \to L^p(\mathbf{R} \times S^1 : h^* TM). \tag{19.5}$$

(We remark that it is more natural to regard the target as $L^p(\mathbf{R}\times S^1:\Lambda^{0,1}(\mathbf{R}\times S^1)\otimes h^*TM)$. But we can identify it with $L^p(\mathbf{R}\times S^1:h^*TM)$ by using a canonical trivialization of $\Lambda^{0,1}(\mathbf{R}\times S^1)$.)

We have

LEMMA 19.6. *There is a map $\mu:\tilde{\mathscr{P}}(H)\to\mathbf{Z}$ such that $D_h\bar\partial_{J,H}$ is a Fredholm operator of index $\mu(\tilde\ell^-)-\mu(\tilde\ell^+)$, if $h\in\mathscr{B}(\tilde\ell^-,\tilde\ell^+)$.*

We omit the proof see [35].

We remark that $\mathbf{R}$ acts on $\mathscr{E}(\tilde\ell^-,\tilde\ell^+)$ and $\mathscr{B}(\tilde\ell^-,\tilde\ell^+)$ by translation of $\tau$-variables and that $h\mapsto\bar\partial_{J,H}h$ is an $\mathbf{R}$-equivariant section.

*Definition* 19.7. $\tilde{\mathscr{M}}(\tilde\ell^-,\tilde\ell^+)=\{h\in\mathscr{B}(\tilde\ell^-,\tilde\ell^+)|\bar\partial_{J,H}h=0\}$. $\mathscr{M}(\tilde\ell^-,\tilde\ell^+)=\tilde{\mathscr{M}}(\tilde\ell^-,\tilde\ell^+)/\mathbf{R}$

Based on McDuff's result, it is proved by Hofer and Salamon in [35] that $\tilde{\mathscr{M}}(\tilde\ell^-,\tilde\ell^+)$ is a smooth manifold of dimension $\mu(\tilde\ell^-)-\mu(\tilde\ell^+)$ for a generic choice of $(J_t,H_t)$. (However we do not use it in this paper.) If $\tilde\ell^-=\tilde\ell^+=\tilde\ell$ then $\tilde{\mathscr{M}}(\tilde\ell,\tilde\ell)$ is a single element. (This is because the equation $\bar\partial_{J,H}h=0$ which is equivalent to (18.3) is a gradient flow equation of $\mathscr{A}_H$.) Otherwise the $\mathbf{R}$ action on $\tilde{\mathscr{M}}(\tilde\ell^-,\tilde\ell^+)$ is free. Hence $\mathscr{M}(\tilde\ell^-,\tilde\ell^+)$ is a smooth manifold for generic $(J_t,H_t)$ also.

However, the space $\mathscr{M}(\tilde\ell^-,\tilde\ell^+)$ is, in general, not compact, and we are forced to investigate its compactification.

If we assume that $M$ is weakly monotone (semi-positive) and $(J_t,H_t)$ is generic, then the space $\mathscr{M}(\tilde\ell^-,\tilde\ell^+)$ is compact if $\mu(\tilde\ell^-)-\mu(\tilde\ell^+)=1$ and is compact upto splitting into two connecting orbits if $\mu(\tilde\ell^-)-\mu(\tilde\ell^+)=2$. (This is because, under the assumption on the weak monotonicity, possibility of bubbling off of pseudoholomorphic sphere is excluded by choosing $(J_t,H_t)$ generically.) (We do not use this fact in this paper.)

These facts enable Hofer and Salamon [35] to define Floer homology for periodic hamiltonian system on weakly monotone symplectic manifolds.

We use a generalization of the notion of stable maps and also we use machinery developed in Chapters 1–3 to define Floer homology for periodic hamiltonian system on general symplectic manifolds.

The following lemma, Theorem 3.3 in [35] (see also Lemma 3.5 of Lê-Ono [43]) as well as Lemma 8.1 is essential to prove the compactness of $C\mathscr{M}(\tilde\ell^-,\tilde\ell^+)$ and to construct a Kuranishi structure on it.

LEMMA 19.8. *There is $\delta'>0$ such that if $\tilde\ell^-\neq\tilde\ell^+$ and if $h\in\mathscr{M}(\tilde\ell^-,\tilde\ell^+)$ then $E_H(h)>\delta'$.*

We will give a proof of it in Section 20 for reader's convenience.

Now we generalize the notion of stable maps as follows. For a natural number $m$ we

$$\underline{m}=\{1,2,\dots,m\}$$

*Definition* 19.9. *A stable connecting orbit* is a triple $((h_1,\dots,h_k),(\sigma_1,\dots,\sigma_I),o)$ such that

(19.9.1) $h_j\in\tilde{\mathscr{M}}(\tilde\ell_j,\tilde\ell_{j+1})$, where $\tilde\ell_j\in\tilde{\mathscr{P}}(H)$, $j=1,\dots,k+1$.

(19.9.2) $\sigma_i\in C\mathscr{M}_{0,1}(M,J,\beta_i)$.

(19.9.3) $o$ is an injective map from $\underline{I}$ to the $k$ copies of $\mathbf{R}\times S^1$.

(19.9.4) Let $\sigma_i = (\Sigma_{\sigma_i}, h_{\sigma_i})$, where $\Sigma_{\sigma_i}$ is a genus zero simistable curve with one marked point and $h_{\sigma_i}: \Sigma_{\sigma_i} \to M$. Let $z_i \in \Sigma_{\sigma_i}$ be the marked point. Let $o(i) = (\tau_i, t_i)$ is on the $j$th copy of $\mathbf{R} \times S^1$. We assume that $h_{\sigma_i}(z_i) = h_j(\tau_i, t_i)$.

(19.9.5) If $\tilde{\ell}_j = \tilde{\ell}_{j+1}$, then there exists $i$ such that $o(i)$ is on the $j$th copy of $\mathbf{R} \times S^1$.

Let $\rho: \underline{I} \to \underline{I}$ be a bijection. We then say that $((h_1, \dots, h_k), (\sigma_1, \dots, \sigma_I), o)$ is isomorphic to $((h_1, \dots, h_k), (\sigma_{\rho(1)}, \dots, \sigma_{\rho(I)}), o \circ \rho^{-1})$. For simplicity we write $((h_1, \dots, h_k), (\sigma_1, \dots, \sigma_I), o)$ for isomorphism class also.

We consider $k$ copies of $\mathbf{R} \times S^1$ plus $\Sigma_{\sigma_i}$, $i = 1, \dots, I$, attached at $o(i) = (\tau_i, t_i)$ to the $j$th copy of $\mathbf{R} \times S^1$. Let us call this space the domain of definition of our stable connecting orbit and write it as $\Sigma$. We say that a point on $\Sigma$ is singular if it corresponds to a singular point of $\Sigma_{\sigma_i}$ or is on the image of $o$. We can define a map $h$ from $\Sigma$ to $M$. Namely we let $h = h_i$ on the $i$th $\mathbf{R} \times S^1$, and $h = h_{\sigma_i}$ on $\Sigma_{\sigma_i}$.

In place of writing $((h_1, \dots, h_k), (\sigma_1, \dots, \sigma_I), o)$, we write $(\Sigma, h)$ sometimes for simplicity.

Let $((h_1, \dots, h_k), (\sigma_1, \dots, \sigma_I), o)$ be a stable connecting orbit and $(r_1, \dots, r_k) \in \mathbf{R}^k$. We define $(r_1, \dots, r_k)((h_1, \dots, h_k), (\sigma_1, \dots, \sigma_I), o)$ as follows. We take $j$th copy $\mathbf{R} \times S^1$, and take its translation of $\mathbf{R}$ factor by $r_j$. We also let $h_i \in \mathcal{M}(\tilde{\ell}^-, \tilde{\ell}_{i+1})$ translate accordingly. Also we translate $o(j)$ by $r_j$ if it is on $j$th component.

We say that $(r_1, \dots, r_k)((h_1, \dots, h_k), (\sigma_1, \dots, \sigma_I), o) \sim ((h_1, \dots, h_k), (\sigma_1, \dots, \sigma_I), o)$. We write $[(h_1, \dots, h_k), (\sigma_1, \dots, \sigma_I), o]$ or $[\Sigma, h]$ the equivalence class containing $((h_1, \dots, h_k), (\sigma_1, \dots, \sigma_I), o)$.

*Definition* 19.10. We say that $[(h_1, \dots, h_k), (\sigma_1, \dots, \sigma_I), o] \in C\mathcal{M}(\tilde{\ell}^-, \tilde{\ell}^+)$ if

(19.10.1) $\tilde{\ell}_{k+1} = \tilde{\ell}^+$.

(19.10.2) $(\beta_1 + \cdots + \beta_I) \# \tilde{\ell}_1 = \tilde{\ell}^-$.

If we define an energy of $[(h_1, \dots, h_k), (\sigma_1, \dots, \sigma_I), o] \in C\mathcal{M}(\tilde{\ell}^-, \tilde{\ell}^+)$ by

$$E_H([(h_1, \dots, h_k), (\sigma_1, \dots, \sigma_I), o]) = \sum E_H(h_i) + \sum E(\sigma_i),$$

where $E(\sigma_i) = \int_{\Sigma_i} h_i^* \omega$, then it is easy to see that

$$E_H([(h_1, \dots, h_k), (\sigma_1, \dots, \sigma_I), o]) = \mathcal{A}_H(\tilde{\ell}^-) - \mathcal{A}_H(\tilde{\ell}^+) \tag{19.11}$$

which depends only on $\tilde{\ell}^-, \tilde{\ell}^+$ and is independent of the choice of elements $[(h_1, \dots, h_k), (\sigma_1, \dots, \sigma_I), o] \in C\mathcal{M}(\tilde{\ell}^-, \tilde{\ell}^+)$.

We can define a topology on $C\mathcal{M}(\tilde{\ell}^-, \tilde{\ell}^+)$ in a way similar to that in Section 10.

THEOREM 19.12. *$C\mathcal{M}(\tilde{\ell}^-, \tilde{\ell}^+)$ is Hausdorff and compact.*

*Proof.* Using Lemmata 19.8 and 8.1, we can prove that there are only finite many possibilities of the number $k$ and $I$ as well as $\beta_i$ in Definition 19.10 for an element $h \in C\mathcal{M}(\tilde{\ell}^-, \tilde{\ell}^+)$. In other words, there are only finitely many possibilities of the combinatorial types of "bubble tree" for stable connecting orbits in $C\mathcal{M}(\tilde{\ell}^-, \tilde{\ell}^+)$.

Theorem 19.12 then can be proved in the same way as Theorem 11.1, by using Gromov

We next need a notion of Kuranishi structure with corners. It is defined by replacing $[0, \infty) \times \mathbf{R}^{n-1}$ in Definition 17.1 by $[0, \infty)^B \times \mathbf{R}^{n-B}$ for some $B$.

Let $X$ be a space with Kuranishi structure with corners. We denote by $S_BX$ the set of all points $x \in X$ such that its Kuranishi neighborhood is $[0, \infty)^B \times \mathbf{R}^{n-B}/\Gamma$ where $x$ corresponds to the origin. $S_BX$ has a "Kuranishi structure" except it is not compact. We put

$$S_{(B)}X = \bigcup_{B' \geqslant B} S_{B'}X.$$

We remark that, for a space $X$ with $n$ dimensional Kuranishi structure with corners, the set $S_BX$ may be nonempty for $B$ with $B > n$. This is similar to the fact that the space with Kuranishi structure of negative virtual dimension can be nonempty. This is the reason we need to introduce Kuranishi structure with corners. (We use only moduli spaces of connecting orbit of virtual dimension 0 or 1 to define and study Floer homology. So in the case when the sympletic manifold is weakly monotone, we only need to study the moduli space which is a closed manifold or a manifold with boundary.)

However when we consider only homology classes, we can prove that the contribution form strata of negative virtual dimension vanishes. (This statement must be made precise. We will do it later in Section 20.)

We can define the notions of orientation and stably almost complex structure of the Kuranishi structure with corners in a way similar to Section 6.

If $X_1, X_2$ are spaces with Kuranishi structure of dimensions $n_1, n_2$, respectively, with corners, it induces a Kuranishi structure of dimension $n_1 + n_2$ with corners on the product $X_1 \times X_2$. Namely if $(U_p, E_p, s_p)$ and $( \ _q, F_q, t_q)$ are the charts of $X_1$ and $X_2$, respectively, then $(U_p \times \ _q, E_p \oplus F_q, s_p \oplus t_q)$ is a chart of $X_1 \times X_2$.

If $X_1, X_2$ are oriented (stably complex) so is $X_1 \times X_2$. We define the following map

$$Glue_{\tilde{\ell}}: \ \ _{b=1}^{B-1} \ C\mathcal{M}(\tilde{\ell}_b, \tilde{\ell}_{b+1}) \to C\mathcal{M}(\tilde{\ell}_1, \tilde{\ell}_B). \tag{19.13}$$

For $(h_1, \ldots, h_{B-1}) \in \ _{b=1}^{B-1} \mathcal{M}(\tilde{\ell}_i, \tilde{\ell}_{i+1})$, we define

$$Glue_{(\tilde{\ell}_1, \ldots, \tilde{\ell}_B)}(h_1, \ldots, h_B) = ((h_1, \ldots, h_B), \emptyset, \emptyset)$$

Here the right-hand side is the case $k = B, I = 0$ in Definition 19.9. It is easy to see that one can extend it to the compactification.

We can now state the following theorem.

Theorem 19.14. *$C\mathcal{M}(\tilde{\ell}^-, \tilde{\ell}^+)$ has a $\mu(\tilde{\ell}^-) - \mu(\tilde{\ell}^+) - 1$-dimensional oriented Kuranishi structure with corners. For $B \geqslant 3$, we have*

$$C\mathcal{M}(\tilde{\ell}^-, \tilde{\ell}^+)_{(B-2)} = \bigcup \mathrm{Im}\, Glue_{(\tilde{\ell}^-, \ldots, \tilde{\ell}_{B-1}, \tilde{\ell}^+)}.$$

*Where the union in the right-hand side is taken over all $\tilde{\ell}_i$ such that $\mathcal{A}_H(\tilde{\ell}^-) > \mathcal{A}_H(\tilde{\ell}_2) > \cdots > \mathcal{A}_H(\tilde{\ell}_{B-1}) > \mathcal{A}_H(\tilde{\ell}^+)$.*

For the proof we first need:

Lemma 19.15. *For each $C_1 < C_2$ there exists only a finitely many $\tilde{\ell}^-, \tilde{\ell}^+ \in \tilde{\mathcal{P}}(H)$ such that $C_2 > \mathcal{A}_H(\tilde{\ell}^-) > \mathcal{A}_H(\tilde{\ell}^+) > C_1$ and that $C\mathcal{M}_m(\tilde{\ell}^-, \tilde{\ell}^+)$ is nonempty, if we identify the pairs $(\tilde{\ell}^-, \tilde{\ell}^+)$ by the equivalence relation $(\tilde{\ell}^-, \tilde{\ell}^+) \sim (\beta \# \tilde{\ell}^-, \beta \# \tilde{\ell}^+) \bmod \ker \varphi_{c_1} \cap \ker \varphi_\omega$, $(\beta \in \pi_2 M / \ker \varphi_{c_1} \cap \ker \varphi_\omega$.)*

The proof of Lemma 19.15 (using Lemma 19.8) is similar to the proof of Lemma 8.9 which is given at the end of Section 11.

In fact, we need to prove somewhat more than stated in Theorem 19.14. Namely, we need to make the Kuranishi structures for various $C\mathcal{M}(\tilde{\ell}^-, \tilde{\ell}^+)$ compatible.

Let us make it more precise what we mean by Kuranishi structures are compatible to the embeddings. To clarify the situation, let us first recall the case when $C\mathcal{M}(\tilde{\ell}_b, \tilde{\ell}_{b+1})$ is smooth (transversal) for each $b = 1, \dots, B$. Namely, we assume that they have Kuranishi structures such that all the obstruction bundles $E_\sigma$ are trivial. Then the neighborhood of the image of $Glue_{\vec{\ell}}$ is also smooth and is diffeomorphic to ${}_{b=1}^{B-1} \mathcal{M}(\tilde{\ell}_b, \tilde{\ell}_{b+1}) \times [0, \varepsilon)^{B-2}$ (Floer [19, Proposition 2d.11]).

In the general case when $C\mathcal{M}(\tilde{\ell}_i, \tilde{\ell}_{i+1})$ may not be smooth, we prove the following:

ADDENDUM 19.16. *A neighborhood of* Im $Glue_{\vec{\ell}}$ *together with oriented Kuranishi structure is diffeomorphic to* $\pm$ ${}_{b=1}^{B-1} C\mathcal{M}(\tilde{\ell}_b^{\,-}, \tilde{\ell}_{b+1}) \times [0, \varepsilon)^{B-2}$. *Here* $\pm$ *depends only on* $\mu(\tilde{\ell}_b) - \mu(\tilde{\ell}_{b+1})$, $b = 1, \dots, B-1$.

*Proof of Theorem* 19.14. The argument goes by the induction on $\mathcal{A}_H(\tilde{\ell}^-) - \mathcal{A}_H(\tilde{\ell}^+)$. More precisely, let $\delta''$ be the minimum of the number $\delta$ in Lemma 8.1 and $\delta'$ in Lemma 19.8.

If $\mathcal{A}_H(\tilde{\ell}^-) - \mathcal{A}_H(\tilde{\ell}^+) \leqslant \delta''$, then either $C\mathcal{M}(\tilde{\ell}^-, \tilde{\ell}^+)$, is empty or $\tilde{\ell}^- = \tilde{\ell}^+$ and $C\mathcal{M}_m(\tilde{\ell}^-, \tilde{\ell}^+)$ is one point. Hence Theorem 19.14, Addendum 19.16 hold.

Assume that we have proved Theorem 19.14, Addendum 19.16 for $\mathcal{A}_H(\tilde{\ell}^-) - \mathcal{A}_H(\tilde{\ell}^+) \leqslant K\delta''$. We prove it for $\mathcal{A}_H(\tilde{\ell}^-) - \mathcal{A}_H(\tilde{\ell}^+) \leqslant (K+1)\delta''$. To construct Kuranishi neighborhood around each point $(\Sigma, h)$, we go in a similar way as in Chapter 3. We consider elliptic operator (19.5) or its generalization similar to (12.1) in the case when $\Sigma$ is singular. We write it

$$D_h \bar{\partial}_{J,H} : L_1^p(\Sigma : h^*TM) \to L^p(\Sigma : h^*TM). \tag{19.17}$$

If it is surjective, we can perform the Taubes' type gluing construction in the same way as Floer [19], to find a neighborhood of $(\Sigma, h)$ which is smooth orbifold as follows.

In order to glue the bubbles, we use the following trick by Gromov [33]. Note that eq. (18.3) is an inhomogeneous Cauchy–Riemann equation. A solution of an inhomogenous Cauchy–Riemann equation can be considered as a solution of (homogeneous) Cauchy–Riemann equation to the product of the domain and the target with an appropriate almost complex structure. Namely we find an almost complex structure $\tilde{J}(J, H)$ on $(\mathbf{R} \times S^1) \times M$ such that a map $h : \mathbf{R} \times S^1 \to M$ satisfies Eq. (18.3) if and only if its graph: $\mathbf{R} \times S^1 \to (\mathbf{R} \times S^1) \times M$ is pseudoholomorphic with respect to the almost complex structure $\tilde{J}(J, H)$ on $(\mathbf{R} \times S^1) \times M$.

We use the following three facts on this almost complex structure $\tilde{J}(J, H)$: The projection from $(\mathbf{R} \times S^1) \times M$ to the $\mathbf{R} \times S^1$ is pseudoholomorphic, all the fibres of $(\mathbf{R} \times S^1) \times M \to \mathbf{R} \times S^1$ are almost complex; the almost complex structures on fibers are the same as the almost complex structure on $M$, the target.

Therefore, a pseudoholomorphic sphere $S$ on $M$ produces a holomorphic sphere $(\tau_0, t_0) \times S$ in $\mathbf{R} \times S^1 \times M$ with $(\tau_0, t_0) \in \mathbf{R} \times S^1$. It is also easy to see that every holomorphic sphere contained in one fiber is obtained in this way.

Therefore, the gluing with connecting orbit and pseudoholomorphic spheres is reduced to the gluing between a pseudoholomorphic map $\mathbf{R} \times S^1 \to (\mathbf{R} \times S^1) \times M$ and pseudoholomorphic spheres. We discussed the later problem in detail already in Chapter 3.

The other type of gluing we need to handle, is the gluing several connecting orbits along the periodic solution, i.e. the gluing related to the map *Glue*. However, in this case, the argument is the same as Floer's [19]. The only points we need to change the argument of [19] is that, there may be a cokernel to the operator (19.17). But we can handle this problem in the same way as Chapter 3, i.e. replacing (19.17) by (19.18) and changing (19.4) accordingly.

If (19.17) is not surjective, we need to take a subspace $E_{(\Sigma,h)}$ satisfying a condition corresponding to (12.7). Especially the map

$$\underset{E_{(\Sigma,h)}}{\circ}(D_h\bar\partial_{J,H}): L^p_1(\Sigma : h^*TM) \to \frac{L^p(\Sigma : h^*TM)}{E_{(\Sigma,h)}} \tag{19.18}$$

is assumed to be surjective. In the case when our $\Sigma$ is in an image of some $Glue_{\vec{\ell}}$ by induction hypothesis, we may assume that $E_{(\Sigma,h)}$ is chosen. Hence we take that one. We next prove that it is well-defined. Namely, if $(\Sigma, h)$ is in an image of two different $Glue_{\vec{\ell}}$ (namely *Glue* defined in different strata), then $E_{(\Sigma,h)}$ is independent of the choice of *Glue*. In fact, if

$$(\Sigma, h) = Glue_\alpha\, x = Glue_\beta\, y$$

for some $x, y, \alpha, \beta$ then there exits some $z$ and $\gamma, \delta$ such that

$$x = Glue_\gamma\, z, \quad y = Glue_\delta\, z.$$

It then follows from induction hypothesis that $E_{(\Sigma,h)}$ induced from $x$ is equal to $E_{(\Sigma,h)}$ induced from $y$.

We next stratify the complement of the image of *Glue* in $C\mathcal{M}(\tilde{\ell}^-, \tilde{\ell}^+)$ in a way similar to Section 8.

We can then extend $E_{(\Sigma,h)}$ to all $(\Sigma, h) \in C\mathcal{M}(\tilde{\ell}^-, \tilde{\ell}^+)$ such that it satisfies a condition corresponding to (12.7) and that they are compatible to those we have already chosen. We thus have made a choice of $E_{(\Sigma,h)}$.

Thus we can repeat the argument of Chapter 3 to construct a Kuranishi structure. We thus have constructed a Kuranishi neighborhood around each point of $C\mathcal{M}(\tilde{\ell}^-, \tilde{\ell}^+)$.

The argument to glue them is the same as the one in Chapter 3, since we have already made a choice of $E_{(\Sigma,h)}$ (which we use in the same way as we use $E_\tau$ in Section 5.) We will discuss the orientation in Section 21.

To complete the proof, let us verify that the virtual dimension of the Kuranishi neighborhood is constant and is equal to $\mu(\tilde{\ell}^-) - \mu(\tilde{\ell}^+) - 1$.

To see this, we need to calculate the index of (19.7). Let $((h_1, \dots, h_k), (\sigma_1, \dots, \sigma_I), o)$ or $(\Sigma, h)$ be a stable connecting orbit determining an element of $C\mathcal{M}_m(\tilde{\ell}^-, \tilde{\ell}^+)$. We recall

$$L^p_1(\Sigma : h^*TM) = \left\{ ((u_j),(u'_i)) \;\middle|\; \begin{array}{l} j = 1, \dots, k, i = 1, \dots I \\ u_j \in L^p_1(\mathbf{R}\times S^1;\ h_j^*TM) \\ u'_i \in L^p_1(\Sigma_{\sigma_i}; h_{\sigma_i}^*TM) \text{ where } \sigma_i = (\Sigma_{\sigma_i}, h_{\sigma_i}) \\ \text{If } o(i) = (\tau_i, t_i) \text{ is in the } j_i\text{th} \\ \quad \text{copy } \mathbf{R}\times S^1, \text{ then } u_{j_i}(\tau_i, t_i) = u'_i(z_i) \\ \quad \text{where } z_i \text{ is the marked point of } \Sigma_{\sigma_i} \end{array} \right\}$$

If we forget the compatibility condition $u_{j_i}(\tau_i, t_i) = u'_i(z_i)$, etc., then the operator is the direct sum of $D_{h_j}\bar\partial_{J,H}$ and $D_{h_{\sigma_i}}\bar\partial_{\Sigma_{\sigma_i}}$ for various irreducible components. Hence its index is a sum of the indices of $D_{h_j}\bar\partial_{J,H}$ and $D_{h_{\sigma_i}}\bar\partial_{\Sigma_{\sigma_i}}$ over various irreducible components. Namely, we have

$$\sum_{j=1}^{k} (\mu(\tilde{\ell}_{j+1}) - \mu(\tilde{\ell}_j) - 1) + 2\sum_{i=1}^{I} \beta_i \cdot c_1 + 2nN, \tag{19.19}$$

where $N$ is the total number of irreducible components of $\Sigma_{\sigma_i}$. The condition $u_{j_i}(\tau_i, t_i) = u_i'(z_i)$ and the similar compatibility conditions on the singular points of $\Sigma_{\sigma_i}$ impose $2n(I + \sum s_i)$ conditions where $s_i$ is the number of singular points on $\Sigma_{\sigma_i}$. Therefore the index of the operator (19.17) is

$$\mu(\tilde{\ell}_1) - \mu(\tilde{\ell}_{k+1}) - k + 2\sum_{i=1}^{I} \beta_i c_1 - 2n(I + \sum s_i) + 2nN. \tag{19.20}$$

To calculate the virtual dimension of the Kuranishi neighborhood around this point, we need to add to (19.20) the number of parameters corresponding to the deformation of complex structures (keeping the combinatorial types) and the parameter of gluing, and also we need to substract the sum of dimensions of the automorphism groups of various $\sigma_i$. We remark that $s_i + 1$ is equal to the number of irreducible component of $\Sigma_{\sigma_i}$. Hence $\sum s_i = N - I$.

Let $\Sigma_{\sigma_i} = \bigcup \Sigma_{\sigma_{i,v}}$ be the decomposition to the irreducible components. Let $t_{i,v}$ be the number of singular points on $\Sigma_{\sigma_{i,v}}$. Here singular point means the point where $\Sigma_{\sigma_{i,v}}$ intersects with other $\Sigma_{\sigma_{i,v'}}$.

The number of parameters (over reals) to deform the complex structures of $\Sigma_{\sigma_i}$ without changing the combinatorial type is $2\sum \max\{0, t_{i,v} - 3\}$. We have 2 more parameters corresponding to the deformation of the position of marked point of $\Sigma_{\sigma_i}$ (i.e. the point where $\Sigma_{\sigma_i}$ is attached to one of the $\mathbf{R} \times S^1$'s).

The dimension of automorphism group of $\Sigma_{\sigma_i}$ is $2\sum \max\{0, 3 - t_{i,v}\}$.

Since the genus of $\Sigma_{\sigma_i}$ is zero, we have a tree with $s_i + 1$ vertices such that vertices have $t_{i,v}$ edges, in a way similar to that in Section 9. Hence by Euler's formula we have

$$s_i + 1 - \frac{1}{2}\sum t_{i,v} = 1. \tag{19.21}$$

We find that the number of parameters for gluing is equal to the sum of $k - 1$ and twice of the number of singular points. Hence it is $2I + 2\sum s_i + (k - 1)$. The number of parameters of the deformation of the singular points which are on the copies of $\mathbf{R} \times S^1$ is $2I$. Summing up, the virtual dimension of our Kuranishi neighborhood is

$$\begin{aligned}
&\mu(\tilde{\ell}_1) - \mu(\tilde{\ell}_{k+1}) - k + 2\sum_{i=1}^{I} \beta_i c_1 - 2n(I + \sum s_i) + 2nN \\
&\quad + 2\sum \max\{0, t_{i,v} - 3\} + 2I \\
&\quad - 2\sum \max\{0, 3 - t_{i,v}\} \\
&\quad + 2I + 2\sum s_i + (k - 1) + 2I.
\end{aligned}$$

Using (19.21) and the fact that the number of irreducible components of $\Sigma_{\sigma_i}$ is $s_i + 1$, we find that this number is equal to

$$\mu(\tilde{\ell}_1) - \mu(\tilde{\ell}_{k+1}) + 2\sum_{i=1}^{I} \beta_i c_i -$$

$$\mu(\tilde{\ell}_1) - \mu(\tilde{\ell}_{k+1}) + 2\sum_{i=1}^{I} \beta_i c_i = \mu(\tilde{\ell}^-) - \mu(\tilde{\ell}^+).$$

Thus the virtual dimension is $\mu(\tilde{\ell}^-) - \mu(\tilde{\ell}^+) - 1$ as required. The proof of Theorem 19.14 and Addendum 19.16 are complete modulo the construction of the orientation and the proof of Lemma 19.8.

## 20. DEFINITION OF FLOER HOMOLOGY

In this section, we will construct Floer homology with coefficient in $\mathbf{Q}$ (or $\Lambda \otimes \mathbf{Q}$) by using Theorem 19.14 and Addendum 19.16.

The definition of the Floer's chain complex as an abelian group is the same as the case of weakly monotone symplectic manifolds which we discussed in Section 18 (Definition 18.11).

We are going to define the boundary operator $\partial_k : C_k(H, J) \to C_{k-1}(H, J)$.

First let us mention the outline of the construction. The detail will follow. We fix Kuranishi structures with corners as in Theorem 19.14 and Addendum 19.16. (We remark that the boundary operator itself may depend on the choice of Kuranishi structure but its chain homotopy class does not depend on it.) Then for each $\tilde{\ell}^-, \tilde{\ell}^+$ with $\mu(\tilde{\ell}^-) - \mu(\tilde{\ell}^+) = 1$, we have a Kuranishi structure with corners on $C\mathcal{M}(\tilde{\ell}^-, \tilde{\ell}^+)$ of dimension 0. However we cannot yet apply Theorem 6.12 directly to define its fundamental class. The reason is that $C\mathcal{M}(\tilde{\ell}^-, \tilde{\ell}^+)$ has a boundary. Hence, though the virtual dimension of the boundary is $-1$ and hence we can take a multisection such that it vanishes only in the interior of $C\mathcal{M}(\tilde{\ell}^-, \tilde{\ell}^+)$, the $\mathbf{Q}$-cycle defined as the zero point set of this multisection *does* depend on the choice of the multisection. So we construct the system of multisections by the same induction as the proof of Theorem 19.14. We then obtain a chain complex. The resulting boundary operators *do* depend on the choice of system of multisections. However the chain homotopy equivalence class of it is independent of it. This is a method first appeared in first named author's paper, [24] Section 12, in the context of Gauge theory Floer homology.

Now let us carry out the plan described above.

We first give some remarks, which can be proved easily. First we remark that Theorem 6.4 and its relative version Lemma 17.8 is generalized to Kuranishi structure with corners. Next we remark that given good coordinate system on $X_i$ then we have an induced good coordinate system on $X_i$. Third given a system of multisections as in Theorem 6.4 on each $X_i$, we get one on $X_i$. Let us make the third statement clearer. For simplicity we consider only the product of two spaces. Let $(U_p, E_p, s_p)$ be a good coordinate system of $X$ and $( {}_q, F_q, t_q)$ be a good coordinate system of $Y$. Let $s_{p,n}$ and $t_{q,n}$ be sequences of systems of multisections satisfying the conclusions of Theorem 6.4. We remark that the good coordinate system of $X \times Y$ is $(U_p \times {}_q, E_p \oplus E_q, s_p \oplus t_q)$. $s_{p,n}$ and $t_{q,n}$ induces multisections $s_{p,n} \oplus 0$ and $0 \oplus t_{q,n}$ of $E_p \oplus E_q$, respectively. We take its sum $s_{p,n} \oplus 0 + 0 \oplus t_{q,n}$ (Definition 3.4) and write it $s_{p,n} \oplus t_{q,n}$. It is straightforward to see that $s_{p,n} \oplus t_{q,n}$ satisfies the conclusion of Theorem 6.4.

Now we start the construction of the good coordinate system and multisections. The construction works on induction on $\mathcal{A}_H(\tilde{\ell}^-) - \mathcal{A}_H(\tilde{\ell}^+)$. We first use Addendum 19.16, to find a good coordinate system on each $C\mathcal{M}(\tilde{\ell}^-, \tilde{\ell}^+)$ such that it is compatible with the maps *Glue*. Namely we assume that given a good coordinate system of the domain of *Glue*, which is a product of the good coordinate system we have by induction hypothesis, then we extend it to its neighborhood by using Addendum 19.16. We then extend it to $C\mathcal{M}(\tilde{\ell}^-, \tilde{\ell}^+)$ in any way. We thus constructed a good coordinate system for each of $C\mathcal{M}(\tilde{\ell}^-, \tilde{\ell}^+)$ by induction.

We next construct the system of multisections on each chart of this good coordinate system, so that it is compatible to the maps *Glue*.

The construction is again by an induction on $\mathscr{A}_H(\tilde{\ell}^-) - \mathscr{A}_H(\tilde{\ell}^+)$. Let us describe one step of the induction. So we assume that multisections are constructed on $C\mathscr{M}_{0,m}(M, J, \beta)$ with $\beta\omega < C$, and on $C\mathscr{M}(\tilde{\ell}^-, \tilde{\ell}^+)$ with $\mathscr{A}_H(\tilde{\ell}^-) - \mathscr{A}_H(\tilde{\ell}^+) < C$, then we construct multisection on $C\mathscr{M}(\tilde{\ell}^-, \tilde{\ell}^+)$ such that $\mathscr{A}_H(\tilde{\ell}^-) - \mathscr{A}_H(\tilde{\ell}^+) \leqslant C + \delta''$. On images of various *Glue* in $C(\tilde{\ell}^-, \tilde{\ell}^+)$, we have already multisections (such as $s_{p,n} \otimes t_{q,n}$). They coincide with each other on the overlapping part by induction hypothesis. (They proof of this fact is the same as the corresponding part in the proof of the Theorem 19.14.) Thus, using Addendum 19.16, we can extend it to a neighborhood of it so that it is transversal to 0. We then can extend it to all of $C\mathscr{M}(\tilde{\ell}^-, \tilde{\ell}^+)$ without changing it on images of *Glue* by using Lemma 17.8.

We thus constructed a good coordinate system and multisections on it, which are compatible with *Glue*. We remark that for $\beta \in \Gamma$ there is a canonical homeomorphism $C\mathscr{M}(\tilde{\ell}^-, \tilde{\ell}^+) \cong C\mathscr{M}(\beta \# \tilde{\ell}^-, \beta \# \tilde{\ell}^+)$. It is easy to see from construction that we can make our Kuranishi structure, good coordinate system and multisections compatible with this homeomorphism.

We now use our multisections to define the boundary operator.

Let us suppose that $\mu(\tilde{\ell}^-) - \mu(\tilde{\ell}^+) = 1$. Then $C\mathscr{M}(\tilde{\ell}^-, \tilde{\ell}^+)$ has a 0 dimensional Kuranishi structure with corners. We have multisections $s_n$ on it. We consider a stratum of $C\mathscr{M}(\tilde{\ell}^-, \tilde{\ell}^+)$ which corresponds to an image of *Glue*. This stratum is a product of moduli spaces $\mathscr{M}(\tilde{\ell}'^-, \tilde{\ell}'^+)$. The virtual dimension of the stratum is negative. Therefore, there exists a factor $C\mathscr{M}(\tilde{\ell}'^-, \tilde{\ell}'^+)$ which admits a Kuranishi structure of negative dimension. So by transversality, our multisection never vanish on that factor. It follows from the compatibility and definition of $s_{p,n}$, the multisection $s_{p,n}$ does not vanish on a neighborhood of those strata. We can also prove by transversality that the set of zeros $(s_{p,n})^{-1}(0)_{\text{set}}$ is not on $C\mathscr{M}(\tilde{\ell}^-, \tilde{\ell}^+) - \mathscr{M}(\tilde{\ell}^-, \tilde{\ell}^+)$.

Therefore the set of zeros $(s_{p,n})^{-1}(0)_{\text{set}}$ is a 0-dimensional compact space, i.e. finitely many points, and is contained in $\mathscr{M}(\tilde{\ell}^-, \tilde{\ell}^+)$. We can define its multiplicity in the same way as Definition 4.5. We thus obtain a rational number and write it as $[(C\mathscr{M}(\tilde{\ell}^-, \tilde{\ell}^+), s_{p,n})]$. Hereafter we write $[C\mathscr{M}(\tilde{\ell}^-, \tilde{\ell}^+)]$ in place of $[(C\mathscr{M}(\tilde{\ell}^-, \tilde{\ell}^+), s_{p,n})]$ when no confusion can occur. We now define

$$\partial_k \delta_{\tilde{\ell}^-} = \sum_{\tilde{\ell}^+;\ \mu(\tilde{\ell}^+) = \mu(\tilde{\ell}^-) - 1} [C\mathscr{M}(\tilde{\ell}^-, \tilde{\ell}^+)]\, \delta_{\tilde{\ell}^+}. \tag{20.1}$$

(Right-hand side in general is an infinite sum.) Using Lemma 19.15, we can prove that (20.1) determines a map $\partial_k : C_k(H, J) \to C_{k-1}(H, J)$ in a way similar to [35].

LEMMA 20.2. $\partial_k \circ \partial_{k-1} = 0$.

*Proof.* The proof is similar to the original argument by Floer. We take $\tilde{\ell}^-, \tilde{\ell}^+$ with $\mu(\tilde{\ell}^-) - \mu(\tilde{\ell}^+) = 2$. It suffices to show that

$$\sum_{\tilde{\ell}^+;\ \mu(\tilde{\ell}^+) = \mu(\tilde{\ell}^-) - 1} [C\mathscr{M}(\tilde{\ell}^-, \tilde{\ell})][C\mathscr{M}(\tilde{\ell}, \tilde{\ell}^+)] = 0. \tag{20.3}$$

(We remark that the sum in (20.3) is a finite sum because of Lemma 19.15.) To show (20.3) we use our multisection $s_{p,n}$ on $C\mathscr{M}(\tilde{\ell}^-, \tilde{\ell}^+)$, which has an oriented Kuranishi structure with corners and of dimension 1.

We consider the stratification of $C\mathscr{M}(\tilde{\ell}^-, \tilde{\ell}^+)$. We consider a stratum corresponding to an element of $C\mathscr{M}(\tilde{\ell}^-, \tilde{\ell}^+) - \mathscr{M}(\tilde{\ell}^-, \tilde{\ell}^+)$. We find easily that the virtual dimension of the domain of these strata are negative except the image of the stratum which is dense in $C\mathscr{M}(\tilde{\ell}^-, \tilde{\ell}) \times C\mathscr{M}(\tilde{\ell}, \tilde{\ell}^+)$ such that $\mu(\tilde{\ell}) = \mu(\tilde{\ell}^-) - 1$.

We remark that both of the factors of $C\mathcal{M}(\tilde{\ell}^-, \tilde{\ell}) \times C\mathcal{M}(\tilde{\ell}, \tilde{\ell}^+)$ have 0 dimensional Kuranishi structure with corners. Hence the zero point sets of our multisections are finite there. Because of the compatibility of multisections and the definition of $[C\mathcal{M}(\tilde{\ell}^-, \tilde{\ell}^+)]$, we find that the order of the zero point set of the multisections $s_{p,n}$ on $C\mathcal{M}(\tilde{\ell}^-, \tilde{\ell}) \times C\mathcal{M}(\tilde{\ell}, \tilde{\ell}^+)$ counted with multiplicity is equal to $[C\mathcal{M}(\tilde{\ell}^-, \tilde{\ell})] \cdot [C\mathcal{M}(\tilde{\ell}, \tilde{\ell}^+)]$.

On the other hand, the zero point set of $s_{p,n}$ on $C\mathcal{M}(\tilde{\ell}^-, \tilde{\ell}^+)$ defines a $\mathbf{Q}$-chain in a way similar to Definition 4.6. Let $s_{p,n}^{-1}(0)$ be this chain. Then in the same way as the proof of Lemma 17.9, we can prove that the boundary of the chain $s_{p,n}^{-1}(0)$ is left-and side of (20.3). (Here we identify 0 dimensional cycle with rational number.) Lemma 20.2 follows. □

We thus have constructed a chain complex $(C_*(H, J), \partial)$. However this chain complex depends on various choices. That is the complex structure $J$, the hamiltonian $H$, the Kuranishi structure on $C\mathcal{M}(\tilde{\ell}^-, \tilde{\ell}^+)$, the multisections on it.

We define Floer homology group:

*Definition* 20.4. $HF_*((M, \omega), J, \Xi) = H_*(C_*(H, J), \partial)$.

Here we write $\Xi$ to show the choices we made.

THEOREM 20.5. *$HF_*((M, \omega), J, \Xi)$ is independent of the choices of $J, H, \Xi$ and depends only on $(M, \omega)$ up to canonical isomorphism induced from chain homotopy equivalence, which is also canonical up to chain homotopy.*

The proof is similar to one by Floer which is discussed in Section 18, by modifying it in the same way as last and this sections. So we discuss it only briefly.

Let $(H_\alpha, J_B)$, $\Xi_\alpha$, $(H_\beta, J_\beta)$, $\Xi_\beta$ be two choices of $J$, $H$, $\Xi$. We join $(H_\alpha, J_\beta)$ and $(H_\beta, J_\beta)$ by homotopy $(H_\tau, J_\tau)$ satisfying (18.12). We obtain a differential equation (18.13). Let $\tilde{\ell}^- \in \tilde{\mathcal{P}}(H_\alpha)$, $\tilde{\ell}^- \in \tilde{\mathcal{P}}(H_\beta)$. Let $\mathcal{M}(para; \tilde{\ell}^-, \tilde{\ell}^+)$ be the set of all solutions of (18.13) such that $\lim_{\tau \to \pm\infty} h(\tau, t) = \ell^{\pm}(t)$. We remark that we do not divide it by $\mathbf{R}$-action since eq. (18.13) is not translation invariant.

We define its compactification $C\mathcal{M}_m(para; \tilde{\ell}^-, \tilde{\ell}^+)$ in the same way in Definition 19.9. More precisely its element is $((h_1, \dots, h_k), (\sigma_1, \dots, \sigma_I), o)$ where $h_i \in \mathcal{M}(\alpha; \tilde{\ell}_i, \tilde{\ell}_{i+1})$ $i < i_0$, $h_i \in \mathcal{M}(\beta; \tilde{\ell}_i, \tilde{\ell}_{i+1})$ $i > i_0$ and $h_{i_0} \in \mathcal{M}(para; \tilde{\ell}_{i_0}, \tilde{\ell}_{i_0+1})$. Here $\mathcal{M}(\alpha; \tilde{\ell}_i, \tilde{\ell}_{i+1})$ is the moduli space $\mathcal{M}(\tilde{\ell}_i, \tilde{\ell}_{i+1})$ defined by using $(H_\alpha, J_\alpha)$. The other conditions for $((h_1, \dots, h_k), (\sigma_1, \dots, \sigma_I), o)$ is similar to Definitions 19.9 and 19.10. We remark that we divide only $\mathcal{M}(\alpha; \tilde{\ell}_i, \tilde{\ell}_{i+1})$, $\mathcal{M}(\beta; \tilde{\ell}_i, \tilde{\ell}_{i+1})$, $i \neq i_0$ by translation symmetry.

We can define a topology on $C\mathcal{M}(para; \tilde{\ell}^-, \tilde{\ell}^+)$ and can prove that it is Hausdorff and compact in the same way as Theorem 19.12. We can then prove that $C\mathcal{M}(\text{para}; \tilde{\ell}^-, \tilde{\ell}^+)$ has a Kuranishi structure with corners and of dimension $\mu(\tilde{\ell}^-) - \mu(\tilde{\ell}^+)$ in the same way as the proof of Theorem 19.14.

There are maps *Glue* similar to one in Section 19. Namely *Glue* is a map such as

$$C\mathcal{M}(\alpha; \tilde{\ell}_1, \tilde{\ell}_2) \times C\mathcal{M}(para; \tilde{\ell}_2, \tilde{\ell}_3) \times C\mathcal{M}(\beta; \tilde{\ell}_3, \tilde{\ell}_4) \to C\mathcal{M}(para; \tilde{\ell}_1, \tilde{\ell}_4).$$

We can then construct the Kuranishi structure compatible to all of *Glue* and the ones we have constructed on $C\mathcal{M}(\alpha; \tilde{\ell}_1, \tilde{\ell}_2)$, $C\mathcal{M}(\beta; \tilde{\ell}_1, \tilde{\ell}_2)$, (i.e. a part of data in $\Xi_\alpha$ and $\Xi_\beta$). We next use the good coordinate systems and multisections on $C\mathcal{M}(\alpha; \tilde{\ell}_1, \tilde{\ell}_2)$, $C\mathcal{M}(\beta; \tilde{\ell}_1, \tilde{\ell}_2)$ and the one on $C\mathcal{M}(para; \tilde{\ell}^-, \tilde{\ell}^+)$. We find good coordinate systems and multisections on $C\mathcal{M}(para; \tilde{\ell}^-, \tilde{\ell}^+)$ which are compatible with *Glue*.

We define $\Phi_{k,\,(H_\tau, J_\tau)} : C_k(H_\alpha, J_\alpha) \to C_k(H_\beta, J_\beta)$ by

$$\Phi_{(H_\tau, J_\tau)}(\delta_{\tilde{\ell}^-}) = \sum_{\substack{\mu(\tilde{\ell}^-) = \mu(\tilde{\ell}^+) \\ \tilde{\ell}^+ \in \tilde{\mathscr{P}}(H_\beta)}} [C\mathscr{M}(para;\, \tilde{\ell}^-, \tilde{\ell}^+)]\delta_{\tilde{\ell}^+}.$$

Here $[C\mathscr{M}(para;\, \tilde{\ell}^-, \tilde{\ell}^+)] \in \mathbf{Q}$ is defined in a similar way to $[C\mathscr{M}(\tilde{\ell}^-, \tilde{\ell}^+)]$. We can then use a similar argument to Lemma 20.2 to show that $\Phi_{k,\,(H_\tau,\, J_\tau)}$ is a chain map.

Next we show that this chain map is, up to chain homotopy, independent of the choice of $(H_\tau, J_\tau)$, Kuranishi structures, good coordinate system, and multisections we used to define it. We write $\Xi_\tau$ for the choice of Kuranishi structures, good coordinate system, and multisections, and write $\Phi_{k,\,(H_\tau,\, J_\tau,\, \Xi_\tau)}$ to show it explicitly.

We choose a homotopy $(H_{\tau,u}, J_{\tau,u})$, $u \in [0, 1]$ such that $(H_{\tau,0}, J_{\tau,0}) = (H_\tau, J_\tau)$ and that $(H_{\tau,1}, J_{\tau,1}) = (H'_\tau, J'_\tau)$. Also $(H_{\tau,u}, J_{\tau,u})$ satisfies condition (18.12) for each $u \in [0, 1]$. Then we consider the union

$$C\mathscr{M}(parapara;\, \tilde{\ell}^-, \tilde{\ell}^+) = \bigcup_{u \in [0,1]} \{u\} \times C\mathscr{M}(para,\, J_{\tau,u};\, \tilde{\ell}^-, \tilde{\ell}^+).$$

Here we write $C\mathscr{M}(para,\, J_{\tau,u};\, \tilde{\ell}^-, \tilde{\ell}^+)$ to show that we use $(H_{\tau,u}, J_{\tau,u})$ to define it. We then can repeat the same argument and show that $C\mathscr{M}(parapara;\, \tilde{\ell}^-, \tilde{\ell}^+)$ has an oriented Kuranishi structure of dimension $\mu(\tilde{\ell}) - \mu(\tilde{\ell}^+) + 1$ such that it is compatible with various *Glue*. Also we may assume that its Kuranishi structure coincides to the one given by $\Xi_\tau$ and $\Xi'_\tau$ at $u = 0, 1$, respectively. Now we consider the case $\mu(\tilde{\ell}^-) - \mu(\tilde{\ell}^+) = -1$. We put

$$\mathscr{H}(\delta_{\tilde{\ell}^-}) = \sum_{\substack{\mu(\tilde{\ell}^-) = \mu(\tilde{\ell}^+) - 1 \\ \tilde{\ell}^+ \in \tilde{\mathscr{P}}(H_\beta)}} [C\mathscr{M}(parapara;\, \tilde{\ell}^-, \tilde{\ell}^+)]\delta_{\tilde{\ell}^+}.$$

Here we use multisection on $C\mathscr{M}_m(parapara;\, \tilde{\ell}^-, \tilde{\ell}^+)$ extending one given by $\Xi_\tau$ and $\Xi'_\tau$. In a way similar to the proof of Lemma 20.2, we obtain

$$\Phi_{(H_\tau,\, J_\tau,\, \Xi_t)} - \Phi_{(H'_\tau,\, J'_\tau,\, \Xi'_t)} = \mathscr{H}\partial + \partial\mathscr{H}.$$

Thus $\Phi_{(H_\tau,\, J_\tau)} : C_*(H_\alpha, J_\alpha) \to C_*(H_\beta, J_\beta)$ is independent of various choices up to chain homotopy. Let

$$\Phi^{\alpha,\beta} : HF_*(H_\alpha, J_\alpha) \to HF_*(H_\beta, J_\beta)$$

be the map induced on homology groups. Next we claim

$$\Phi^{\beta,\gamma} \circ \Phi^{\alpha,\beta} = \Phi^{\alpha,\gamma}. \tag{20.6}$$

The proof is the same as Floer's and is omitted. On the other hand, we can prove that

$$\Phi^{\alpha,\alpha} = id. \tag{20.7}$$

To show (20.7) we remark that we can use trivial homotopy $(H_\tau, J_\tau) \equiv (H_\alpha, J_\alpha)$ to calculate $\Phi^{\alpha,\alpha}$. Then we find that there exists an $\mathbf{R}$ action on $C\mathscr{M}(para;\, \tilde{\ell}^-, \tilde{\ell}^+)$, the translation along $\tau$ coordinate. This action is free if $\tilde{\ell}^- \neq \tilde{\ell}^+$. In fact

$$\frac{\overline{C\mathscr{M}(para;\, \tilde{\ell}^-, \tilde{\ell}^+)}}{\mathbf{R}} = \overline{C\mathscr{M}(\alpha;\, \tilde{\ell}^-, \tilde{\ell}^+)}.$$

We remark also that our Kuranishi structure, good coordinate system, and multisections are the same for $\tau = -\infty, +\infty$. Hence we may assume that the Kuranishi structure, good coordinate system, and multisection are products of the ones for $C\mathcal{M}(\alpha; \tilde{\ell}^-, \tilde{\ell}^+)$ and $\mathbf{R}$. Hence if $\tilde{\ell}^- \neq \tilde{\ell}^+$ and $\mu(\tilde{\ell}^-) = \mu(\tilde{\ell}^+)$ then the multisection does not vanish on $C\mathcal{M}(para; \tilde{\ell}^-, \tilde{\ell}^+)$. On the other hand, if $\tilde{\ell}^- = \tilde{\ell}^+$, there is one element in $C\mathcal{M}(para; \tilde{\ell}^-, \tilde{\ell}^+)$, $h(\tau, t) \equiv \tilde{\ell}^-(t) = \tilde{\ell}^+(t)$. Using the fact that all the periodic orbits are nondegenerate $h(\tau, t) \equiv \tilde{\ell}^+(t)$ is isolated in $C\mathcal{M}(para; \tilde{\ell}^+, \tilde{\ell}^+)$ and is transversal. On the complement of $h(\tau, t) \equiv \tilde{\ell}^+(t)$ in $C\mathcal{M}(para; \tilde{\ell}^+, \tilde{\ell}^+)$, we have again an $\mathbf{R}$-action, hence we can show that there is no zero of multisection there. Thus we have $\Phi^{\alpha,\alpha} = id$. The proof of Theorem 20.5 is now complete. □

We close this section by proving Lemma 19.8. The proof is by contradiction. We assume that there exists a sequence $h_i : \mathbf{R} \times S^1 \to M$ such that

(20.8.1) $h_i$ is a solution of (1.94),

(20.8.2) $\lim_{i\to\infty} E_H(h_i) = 0$,

(20.8.3) $E_H(h_i) \neq 0$ for every $i$,

(20.8.4) $\lim_{\tau\to\pm\infty} \left|\frac{\partial h_i}{\partial \tau}\right| = 0$ for each $i$,

and will deduce a contradiction.

Sublemma 20.9. $\limsup_{i\to\infty}\left(\left|\frac{\partial h_i}{\partial \tau}\right| + \left|\frac{\partial h_i}{\partial t}\right|\right) \leqslant C$, *where* $C$ *is independent of* $i$.

*Proof.* If not there exists a subsequence, and $(\tau_i, t_i) \in \mathbf{R} \times S^1$ such that

$$\lim_{i\to\infty}\left(\left|\frac{\partial h_i}{\partial \tau}\right| + \left|\frac{\partial h_i}{\partial t}\right|\right)(\tau_i, t_i) = \infty.$$

By (20.8.4), we may assume that $|\partial h_i/\partial \tau|$ is maximal at $(\tau_i, t_i)$. By scaling we have a pseudoholomorphic map $h : \mathbf{C} \to M$ such that

$$\int_C h^*\omega \leqslant \limsup_{i\to\infty} E_H(h_i).$$

By Lemma 8.1 and removable singularity, the left-hand side is not smaller than $\delta$. This

Sublemma 20.10. $\limsup_{i\to\infty}\left|\frac{\partial h_i}{\partial \tau}\right| =$

*Proof.* If not, there exists a subsequence, and $(\tau_i, t_i) \in \mathbf{R} \times S^1$ such that $|\partial h_i/\partial \tau|(\tau_i, t_i) > \delta_0 > 0$. By (20.8.4), we may assume that $|\partial h_i/\partial \tau|$ is maximal at $(\tau_i, t_i)$. By translating the solution in $\tau$ direction, we may assume $\tau_i = 0$. By Sublemma 20.8 and elliptic regularity, we have a subsequence which converges in $C^\infty$-topology on any compact set.

Hence we have a limit of $\lim_{i\to\infty} h_i = h$ such that $E_H(h) \leqslant \liminf_{i\to\infty} E_H(h_i) = 0$. On the other hand, we have $|\partial h/\partial\tau|(0, t) \geqslant \delta_0 > 0$. Since $C^2$-norm of $h_i$ is bounded by Sublemma 20.9 and elliptic regularity. (Here we put $t = \lim_{i\to\infty} t_i$.) Since $h$ is a zero of (19.4), it follows from $|\partial h/\partial\tau|(0, t) > 0$ that $E_H(h) > 0$, a contradiction. □

SUBLEMMA 20.11. *For sufficiently large i, there exists* $\ell_i \in \mathscr{P}(H)$ *such that*

$$\lim_{i\to\infty} \sup_{\tau,t} dist(h_i(\tau, t), \ell_i(t)) = 0.$$

*Proof.* Let $\phi : M \to M$ be the time one map. By Sublemma 20.10 we find that

$$\lim_{i\to\infty} \sup dist(\phi h_i(\tau, 0), h_i(\tau, 0)) = 0.$$

Using the fact that there are only a finite number of fixed points of $\phi$, Sublemma 20.11 follows easily. □

Since $\mathscr{P}(H)$ is a finite set, we may assume that $\ell_i = \ell$ is independent of $i$. The rest of the proof is similar to (and easier than) the proof of Lemma 11.2 in Section 14.

We put

$$h_i(\tau, t) = \exp_{\ell(t)} a_i(\tau, t).$$

In a way similar to Sublemma 14.3, we can prove that

$$\frac{da_i}{d\tau}(t) = -J(\ell(t))\frac{Da_i}{dt} - \nabla^2_{\ell(t)} H_t(a_i) + O_2(a_i) \tag{20.12}$$

$$\|O_2(a)\|_{L^2_{m-1}([\tau)\times S^1)} \leqslant \|a\|^2_{L^2_m((\tau-1/2,\tau+1/2)\times S^1)}. \tag{20.13}$$

We remark that the nondegeneracy of $\ell$ implies that the operator

$$a \mapsto -J(\ell(t))\frac{Da_i}{dt} - \nabla^2_{\ell(t)} H_t(a) \tag{20.14}$$

is invertible. By (20.8.4), there is a point $\tau_i$ where $\|a_i(\tau)\|_{L^2(S^1)}$ is maximal. By Sublemma 20.10 we may assume that $\lim_{i\to\infty} \|a_i(\tau_i)\|_{L^2(S^1)} = 0$. Using it and elliptic regularity we find that

$$\|O_2(a_i(\tau))\|_{L^2} \leqslant \varepsilon_i \|a_i(\tau)\|_{L^2} \quad \text{and} \quad \lim_{i\to\infty} \varepsilon_i = 0. \tag{20.15}$$

On the other hand, by (20.11) and

$$\left.\frac{d}{d\tau}\|a_i(\tau)\|^2_{L^2(S^1)}\right|_{\tau=} = 0,$$

$$\left|\left\langle -J(\ell(t))\frac{Da_i(\tau_i)}{dt} - \nabla^2_{\ell(t)} H_t(a_i(\tau_i)), a_i(\tau_i)\right\rangle\right| < \langle O_2(a_i)(\tau_i), a_i(\tau_i)\rangle. \tag{20.16}$$

(20.15) and (20.16) contradicts the invertibility of (20.14).

The proof of Lemma 19.8 is now complete.

## 21. COHERENT ORIENTATION

In this section, we are going to define an orientation of our Kuranishi structure constructed in Section 19. The existence of tangent bundle can be proved in the same way as in the proof of Proposition 16.1. In view of the argument of Section 16, we are only to show the following Lemma 21.4 to construct an orientation.

Let $\mathscr{B}(\tilde{\ell}^-, \tilde{\ell}^+)$ be as in the beginning of Section 19. But in this section we consider its subset consisting of smooth maps (for simplicity), and denote it by the same symbol. For each element $h \in \mathscr{B}(\tilde{\ell}^-, \tilde{\ell}^+)$ we have an elliptic complex (19.5). It gives a family of elliptic operators parametrized by $\mathscr{B}(\tilde{\ell}^-, \tilde{\ell}^+)$. We denote it by $D_h\bar{\partial}_{J,H}$. Hence its index gives an element

$$Index(D_h\bar{\partial}_{J,H}, \tilde{\ell}^-, \tilde{\ell}^+) \in KO(\mathscr{B}(\tilde{\ell}^-, \tilde{\ell}^+)). \tag{21.1}$$

The orientation corresponds to a lift of (21.1) to $KSO(\mathscr{B}(\tilde{\ell}^-, \tilde{\ell}^+))$. We need to make sure that the lift satisfies various compatibility conditions for gluing and bubble. To describe it we need some notations.

We take an arbitrary compact subset of $\mathscr{K}(\tilde{\ell}_i, \tilde{\ell}_{i+1}) \subseteq \mathscr{B}(\tilde{\ell}_i, \tilde{\ell}_{i+1})$. We then have a gluing map

$$Pat: \mathscr{K}(\tilde{\ell}_1, \tilde{\ell}_2) \times \mathscr{K}(\tilde{\ell}_2, \tilde{\ell}_3) \to \mathscr{B}(\tilde{\ell}_1, \tilde{\ell}_3). \tag{21.2}$$

We define it by shifting elements of $\mathscr{K}(\tilde{\ell}_i, \tilde{\ell}_{i+1})$ so that their supports are almost disjoint and gluing them by using a partition of unity. We do not specify the map since we only need its homotopy class.

We next consider the compatibility with bubble. For $\beta \in \Gamma$, let $\mathscr{B}(\beta)$ be the set of all smooth maps from $S^2$ to $M$ representing $\beta$. We define $ev: \mathscr{B}(\beta) \to M$ by $ev(h) = h(p_0)$. Here we fix $p_0 \in S^2$. Let $\mathscr{K}(\beta) \subseteq \mathscr{B}(\beta)$ and $\mathscr{K}(\tilde{\ell}^-, \tilde{\ell}^+) \subseteq \mathscr{B}(\tilde{\ell}^-, \tilde{\ell}^+)$ be compact subsets. Choose sufficiently small $\varepsilon$ and put

$$\mathscr{K}(\beta) \times_\varepsilon \mathscr{K}(\tilde{\ell}^-, \tilde{\ell}^+) = \{(h, h', (\tau, t)) \mid dist(ev(h), h'(\tau, t)) < \varepsilon\}.$$

By choosing $\varepsilon$ enough small we find a map

$$Pat: \mathscr{K}(\beta) \times_\varepsilon \mathscr{K}(\tilde{\ell}^-, \tilde{\ell}^+) \to \mathscr{B}(\beta \# \tilde{\ell}^-, \tilde{\ell}^+),$$

whose homotopy class is well defined. For each $h \in \mathscr{B}(\beta)$, we consider the linearization of pseudoholomorphic curve equation:

$$D_h\bar{\partial}: L^p_1(S^2, h^*TM) \to L^p(S^2, \Lambda^{0,1}(S^2) \otimes h^*TM). \tag{21.3}$$

(21.3) is a family of elliptic operators. Hence we have its index

$$Index(D_h\bar{\partial}; \beta) \in KO(\mathscr{B}(\beta)).$$

The symbol of (21.3) is complex linear and $S^2$ is closed. Hence we find an element $Index(D_h\bar{\partial}) \in K(\mathscr{B}(\beta))$ which is a lift of $Index(D_h\bar{\partial})$. It induces $Ori(\beta) \in KSO(\mathscr{B}(\beta))$.

LEMMA 21.4. *There exists an element $Ori(\tilde{\ell}^-, \tilde{\ell}^+) \in KSO(\mathscr{B}(\tilde{\ell}^-, \tilde{\ell}^+))$ with the following properties.*

(21.4.1) *$Ori(\tilde{\ell}^-, \tilde{\ell}^+)$ goes to $Index(D_h\bar{\partial}_{J,H}, \tilde{\ell}^-, \tilde{\ell}^+)$ by the natural projection $KSO(\mathscr{B}(\tilde{\ell}^-, \tilde{\ell}^+)) \to KO(\mathscr{B}(\tilde{\ell}^-, \tilde{\ell}^+))$*

(21.4.2) *Let* $\mathscr{K}(\tilde{\ell}_i, \tilde{\ell}_{i+1}) \subseteq \mathscr{B}(\tilde{\ell}_i, \tilde{\ell}_{i+1})$ *be compact subsets and incl denotes the inclusion map. We have*

$$Pat^*(Ori(\tilde{\ell}_i, \tilde{\ell}_3)) = incl^*Ori(\tilde{\ell}_1, \tilde{\ell}_2) \oplus incl^*Ori(\tilde{\ell}_2, \tilde{\ell}_3).$$

(21.4.3) *Let* $\mathscr{K}(\tilde{\ell}^-, \tilde{\ell}^+) \subseteq \mathscr{B}(\tilde{\ell}^-, \tilde{\ell}^+)$, $\mathscr{K}(\beta) \subseteq \mathscr{B}(\beta)$ *be compact subsets and incl be inclusion maps. Then we have*

$$Pat^*(Ori(\beta \# \tilde{\ell}^-, \tilde{\ell}^+)) = incl^*Ori(\beta) \oplus incl^*Ori(\tilde{\ell}^-, \tilde{\ell}^+).$$

*Remark* 21.5. The reader may feel cumbersome to go back and check the arguments of Sections 16 and 19 to see that Lemma 21.4 implies the existence of the orientation on our Kuranishi structure, though it is in fact quite immediate.

However, in fact, we need orientation only on the main stratum and on the stratum corresponding to the connecting orbits consisting of two maps $\mathbf{R} \times S^1 \to M$ glued by a periodic solution; for the argument in Section 20. (This is because we only need the existence of tangent bundle to make the multisection transversal to the strata of negative virtual dimension.) In that case, the existence of orientation is immediate from Lemma 21.4.

*Proof.* The argument below is indicated in Floer's original paper [19] (see also [21]). Let $\tilde{\ell}^-, \tilde{\ell}^+ \in \tilde{\mathscr{P}}(H)$. We write $\tilde{\ell}^{\pm} = [\ell^{\pm}, u^{\pm}]$.

We take $[0, \infty) \times S^1$ and $D^2$ and glue them at $\{0\} \times S^1 \cong \partial D^2$. Let $Y$ be the 2 manifold we obtain. We call it "a cap with half infinite cylinder". Note that $Y$ has a natural conformal structure.

We glue the map $u^{\pm}: D^2 \to M$ with the map $[0, \infty) \times S^1 \to M$ which is a composition of the projection to the second factor and $\ell^{\pm}$. We obtain a map from $Y$ to $M$. We denote this map also by $u^{\pm}: Y \to M$.

Let $u: D^2 \to M$ be a map which restricts to an element of $\mathscr{P}(H)$ at $\partial D^2$. Choose a cut-off function $\chi: [0, \infty) \to [0, 1]$ which is 0 in a neighborhood of 0 and which is 1 if $\tau > R$. We define the operator $P^-(u): L^p_1(Y, u^*TM) \to L^p(Y, \Lambda^{0,1}(Y) \otimes u^*TM)$ by

$$P^-(u) = \begin{cases} (1 - \chi(\tau))(D_u\bar{\partial}) + \chi(\tau)(D_u\bar{\partial}_H) & \text{on } [0, \infty) \times S^1, \\ D_u\bar{\partial} & \text{on } D^2. \end{cases} \tag{21.7}$$

Gluing $(-\infty, 0] \times S^1$ and $D^2$ at $\{0\} \times S^1$ we get a 2 manifold $\bar{Y}$. Here we choose the orientation of $D^2$ opposite to the standard one. Note that $Y$ and $\bar{Y}$ are canonically diffeomorphic but have the opposite complex structure. For a map $u: D^2 \to M$, we can extend it to $\bar{Y}$ in a similar way. We can define an operator $P^+(u): L^p_1(\bar{Y}, u^*TM) \to L^p(\bar{Y}, \Lambda^{0,1}(\bar{Y}) \otimes u^*TM)$ in way similar to (21.7).

Using the fact that $\ell$ is a nondegenerate periodic orbit, we can prove that $P^+(u)$, $P^-(u)$ are Fredholm.

Let $\theta \in \pi_2(M, \ell)$ with $\ell \in \mathscr{P}(H)$. We have a homotopy class of maps $u: D^2 \to M$ which restricts to $\ell$. By gluing $h(\tau, t) = \ell(t)$, and resolving singularity at $S^1$ by using partition of unity, we regard $u: D^2 \to M$ as a map $u: Y \to M$.

We let $\mathscr{B}(\theta, \ell)$ be the Banach manifold consisting of smooth maps in this homotopy class such that it converges to $\ell$ at infinity in $L^p_1$ norm. We can define $P^+(u)$ and $P^-(u)$ by Formula (21.7) for $u \in \mathscr{B}(\theta, \ell)$. Taking the indices of $P^+(u)$ and $P^-(u)$, we get elements of $KO(\mathscr{B}(\theta, \ell))$ which we denote by *index* $P^+(\theta)$ and *index* $P^-(\theta)$.

SUBLEMMA 21.8. *There is an isomorphism canonical up to homotopy between* det *Index* $P^+(\theta)$ *and* det *Index* $P^-(\theta)$. *Here* det *denotes the determinant line bundle of a virtual vector bundle considered as a (virtual) bundle over* $\mathbf{R}$.

*Proof.* Fix a positive real number $R$. Gluing $Y$–$(R,\infty)\times S^1$ and $\bar{Y}-(-\infty,-R)\times S^1$ along boundaries, we get an oriented 2-manifold with conformal structure, which is isomorphic to $\mathbf{C}P^1$. Using partition of unity, we can glue the operators $P^+(u)$ and $P^-(u)$ to obtain an operator $P^+(u)\#\, P^-(u)$. It is parametrized by $u\in\mathscr{B}(\theta,\ell)$ and we can consider the index of this family. Since the symbol of the operators are complex linear it follows that this family of operators can be deformed to a family of complex linear elliptic operators on $\mathbf{C}P^1$. In particular it has a canonical orientation on its determinant line bundle, hence its determinant line bundle is trivial.

By the sum formula for the index of a family of elliptic differential operators (see, for example, [24, Section 4]), index bundle for $P^+(u)\#\, P^-(u)$ is the sum of the index bundles for $P^+(u)$ and $P^-(u)$ in $KO$ group. Therefore we have

$$\det P^+(u)\# P^-(u)=\det P^+(u)\otimes\det P^{-1}(u).$$

Since the left-hand side is a trivial bundle, $\det P^+(u)$ is isomorphic to $\det P^-(u)$.

Ambiguity of isomorphisms between real-line bundles are multiplication by a non-vanishing real-valued functions. Since $\det P^+(u)\otimes\det P^-(u)$ has a canonical orientation, the isomorphism coming from isomorphism of this bundle is a well-defined positive real-valued function. Hence the isomorphisms are given uniquely up to homotopy. □

SUBLEMMA 21.9. det *Index* $P^-(\theta)$ *is a trivial bundle.*

*Proof.* Fix $u_0$ representing $\theta\in\pi_2(M,\ell)$. We replace $P^+(u)$ in the proof of Sublemma 21.8 by $P^+(u_0)$. Using a partition of unity, we glue the elliptic differential operators $P^-(u)$ and $P^+(u_0)$ to get an elliptic differential operator on $\mathbf{C}P^1$ with complex linear symbol. Hence its index bundle has a canonical orientation. By the sum formula for a family of index, we have

$$\det \mathit{Index}(P^-(u)\#\; P^+(u_0))=\det \mathit{Index}(P^-(u))\otimes\det \mathit{Index}(P^+(u_0)).$$

Since the left-hand side is trivial and since $P^+(u_0)$ is a fixed operator, it follows that det *Index* $P^-(\theta)$ is a trivial bundle. □

*Remark* 21.10. Conley–Zehnder index is regarded as Atiyah–Patodi–Singer type index for $P^-(u)$. (See Appendix of [42].)

Now for each $\tilde{\ell}=[\ell,u]\in\tilde{\mathscr{P}}(H)$, we fix for a moment $\theta_{\tilde{\ell}}\in\pi_2(M,\ell)$ such that $\theta_{\tilde{\ell}}\#(-u)\in\ker\phi_\omega\cap\ker\phi_{c_1}$. (This condition is independent of the representative $(\ell,u)$.) We take the representative so that $u_{\tilde{\ell}}\in\theta_{\tilde{\ell}}$ and fix it for a moment. We next fix a trivialization of the bundle *Index* $P^+(\theta_{\tilde{\ell}})$. Then the trivialization of *Index* $P^-(\theta_{\tilde{\ell}})$ is induced by Sublemma 21.8. We are going to discuss later the effect of the change of the choices of $\theta_{\tilde{\ell}}\in\pi_2(M,\ell)$, the trivialization of *Index* $P^+(\theta_{\tilde{\ell}})$.

Once we fix them we can find $Ori(\tilde{\ell}^-,\tilde{\ell}^+)\in KSO(\mathscr{B}(\tilde{\ell}^-,\tilde{\ell}^+))$ as follows. We take a compact subset $\mathscr{K}(\tilde{\ell}^-,\tilde{\ell}^+)\subseteq\mathscr{B}(\tilde{\ell}^-,\tilde{\ell}^+)$ and $\mathscr{K}(\theta_{\tilde{\ell}^\pm},\ell^\pm)\subseteq\mathscr{B}(\theta_{\tilde{\ell}^\pm},\ell^\pm)$. We then define

$$Pat:\mathscr{K}(\theta_{\tilde{\ell}^-},\ell^-)\times\mathscr{K}(\tilde{\ell}^-,\tilde{\ell}^+)\times\mathscr{K}(\theta_{\tilde{\ell}^+},\ell^+)\to\mathscr{B}(0)$$

by using partition of unity. By fixing the first and third component to $u_{\tilde{\ell}^{\pm}}$ we obtain a map $Pat: \mathscr{K}(\tilde{\ell}^-, \tilde{\ell}^+) \to \mathscr{B}(0)$. By sum formula for family of indices, we have

$$Index(D_h\bar{\partial}, \beta) = Pat^*(Index(P^-(\theta_{\tilde{\ell}^-})) + incl^*Index(P^+(\theta_{\tilde{\ell}^+}))$$

$$+ incl^*Index(D_h\bar{\partial}_{J,H}, \tilde{\ell}^-, \tilde{\ell}^+).$$

Hence the trivializations of $\det Index(D_h\bar{\partial}, \beta)$, $\det Index P^-(u_{\tilde{\ell}^-})$ and $\det Index P^+(u_{\tilde{\ell}^+})$ induce an orientation of $incl^*\, Index(D_h\bar{\partial}_{J,H}, \tilde{\ell}^-, \tilde{\ell}^+)$. Since it is natural with inclusions, we obtain an orientation of $Index(D_h\bar{\partial}_{J,H}, \tilde{\ell}^-, \tilde{\ell}^+)$ namely the lift $Ori(\tilde{\ell}^-, \tilde{\ell}^+) \in KSO(\mathscr{B}(\tilde{\ell}^-, \tilde{\ell}^+))$.

We next discuss the effect of the change of $\theta_{\tilde{\ell}} \in \pi_2(M, \ell)$.

SUBLEMMA 21.11. *Let $u_1, u_2: D^2 \to M$ such that $[\ell, u_1] = [\ell, u_2]$. Let $\theta_i \in \pi_2(M, \ell)$ be the homotopy class of $u_i$. Then there is a canonical one-to-one correspondence between the trivialization of $Index\, P^-(\theta_1)$ and $Index\, P^-(\theta_2)$.*

*Proof.* We choose $u: D^2 \to M$ such that $[\ell, u_1] = [\ell, u_2] = [\ell, u]$. Then $h \mapsto h \# u$ defines a map $\mathscr{K}(\theta_i, \ell_i) \to \mathscr{B}(0)$. (The latter is the space of maps $h: S^2 \to M$ such that $h^*c_1 = h^*\omega = 0$.)

Thus, by index sum formula, the orientation of $Index(D_h\bar{\partial}; 0) \in KO(\mathscr{B}(0))$ and orientation of $\det Index\, P^+(u)$ determines orientation of both of $\det Index\, P^-(u_1)$ and $\det Index\, P^-(u_2)$. Sublemma 21.11 follows. □

*Remark* 21.12. The space $\mathscr{B}(0)$ is disconnected in general. The trivialization of it may not be unique. However we can fix it as follows. Each element of $h \in \mathscr{B}(0)$ induces an elliptic complex $Index(D_h\bar{\partial}; 0)$ on $\mathbf{C}P^1$ whose symbol is the same as the Dolbeault complex over $\mathbf{C}P^1$. The space of elliptic operator on $\mathbf{C}P^1$ whose symbol is the same as the Dolbeault complex is connected and there is a canonical orientation on it, which is induced by the complex orientation. This argument works for $\mathscr{B}(\beta)$ also. This orientation is one we used to define $Ori(\beta) \in KSO(\mathscr{B}(\beta))$.

Using Sublemma 21.11 we find that we need only once to choose an orientation of $Index\, P^-(\theta_1)$ for each $\tilde{\ell} \in \tilde{\mathscr{P}}(H)$. (Then the others are induced automatically.)

We furthermore will make the orientation compatible for the action of $\Lambda$ on $\tilde{\mathscr{P}}(H)$. Let $\beta \in \Lambda$, we represent it by a map $h: \mathbf{C}P^1 \to M$. Choose $[\ell, u] \in \tilde{\mathscr{P}}(H)$. We may assume that $u(0) = h(0)$. Hence by gluing we have

$$\det Index\, P^+(\beta \# u) = \det Index\, P^+(u) \otimes \det Index(D_h\bar{\partial}; \beta) \otimes \det T_{u(0)}M^*.$$

Hence using trivialization of $\det Index(D_h\bar{\partial}; \beta)$ and orientation of $M$ we find that the orientation of $Index\, P^+(u)$ induces orientation on $Index\, P^+(u \# \beta)$. By Sublemma 21.11 and its proof, this correspondence is independent of the choice of $h$ and is compatible by the identification of the orientations for different choice of $u$.

Therefore, if we fix a choice of trivialization of $\det Index\, P^+(u)$ such that $[\ell, u] \in \tilde{\mathscr{P}}(H)$, then for any $[\ell, u'] \in \tilde{\mathscr{P}}(H)$, we have a trivialization of $\det Index\, P^-(u')$. In other words, we are only to choose orientations for each element of $\mathscr{P}(H)$. (This is a finite set.)

If we choose orientations of $\det Index\, P^-(u)$ in this way, (21.4.3) is immediate from the definition.

We next prove (21.4.2). Let us consider a compact set $\mathscr{K}(\tilde{\ell}^-, \tilde{\ell}^+) \subseteq \mathscr{B}(\tilde{\ell}^-, \tilde{\ell}^+)$ and $\mathscr{K}(\theta_{\tilde{\ell}^{\pm}}, \ell^{\pm}) \subseteq \mathscr{B}(\theta_{\tilde{\ell}^{\pm}}, \ell^{\pm})$. We then define a map

$$Pat : \mathscr{K}(\theta_{\tilde{\ell}^-}, \ell^-) \times \mathscr{K}(\tilde{\ell}^-, \tilde{\ell}^+) \to \mathscr{B}(\theta_{\tilde{\ell}^+}, \ell^+) \tag{21.13}$$

by gluing. Then by index sum formula we have the following isomorphism.

$$Pat^*(\det Index(P^+(\theta_{\tilde{\ell}^+}))) = \det Index(P^+(\theta_{\tilde{\ell}^-})) \otimes \det incl^* Index(D_h\bar{\partial}_{J,H}, \tilde{\ell}^-, \tilde{\ell}^+). \tag{21.14}$$

Then by definition and the proof of Sublemma 21.11, we find that isomorphism (21.14) is compatible with trivializations.

We now prove (21.4.2). We choose $\mathscr{K}(\tilde{\ell}_i, \tilde{\ell}_{i+1}) \subseteq \mathscr{B}(\tilde{\ell}_i, \tilde{\ell}_{i+1})$ such that

$$Pat(\mathscr{K}(\tilde{\ell}_1, \tilde{\ell}_2) \times \mathscr{K}(\tilde{\ell}_2, \tilde{\ell}_3)) \subseteq \mathscr{K}(\tilde{\ell}_1, \tilde{\ell}_3).$$

Furthermore we choose $\mathscr{K}(\theta_{\tilde{\ell}_i}, \ell) \subseteq \mathscr{B}(\theta_{\tilde{\ell}_i}, \ell)$ such that

$$Pat(\mathscr{K}(\theta_{\tilde{\ell}_1}, \ell_1) \times \mathscr{K}(\tilde{\ell}_1, \tilde{\ell}_2)) \subseteq \mathscr{K}(\theta_{\tilde{\ell}_2}, \ell_2)$$

$$Pat(\mathscr{K}(\theta_{\tilde{\ell}_1}, \ell_1) \times \mathscr{K}(\tilde{\ell}_1, \tilde{\ell}_3)) \subseteq \mathscr{K}(\theta_{\tilde{\ell}_3}, \ell_3)$$

$$Pat(\mathscr{K}(\theta_{\tilde{\ell}_2}, \ell_2) \times \mathscr{K}(\tilde{\ell}_2, \tilde{\ell}_3)) \subseteq \mathscr{K}(\theta_{\tilde{\ell}_3}, \ell_3).$$

We then find that the following diagram is homotopy commutative:

$$\begin{array}{ccc}
\mathscr{K}(\theta_{\tilde{\ell}_1}, \ell_1) \times \mathscr{K}(\tilde{\ell}_1, \tilde{\ell}_2) \times \mathscr{K}(\tilde{\ell}_2, \tilde{\ell}_3) & \xrightarrow{1 \times Pat} & \mathscr{K}(\theta_{\tilde{\ell}_1}, \ell_1) \times \mathscr{K}(\tilde{\ell}_1, \tilde{\ell}_3) \\
\downarrow_{Pat \times 1} & & \downarrow_{Pat} \\
\mathscr{K}(\theta_{\tilde{\ell}_2}, \ell_2) \times \mathscr{K}(\tilde{\ell}_2, \tilde{\ell}_3) & \xrightarrow{Pat} & \mathscr{K}(\theta_{\tilde{\ell}_3}, \ell_3)
\end{array}$$

Diagram 21.15.

(21.4.2) follows immediately from Diagram 21.15, (21.14) and the index sum formula.

We finally remark what happens when we change the choice of the orientations of $Index\, P^+(u)$ for $[\ell, u] = \tilde{\ell}$. Suppose that we have two choices. Put $\varepsilon_{\tilde{\ell}} = 1$ if the two choices gives the same orientation and $\varepsilon_{\tilde{\ell}} = -1$ otherwise. Then we find that the orientation class changes as $Ori(\tilde{\ell}^-, \tilde{\ell}^+) \mapsto \varepsilon_{\tilde{\ell}^-} \varepsilon_{\tilde{\ell}^+} Ori(\tilde{\ell}^-, \tilde{\ell}^+)$. Here we let $\{\pm 1\}$ act on $KSO$ by reversing the orientation. Thus we can find a chain isomorphism between corresponding Floer's chain complex by putting $\delta_{\tilde{\ell}} \mapsto \varepsilon_{\tilde{\ell}} \delta_{\tilde{\ell}}$.

## 22. COMPUTATION OF FLOER HOMOLOGY

In this section, we complete the proof of Theorem 1.1. We prove

THEOREM 22.1. *There exists an isomorphism*

$$HF_*(M, \omega) \cong H_{*+n}(M; \mathbf{Q}) \otimes \Lambda. \tag{22.1}$$

Theorem 1.1 will follow from Theorem 22.1 immediately, [35, 52].

By Theorem 20.5, we may take any hamiltonian such that all periodic solutions are nondegenerate, to calculate the Floer homology. We take a time independent map $h: M \to \mathbf{R}$ as our hamiltonian. Namely we put $H(p, t) = h(p)$. We assume that $h: M \to \mathbf{R}$ is a Morse function and we put

$$Crit(h) = \{x \in M \mid dh(x) = 0\}.$$

For a point $x \in Crit(h)$, we put $\ell_x(t) \equiv x$, hence $\ell_x$ is an element of $LM$. We attach a trivial disk $(u_x(p) \equiv x)$ and put $\tilde{\ell}_x = [\ell_x, u_x]$. We remark that $\ell_x$ is a periodic solution of $X_H$. Since $\nabla_x^2 h$ is nondegenerate it follows that $\ell_x$ is a nondegenerate periodic solution.

We next perturb $h$ a bit and assume:

(22.2.1) Gradient vector field $\nabla h$ is Morse–Smale. Namely, the stable and unstable manifolds of $\nabla h$ are transversal.

We next assume that $h$ is $C^2$-small, it then follows that:

(22.2.2) There is no 1 periodic orbit other than $\ell_x$, $x \in Crit(h)$.

Using the fact that $h$ is $C^2$-small, we find that Conley–Zehnder index for a critical point $x$ of $H$ satisfies $\mu(\tilde{\ell}_x) = \eta(x) - n$, here $\eta(x)$ is the Morse index of $x$.

We are going to show that, for this hamiltonian, the Floer's chain complex $C_*(H, J)$ is isomorphic to Morse–Witten complex of $h$ tensored with $\Lambda$. Let us recall here very briefly the definition of Morse–Witten complex.

For $x^-, x^+ \in Crit(h)$ we put

$$\tilde{\mathcal{M}}_h(x^-, x^+) = \left\{ \gamma : \mathbf{R} \to M \;\middle|\; \begin{array}{l} \dfrac{d\gamma}{d\tau} + \nabla_{\gamma(\tau)} h = 0 \\ \lim\limits_{\tau \to \pm\infty} \gamma(\tau) = x^{\pm} \end{array} \right\}.$$

We divide it by an $\mathbf{R}$ action and denote the quotient space by $\mathcal{M}_h(x^-, x^+)$. (22.2.1) implies that $\mathcal{M}_h(x^-, x^+)$ is a smooth manifold. Its dimension is calculated by using Morse index $\eta(x)$ such that

$$\dim \mathcal{M}_h(x^-, x^+) = \eta(x^-) - \eta(x^+) - 1. \tag{22.3}$$

$\mathcal{M}_h(x^-, x^+)$ has an orientation. We will discuss the way to take it later in this section. Now we define Morse–Witten complex as follows.

$$C_k(M, h) = \bigoplus_{\substack{\eta(x) = k \\ x \in Cr(H)}} \mathbf{Q}\delta_x$$

$$\partial \delta_x = \sum_{\substack{y \in Cri(H) \\ \eta(y) = \eta(x) - 1}} [\mathcal{M}_h(x, y)]\delta_y.$$

Here $[\mathcal{M}_h(x, y)] \in \mathbf{Q}$ is "order counted with orientation" as usual. It is known that $\partial \circ \partial = 0$ and

$$H_*(C_*(M, H), \partial) = H_*(M; \mathbf{Q}).$$

(See for example [65]).

We now start the proof of Theorem 22.1. The key observation (which is due to Floer) is that, since our hamiltonian is time independent, it follows that there is an $S^1$-action on our moduli space $C\mathcal{M}(\tilde{\ell}_{x^-}, \tilde{\ell}_{x^+})$. Let $C\mathcal{M}(\tilde{\ell}_{x-}, \tilde{\ell}_{x+})^{S^1}$ be a fixed point set. We remark that $\mathcal{M}(\tilde{\ell}_{x-}, \tilde{\ell}_{x+})^{S^1} = \mathcal{M}_H(x^-, x^+)$ set theoretically. For $\gamma = \mathcal{M}_H(x^-, x^+)$, we denote by $h_\gamma$ the corresponding element in $C\mathcal{M}(\tilde{\ell}_{x^-}, \tilde{\ell}_{x^+})^{S^1}$. (Namely $h_\gamma(\tau, t) = \gamma(\tau)$.)

Mainlemma 22.4. *We can choose system of multisections $s_{p,n}$ transversal to 0 and satisfying other properties we assumed in Section 21 and that*

(22.4.1) *If* $\mu(\tilde{\ell}_1) = \mu(\tilde{\ell}_2) + 1$, *then* $s_{p,n}^{-1}(0) = C\mathcal{M}(\tilde{\ell}_1, \tilde{\ell}_2)^{S^1}$.
(22.4.2) *If* $\sigma \in C\mathcal{M}(\tilde{\ell}_{x^-}, \tilde{\ell}_{x^+})^{S^1}$ *then the obstruction bundle* $E_\sigma$ *is trivial.*
(22.4.3) *The orientation is preserved by the diffeomorphism* $C\mathcal{M}(\tilde{\ell}_{x^-}, \tilde{\ell}_{x^+})^{S^1} = \mathcal{M}_h(x^-, x^+)$.

We first prove Theorem 22.1 assuming Mainlemma 22.4. We first remark that $C\mathcal{M}(\tilde{\ell}_1, \tilde{\ell}_2)^{S^1}$ is empty unless $\tilde{\ell}_1 = \beta \# \tilde{\ell}_{x^-}$, $\tilde{\ell}_2 = \beta \# \tilde{\ell}_{x^+}$ for some $x^-$, $x^+$, $\beta$. It then follows from Mainlemma 22.4 that if $\mu(\tilde{\ell}_1) = \mu(\tilde{\ell}_2) + 1$ then

$$[C\mathcal{M}(\tilde{\ell}_1, \tilde{\ell}_2)] = \begin{cases} [\mathcal{M}_h(x^-, x^+)] & \tilde{\ell}_1 = \beta \# \tilde{\ell}_{x^-},\ \tilde{\ell}_2 = \beta \# \tilde{\ell}_{x^+} \\ 0 & \text{otherwise.} \end{cases}$$

Hence, by using (22.2.2) also, isomorphism (22.1) holds in the level of chain complex if we choose multisection, etc. as in Mainlemma 22.4. Theorem 22.1 follows. □

We now prove Mainlemma 22.4. We first remark that (22.4.2) is a consequence of (22.2.1) as we will discuss later, together with the proof of (22.4.3). Let $\tilde{\ell}_1$, $\tilde{\ell}_2$ be as in (22.4.1). By (22.4.2) we find that $C\mathcal{M}(\tilde{\ell}_1, \tilde{\ell}_2)^{S^1}$ is isolated from other part of $C\mathcal{M}(\tilde{\ell}_1, \tilde{\ell}_2)$. We remark that the action of $S^1$ on $C\mathcal{M}(\tilde{\ell}_1, \tilde{\ell}_2) - C\mathcal{M}(\tilde{\ell}_1, \tilde{\ell}_2)^{S^1}$ is locally free. Hence we can define a Kuranishi structure of negative dimension on

$$\frac{C\mathcal{M}(\tilde{\ell}_1, \tilde{\ell}_2) - C\mathcal{M}(\tilde{\ell}_1, \tilde{\ell}_2)^{S^1}}{S^1}.$$

We then choose a multisection so that it is transversal to 0. Therefore the zero point set on

$$\frac{C\mathcal{M}(\tilde{\ell}_1, \tilde{\ell}_2) - C\mathcal{M}(\tilde{\ell}_1, \tilde{\ell}_2)^{S^1}}{S^1}.$$

is empty. We lift this multisection to $C\mathcal{M}(\tilde{\ell}_1, \tilde{\ell}_2) - C\mathcal{M}(\tilde{\ell}_1, \tilde{\ell}_2)^{S^1}$, then $s_{p,n}^{-1}(0) = C\mathcal{M}(\tilde{\ell}_1, \tilde{\ell}_2)^{S^1}$ is satisfied.

However we need to be a bit more careful so that our multisections satisfy compatibility conditions we assumed in Section 21. In order to do so, we have to work on the induction on $\mathcal{A}_H(\tilde{\ell}_1) - \mathcal{A}_H(\tilde{\ell}_2)$ hence to study the moduli space of virtual dimension 1 or less is not enough even when we only need to construct a Kuranishi structure on the moduli space of virtual dimension 1 or less. So we proceed as follows.

First as we remarked that (22.4.2) is valid for any index. Therefore again $C\mathcal{M}(\tilde{\ell}_1, \tilde{\ell}_2)^{S^1}$ is isolated from other part of $C\mathcal{M}(\tilde{\ell}_1, \tilde{\ell}_2)$. Because of (22.4.2), we may choose the Kuranishi structure and multisection on $C\mathcal{M}(\tilde{\ell}_1, \tilde{\ell}_2)^{S^1}$ so that $E_\sigma$ is 0 (and hence it is automatically $S^1$-invariant.)

Once we observe it, we construct the $S^1$-invariant multisection and Kuranishi structure, and good coordinate system on $C\mathcal{M}(\tilde{\ell}_1, \tilde{\ell}_2) - C\mathcal{M}(\tilde{\ell}_1, \tilde{\ell}_2)^{S^1}$ by induction on $\mathcal{A}_H(\tilde{\ell}_1) - \mathcal{A}_H(\tilde{\ell}_2)$, in the way we discussed above. (That is going down to

$$\frac{C\mathcal{M}(\tilde{\ell}_1, \tilde{\ell}_2) - C\mathcal{M}(\tilde{\ell}_1, \tilde{\ell}_2)^{S^1}}{S^1}.$$

We finally prove (22.4.2) and (22.4.3). We include here an explanation of coherent orientation for $\mathcal{M}_h(x^-, x^+)$. Let $W^u(x)$ and $W^s(x)$ be the unstable and stable manifolds of the gradient flow $\nabla h$ at $x$. We fix an orientation of each of $T_x W^u(x)$.

Let $\gamma \in \mathcal{M}_h(x^-, x^+)$. We choose an orientation of $T_{x^-}M$. Hence using $\gamma$, we obtain an orientation of $T_{x^+}M$. Therefore we obtain orientations of $T_{x^\pm} W^S(x^\pm)$. We may identify a neighborhood of $\gamma \in \mathcal{M}_h(x^-, x^+)$ with a neighborhood of $\gamma(\tau_0) \in W^s(x^-) \cap W^u(x^+) \cap h^{-1}(c)$ in $W^s(x^-) \cap W^u(x^+) \cap h^{-1}(c)$. (Here $h(x^-) < c < h(x^+)$). Since we obtain an orientation of $T_{\gamma(\tau_0)} h^{-1}(c)$ by using orientation of $T_{x^-}M$ we gave and using the parallel transport along the path $\gamma$, we obtain an orientation of $W^s(x^-) \cap W^u(x^+) \cap h^{-1}(c)$ at $\gamma(\tau_0)$. If we change the orientation of $T_{x^-}M$, then the orientations of $W^s(x^-)$ and $T_{\gamma(\tau_0)} h^{-1}(c)$ change while the orientation of $W^u(x^+) \cap h^{-1}(c)$ does not change. Hence the orientation of $W^s(x^-) \cap W^u(x^+) \cap h^{-1}(c)$ does not change. Namely the orientation we put on the neighborhood of $W^s(x^-) \cap W^u(x^+) \cap h^{-1}(c)$ is independent of the orientation of $T_{x^-}M$ and depends only on orientations of $W^s(x^\pm)$. It is easy to see that this orientation is compatible with the gluing map in Morse theory:

$$\mathcal{M}_h(x_1, x_2) \times \mathcal{M}_h(x_2, x_3) \to \mathcal{M}_h(x_1, x_3).$$

*Remark* 22.5. The discussion above works for nonorientable manifolds also. (But in this paper we are studying symplectic manifold which is oriented.)

In order to compare this orientation to the one in the connecting orbit $C\mathcal{M}(\tilde{\ell}_{x^-}, \tilde{\ell}_{x^+})^{s^1}$, we rewrite it by using a similar method to Section 21.

Let $\gamma \in \mathcal{M}_h(x^-, x^+)$. We consider the linearized operator $L_\gamma$ of the gradient flow equation

$$\frac{d\gamma}{d\tau} + \nabla h(\gamma(\tau)) = 0.$$

We regard $L_\gamma$ as

$$L_\gamma : L^2_1(\mathbf{R}; \gamma^* TM) \to L^2(\mathbf{R}; \gamma^* TM). \tag{22.6}$$

$L_\gamma$ is a differential operator of first order and is asymptotic to

$$\frac{d}{d\tau} + \nabla^2_{x^\pm} h \quad \text{as } \tau \to \pm\infty.$$

Since $x^\pm$ is a nondegenerate critical point, it follows that $\nabla^2_x \pm h$ is invertible. Therefore, since $L_\gamma$ is an ordinary differential operator, (22.6) is a Fredholm operator.

Let $\chi : \mathbf{R} \to [0, 1]$ be a smooth function such that

$$\chi(\tau) = \begin{cases} 0 & -1 < \tau < 1 \\ 1 & |\tau| > 2. \end{cases}$$

$$L^+(x) : L^2_1([0, \infty), T_x M) \to L^2([0, \infty), T_x M)$$

$$L^-(x) : L^2_1((-\infty, 0], T_x M) \to L^2((-\infty, 0], T_x M)$$

by

$$L^{\pm}(x) = \frac{\partial}{\partial\tau} + \chi(\tau)\nabla_x^2 h. \tag{22.7}$$

$L^{\pm}(x)$ is again a Fredholm operator. It is easy to see that

$$\ker L^{-}(x) \cong T_x W^s(x)$$

$$\ker L^{+}(x) \cong T_x W^u(x) \tag{22.8}$$

and also these operators are surjective. Let $\gamma \in \mathcal{M}_h(x^-, x^+)$. We can glue the operators $L^-(x^-)$, $L_\gamma$, and $L^+(x^+)$ to obtain an operator $L^-(x^-)\# L_\gamma \# L^+(x^+)$ on a long interval $[-R, R]$.

Now we recall that we fixed orientation of each $T_x W^u(x)$. We fix an orientation of $T_{x^-}M$ for a moment. We then obtain an orientation of $\det \mathit{Index}\, L^-(x^-)$ and $\det \mathit{Index}\, L^+(x^+)$. On the other hand, the operator $L^-(x^-)\# L_\gamma \# L^+(x^+)$ is deformed to the operator $D/d\tau$ (covariant derivative). The index of $D/d\tau$ (regarded as an operator of interval of finite length), is identified to $T_{x^+}M$ or $T_{x^-}M$. (They are identified to each other by the parallel transport along $\gamma$.) Hence the orientation of $T_{x^-}M$ induces one on $\mathit{Index}(L^-(x^-)\# L_\gamma \# L^+(x^+))$.

Thus the trivialization of $\det \mathit{Index}\, L_\gamma$ is induced from the trivialization of $\det \mathit{Index}\, L^-(x^-)$, $\det \mathit{Index}\, L^+(x^+)$, $\det \mathit{Index}\,(L^-(x^-)\# L_\gamma \# L^+(x^+))$.

If we change the orientation of $T_{x^-}M$ then the trivialization of $\det \mathit{Index}\, L^+(x^+)$ and $\det \mathit{Index}\,(L^-(x^-)\# L_\gamma \# L^+(x^+))$ changes and the trivialization of $\det \mathit{Index}\, L^-(x^-)$ does not change. Hence the trivialization of $\det \mathit{Index}\, L_\gamma$ is independent of the orientation of $T_{x^-}M$.

It is straightforward to see that the orientation of $\mathcal{M}_h(x^-, x^+)$ we discussed above coincides to one we discussed first.

Now let $h_\gamma \in C\mathcal{M}_h(\tilde{\ell}_{x^-}, \tilde{\ell}_{x^+})^{S^1}$ be the element of corresponding to $\gamma \in \mathcal{M}_h(x^-, x^+)$. We find that there is an $S^1$ action on $L^2_1(\mathbf{R}\times S^1; h_\gamma^* TM)$ and $L^2(\mathbf{R}\times S^1; \Lambda^{0,1}(\mathbf{R}\times S^1)\otimes h_\gamma^* TM)$ and that $D_{h_\gamma}\bar\partial_H$ is $S^1$-invariant. We decompose

$$L^2_1(\mathbf{R}\times S^1; h_\gamma^* TM) = \bigoplus_m L^2_1(\mathbf{R}\times S^1; h_\gamma^* TM)_m$$

$$L^2(\mathbf{R}\times S^1; \Lambda^{0,1}(\mathbf{R}\times S^1)\otimes h_\gamma^* TM) = \bigoplus_m L^2(\mathbf{R}\times S^1; \Lambda^{0,1}(\mathbf{R}\times S^1)\otimes h_\gamma^* TM)_m$$

such that $S^1$ acts on $L^2_1(\mathbf{R}\times S^1; h_\gamma^* TM)_m$ and $L^2(\mathbf{R}\times S^1; \Lambda^{0,1}(\mathbf{R}\times S^1)\otimes h_\gamma^* TM)_m$ by $z \mapsto z^m$. The operator $D_{h_\gamma}\bar\partial_H$ induces

$$(D_{h_\gamma}\bar\partial_H)_m : L^2_1(\mathbf{R}\times S^1; h_\gamma^* TM)_m \to L^2(\mathbf{R}\times S^1; \Lambda^{0,1}(\mathbf{R}\times S^1)\otimes h_\gamma^* TM)_m.$$

$$(D_{h_\gamma}\bar\partial_H)_m = \frac{D}{\partial\tau} - 2\pi m + A(\tau),$$

such that the operator norm of $A(\tau)$ is close to 0. This is because we assume our function $h$ to be $C^2$-close to 0. It follows that the operator $(D_{h_\gamma}\bar\partial_H)_m$ is invertible for $m \neq 0$. Hence the kernel and the cokernel of $D_{h_\gamma}\bar\partial_H$ are the same as that of $(D_{h_\gamma}\bar\partial_H)_0$. We find also that $(D_{h_\gamma}\bar\partial_H)_0$ is the same as $L_\gamma$. (See Appendix B in [43, 62]). (22.4.2) follows.

To show (22.4.3) we choose the map $u_{\pm}: D^2 \to M$ such that $u_{\pm}(p) = x^{\pm}$. We obtain an operator $P^{\pm}(u^{\pm})$ as in Section 21. We find that this is also $S^1$-invariant. Moreover we can glue it with $D_{h_\gamma}\bar\partial_H$ in $S^1$-invariant way. Then when we deform $P^-(u^-)\# D_{h_\gamma}\bar\partial_H \# P^+(u^+)$ to the Cauchy–Riemann operator, we can do it while keeping $S^1$-invariance. Namely we can take $P(a)$ such that $P(0) = P^-(u^-)\# D_{h_\gamma}\bar\partial_H \# P^+(u^+)$, $P(1) = \bar\partial$, and that $P(a)$ is $S^1$-equivariant. We may choose $P(a)$ so that the index of $(P(a))_m$ (the part where $S^1$ acts by $z \mapsto z^m$) is always zero during perturbation for $m \neq 0$. Furthermore we can choose $P(a)$ so that $(P(a))_0$ can be identified to the deformation of $L^-(x^-)\# L_\gamma \# L^+(x^+)$ to $D/d\tau$. (We remark that the choice of orientation of $Index\, P^-(u_x)$ corresponds one to one to the choice of orientation of $T_{x^-}W^u(x^-)$.) (22.4.3) then follows from construction.

The proofs of Mainlemma 22.4, Theorems 22.2 and 1.1 are now complete. □

## 23. KONTSEVICH–MANIN'S AXIOMS

In this section, we state and verify the axioms formulated in [39] on the Gromov–Witten invariant constructed in Section 17. In fact once the machinery we developed in this paper are given, the argument below is a minor modification of one in [39].

THEOREM 23.1.1. *$I^M_{g,m,\beta}$ is invariant by the action of symmetric group (by exchanging the factors for $H^*(M,\mathbf{Q})^{\otimes m}$ and by renumbering the marked points for $C\mathcal{M}_{g,m}$).*

We remark that we can make our Kuranishi structure and multisections invariant of the action of the symmetric group acting by exchanging the factors, since this group is a finite group. Theorem is then obvious from construction. □

The next axiom by Kontsevich–Manin is that the degree of $\mu$ is equal to $2n(g-1) - 2\beta c_1$. This is immediate from Definition 7.12 or from the fact that the dimension of our Kuranishi structure on $C\mathcal{M}_{g,m}(M,J,\beta)$ is $2m + 2\beta c_1(M) + 2(3-n)(g-1)$.

We use the terminology, basic, in the same way as [39]. Namely, we say that $I^M_{g,m,\beta}$ is basic if $(g,m) = (0,3), (1,1), (g,0)$. Let $e^0_M \in H^0(M;\mathbf{Q})$ be the Poincaré dual to the fundamental class.

THEOREM 23.1.2. *If $I^M_{g,m,\beta}$ is not basic, then we have*

$$I^M_{g,m,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_{m-1} \otimes e^0_M) = \pi_m! I^M_{g,m,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_{m-1})$$

*here $\pi_m : C\mathcal{M}_{g,m} \to C\mathcal{M}_{g,m-1}$ is the map forgetting the last marked points and $\pi_m!: H^*(C\mathcal{M}_{g,m};\mathbf{Q}) \to H^{*-2}(C\mathcal{M}_{g,m-1};\mathbf{Q})$.*

For the proof we construct a map $\tilde\pi_m : C\mathcal{M}_{g,m}(M,J,\beta) \to C\mathcal{M}_{g,m-1}(M,J,\beta)$. For a moment we do not assume that $I^M_{g,m,\beta}$ is not basic. Let $C\mathcal{M}_{g,m}(M,J,\beta)_0$ be the set of all stable maps $(\Sigma, h)$ which is still stable after removing the last marked point. Obviously, there exists a map $\tilde\pi_m : C\mathcal{M}_{g,m}(M,J,\beta)_0 \to C\mathcal{M}_{g,m-1}(M,J,\beta)$.

LEMMA 23.2. *If we assume either $I^M_{g,m,\beta}$ is not basic or $\beta \neq 0$, $m \neq 0$ then the map $\tilde\pi_m : C\mathcal{M}_{g,m}(M,J,\beta)_0 \to C\mathcal{M}_{g,m-1}(M,J,\beta)$ is extended to $\tilde\pi_m : C\mathcal{M}_{g,m}(M,J,\beta) \to C\mathcal{M}_{g,m-1}(M,J,\beta)$.*

*If $I^M_{g,m,\beta}$ is not basic, then the following diagram commutes.*

$$\begin{array}{ccc} C\mathcal{M}_{g,m}(M,J,\beta) & \xrightarrow{\tilde\pi_m} & C\mathcal{M}_{g,m-1}(M,J,\beta) \\ \downarrow & & \downarrow \\ C\mathcal{M}_{g,m} & \xrightarrow{\pi_m} & C\mathcal{M}_{g,m-1} \end{array}$$

Diagram 23.3.

*Proof.* Let $(\Sigma, h)$ be an element of $C\mathcal{M}_{g,m}(M,J,\beta) - C\mathcal{M}_{g,m}(M,J,\beta)_0$. Let $\Sigma_0$ be the component of $\Sigma$ containing the $m$th marked point. Then $h$ is constant on $\Sigma_0$ and $2g_0 + m_0 = 3$ where $g_0$ is the genus of $\Sigma_0$ and $m_0$ is the number of points on $\Sigma_0$ which is marked or singular.

We claim that $g_0 = 0$. In fact we know that there is one ($m$th) marked point on $\Sigma_0$. Hence if $g_0 = 1$ then there is no singular point on $\Sigma_0$. Hence $\Sigma$ is nonsingular and $(g, m) = (1,1)$. If $I^M_{g,m,\beta}$ is not basic, this is impossible. If $\beta \neq 0$ this is impossible also since $h$ is constant on $\Sigma_0$ and there is no other component.

Now we have $g_0 = 0$ and $m_0 = 1, 2, 3$. It then follows from the assumption that $I^M_{g,m,\beta}$ is not basic or $\beta \neq 0$ that $\Sigma \neq \Sigma_0$. Namely there exists at least one singular point on $\Sigma_0$. Hence there are two possibilities.

(23.4.1) $\Sigma_0$ has one singular point and there is one marked point (say the $k$th one) on it other than $m$th one.

(23.4.2) $\Sigma_0$ has two singular points and there is only one marked point, the $m$th one.

In case (23.4.1), we remove $\Sigma_0$ from $\Sigma$ and put $k$th marked points at the position where $\Sigma_0$ was attached. In case (23.4.2), we remove $\Sigma_0$ from $\Sigma$ and glue it at the two points where $\Sigma_0$ was attached. (Since $h$ is constant on $\Sigma_0$, $h$ induces a map after modification.) Thus we have defined $\tilde\pi_m : C\mathcal{M}_{g,m}(M,J,\beta) \to C\mathcal{M}_{g,m-1}(M,J,\beta)$. □

We remark that this map $\tilde\pi_m : C\mathcal{M}_{g,m}(M,J,\beta) \to C\mathcal{M}_{g,m-1}(M,J,\beta)$. can be regarded as a universal family. Namely the fiber of $\tilde\pi_m$ at $(\Sigma_\sigma, h_\sigma)$ is identified to $\Sigma_\sigma$ itself divided by the group $Aut(\Sigma_\sigma, h_\sigma)$. For the point of $\Sigma_\sigma$ which is neither singular nor marked it is immediate to find the corresponding point in $\tilde\pi_m^{-1}(\Sigma_\sigma, h_\sigma)$. (Regard that point as $m$th marked point). If $x \in \Sigma_\sigma$ coincides to the $k$th marked point, then to find the corresponding element of $\tilde\pi_m^{-1}(\Sigma_\sigma, h_\sigma)$, we attach $\mathbf{C}P^1$ at $x$ and put $k$th and $m$th marked points on $\mathbf{C}P^1$. (The map will be constant there.) Such an element is unique and in $\tilde\pi_m^{-1}(\Sigma_\sigma, h_\sigma)$. Finally, if $x \in \Sigma_\sigma$ is a singular point, where two components $\Sigma_v$ and $\Sigma_w$ meet, then we take $\mathbf{C}P^1$ and glue $\Sigma_v$, $\Sigma_w$ at $x_v$, $x_w$, respectively, to $\mathbf{C}P^1$. We put also $m$th marked point on this $\mathbf{C}P^1$. We again find an element of $\tilde\pi_m^{-1}(\Sigma_\sigma, h_\sigma)$. We remark that these two cases correspond to (23.4.1) and (23.4.2), respectively.

*Proof of Theorem* 23.1.2. Using the above description, we find that the Kuranishi structure on $C\mathcal{M}_{g,m-1}(M,J,\beta)$ induces one on $C\mathcal{M}_{g,m}(M,J,\beta)$ as follows.

We assume that $I^M_{g,m,\beta}$ is not basic. Let $\sigma = (\Sigma_\sigma, h_\sigma) \in C\mathcal{M}_{g,m-1}(M,J,\beta)$. We take a chart $(U_\sigma, \Gamma_\sigma, E_\sigma, s_\sigma)$ around it. $(U_\sigma = {}_\sigma/\Gamma_\sigma)$. Let $[\Sigma_\sigma] \in W_\sigma/\Lambda_\sigma$, be the chart of $C\mathcal{M}_{g,m-1}$ around $[\Sigma_\sigma]$. There is a homomorphism $\Gamma_\sigma \to \Lambda_\sigma$ and a map ${}_\sigma \to W_\sigma$ which is $\Gamma_\sigma \to \Lambda_\sigma$ equivariant. There is a universal family of Riemann surface $\hat W_\sigma \to W_\sigma$ (namely the fiber of $x \in W_\sigma$ is identified to the Riemann surface represented by $x$), on which $\Lambda_\sigma$ acts. The inverse image $\pi_m^{-1}(W_\sigma/\Lambda_\sigma)$ is equal to $\hat{W}_\sigma/\Lambda_\sigma$. Now we put

$$\hat{U}_\sigma = \frac{{}_\sigma \times_{W_\sigma} \hat{W}_\sigma}{\Gamma_\sigma}.$$

This is an orbifold. On $\hat{U}_\sigma$, we have an orbibundle $\hat{E}_\sigma$ and its section $\hat{s}_\sigma$ induced by $E_\sigma$ and $s_\sigma$, respectively. It is straightforward to verify that they define a Kuranishi structure on $C\mathcal{M}_{g,m}(M, J, \beta)$.

Using Lemma 17.8 in the same way as the proof of Theorem 17.11, we can prove that we can use this Kuranishi structure to define $I^M_{g,m,\beta}$. To see this we reinterpret the above Kuranishi structure as follows. Let $E_\sigma$ be the subspace of $C^\infty(\Sigma_\sigma; \Lambda^{0,1}(\Sigma_\sigma) \otimes h_\sigma^* TM)$ used to define Kuranishi structure on $C\mathcal{M}_{g,m-1}(M, J, \beta)$. We may choose so that the support of elements of $E_\sigma$ is disjoint from *marked* or singular points. For each element of $(\Sigma_\sigma^+, h_\sigma^*) \in \tilde{\pi}_m^{-1}(\Sigma_\sigma, h_\sigma)$ we can regard $E_\sigma$ as a subspace of $C^\infty(\Sigma_\sigma^+; \Lambda^{0,1}(\Sigma_\sigma^+) \otimes h_\sigma^* TM)$. We use it to define a Kuranishi structure. It is easy to see that the structure we obtain is the one described above. We can therefore apply the proof of Theorem 17.11.

Next we take multisection $s'_\sigma$ and lift it to a multisection $\hat{s}'_\sigma$ of $C\mathcal{M}_{g,m-1}(M, J, \beta)$. We then have

$$\hat{s}'^{-1}_\sigma(0) = s'^{-1}_\sigma(0) \times_{C\mathcal{M}_{g,m-1}} C\mathcal{M}_{g,m} \tag{23.5}$$

as a chain over $\mathbf{Q}$. Since the composition of Poincaré duals and $\pi_m!$ is realized by taking a fibre product $\times_{C\mathcal{M}_{g,m-1}} C\mathcal{M}_{g,m}$, Theorem 23.1.2 follows. □

THEOREM 23.1.3.

$$I^M_{0,3,\beta}(\gamma_1 \otimes \gamma_2 \otimes e^0_M) = \begin{cases} 0 & \text{if } \beta \neq 0 \\ \int \gamma_1 \wedge \gamma_2 & \text{if } \beta = 0. \end{cases}$$

*Proof.* The proof is similar to the discussion of Section 22. We choose a cycle $C_i$, $i = 1, 2$, dual to $\gamma_i$ and that $C_1$ is transversal to $C_2$. By dimension counting we need to consider only the case $\operatorname{codim} C_i + \operatorname{co dim} C_2 = 2n + 2\beta c_1$.

We consider the moduli space $C\mathcal{M}_{0,2}(M, J, \beta)$. It has a Kuranishi structure of dimension $2n + 2\beta c_1 - 2$. We have the following commutative diagram similar to Diagram 23.3

Diagram 23.6.

We can choose multisections $s'_\sigma$ on $C\mathcal{M}_{0,2}(M, J, \beta)$ so that $(ev_1 \times ev_2) s'^{-1}_\sigma(0) \cap (C_1 \times C_2) = \emptyset$.

Now we show that the Kuranishi structure and multisection $s'_\sigma$ on $C\mathcal{M}_{0,2}(M, J, \beta)$ induce ones on $C\mathcal{M}_{0,3}(M, J, \beta)$. We can do it by modifying the argument of the proof of Theorem 23.1.2. The argument there itself cannot be applied directly since $C\mathcal{M}_{0,2}$ is empty. However the reinterpreted way works. Namely, we take subspace $E_\sigma$ of $C^\infty(\Sigma_\sigma; \Lambda^{0,1}(\Sigma_\sigma) \otimes h_\sigma^* TM)$ for $\sigma \in C\mathcal{M}_{0,2}(M, J, \beta)$, lift it and use it to construct a Kuranishi structure on $C\mathcal{M}_{0,3}(M, J, \beta)$. Then the multisection $s'_\sigma$ can be lifted to a multisection $s'_\sigma$ on $C\mathcal{M}_{0,3}(M, J, \beta)$ such that $(ev_1 \times ev_2) s'^{-1}_\sigma(0) \cap (C_1 \times C_2) = \emptyset$. Hence $I^M_{0,3,\beta}(\gamma_1 \otimes \gamma_2 \otimes e^0_v) = 0$ if $\beta \neq 0$.

If $\beta = 0$, the map $C\mathcal{M}_{0,3}(M, J, 0) \xrightarrow{\bar{\pi}_m} C\mathcal{M}_{0,2}(M, J, 0)$ is not well-defined. So the discussion above breaks down. In this case, however, the moduli space $C\mathcal{M}_{0,3}(M, J, 0)$ is identified to $M$ itself and is transversal. Thereore the theorem holds in this case also. □

THEOREM 23.1.4. *If* $\deg \gamma_m = 2$ *and if* $I^M_{g,m,\beta}$ *is not basic then we have*

$$\pi_m!(I^M_{g,m,\beta}\ (\gamma_1 \otimes \cdots \otimes \gamma_m)) = \int_\beta \gamma_m \cdot I^M_{g,m-1,\beta}\ (\gamma_1 \otimes \cdots \otimes \gamma_{m-1}).$$

*Proof.* We again use Lemma 23.2 and (23.5). We choose Poincaré dual $C_i$ of $\gamma_i$ and a cycle $B$ of $C\mathcal{M}_{g,m-1}$ of codimension $\deg I^M_{g,m-1,\beta}\ (\gamma_1 \otimes \cdots \otimes \gamma_{m-1})$. We choose a multisection $s'_\sigma$ such that $s'^{-1}_\sigma(0)$ is transversal to $C_i$ and $B$. Hence the (set theoretical) intersection $s'^{-1}_\sigma(0) \cap (C_1 \times \cdots \times C_{m-1} \times B)$ consists of finitely many points. We can then choose Poincaré dual $C_m$ to $\gamma_m$ so that it is transversal to all the maps represented by this finitely many points $s'^{-1}_\sigma(0) \cap (C_1 \times \cdots \times C_{m-1} \times B)$. Using (23.5) we find that

$$\hat{s}'^{-1}_\sigma(0) \cap (C_1 \times \cdots \times C_m \times \pi_m^{-1} B) = \{((\Sigma_\sigma, h_\sigma), x) \in s'^{-1}_\sigma(0) \cap (C_1 \times \cdots \times C_{m-1} \times B)) \times C_m \mid x \in \operatorname{Im} h_\sigma\}. \tag{23.6}$$

We remark that the left- and right-hand sides of the conclusion are obtained by counting the order of the left- and right-hand sides of (23.6) together with sign and multiplicity, respectively. By using the fact that (23.6) is the identity of $\mathbf{Q}$-cycles, we can check that the multiplicity and sign coincide. Theorem 23.1.4 follows. □

THEOREM 23.1.5.

(23.1.5.1) $$I^M_{0,m,0}(\gamma_1 \otimes \cdots \otimes \gamma_m) = \int_M \gamma_1 \wedge \cdots \wedge \gamma_m e^0_{C\mathcal{M}_{0,m}}$$

(23.1.5.2) $$I^M_{1,1,0}(e^0_M) = \chi(M) e^0_{C\mathcal{M}_{1,1}}$$

(23.1.5.3) $$\mathrm{I}^M_{1,1,0}(\gamma) = c \int_M (c_{n-1}(M) \wedge \gamma) e^2_{C\mathcal{M}_{1,1}}.$$

*Here* $e^2_{C\mathcal{M}_{1,1}} = c^1(O(1))$ *is the canonical generator of second cohomology of* $C\mathcal{M}_{1,1} = \mathbf{C}P^1$ *(homeomorphism),* $\chi(M)$ *is the Euler number and* $c$ *is a universal constant we define later.*

*Proof.* (23.1.5.1) is immediate from the fact that $C\mathcal{M}_{0,m}(M, J, 0) = M \times C\mathcal{M}_{0,m}$ and is transversal.

To show other two formulas, we consider $C\mathcal{M}_{1,1}(M, J, 0)$. Set theoretically it coincides with $M \times C\mathcal{M}_{1,1}$. However this moduli space is not transversal. The cokernel of the linearized operator, in this case, coincides to $H^{0,1}((T^2, J_{T^2}); \mathbf{C}) \otimes T_x M$ at $(x, (T^2, J_{T^2}))$. This consists of an orbibundle on $M \times C\mathcal{M}_{1,1}$. Hence, by definition, we are only to calculate the orbifold Euler number of this bundle over an orbifold $M \times C\mathcal{M}_{1,1}$.

We find that

$$c_n(H^{0,1}((T^2, J_{T^2}); \mathbf{C}) \otimes T_x M) = c_n(T_x M) \times e^0_{C\mathcal{M}_{1,1}} + c_{n-1}(T_x M) \times c_1(H^{0,1}((T^2, J_{T^2}); \mathbf{C}).$$

Hence by putting $c_1(H^{0,1}((T^2, J_{T^2}); \mathbf{C}) = c e^2_{C\mathcal{M}_{2,1}}$, we obtain (23.1.5.2) and (23.1.5.3). □

To state the next result, we need a notation. Let $\Delta_a$ be a homogeneous basis of $H^*(M;\mathbf{Q})$. We put $g_{ab} = \int_M \Delta_a \wedge \Delta_b$ and let $(g^{ab}) = (g_{ab})^{-1}$.

Let $g_1 + g_2 = g$, $m_1 + m_2 = m$ and $\rho = \rho_1 \amalg \rho_2 : \underline{m_1} \amalg \underline{m_2} \to \underline{m}$ be a bijection. We obtain $\varphi_\rho : C\mathcal{M}_{g_1,m_1+1} \times C\mathcal{M}_{g_2,m_2+1} \to C\mathcal{M}_{g,m}$ as follows. Let $(\Sigma_i, z_i) \in C\mathcal{M}_{g_i,m_i}$. We glue $\Sigma_1$ and $\Sigma_2$ at $m_1 + 1$th and $m_2 + 1$th marked points and obtain $\Sigma$. We regard $k$th marked point of $(\Sigma_i, z_i)$ as $\rho_i(k)$th marked point of $\Sigma$. We have $(\Sigma_i, z) \in C\mathcal{M}_{g,m}$ in this way. We put $\varphi_\rho((\Sigma_1, z_1), (\Sigma_2, z_2)) = (\Sigma, z)$.

Using $\rho_i$ we obtain, $\rho_i^* : H^*(M;\mathbf{Q})^{\otimes m} \to H^*(M;\mathbf{Q})^{\otimes m_i}$.

THEOREM 23.1.6.

$$\varphi^*_{\rho_1,\rho_2}(I^M_{g,m,\beta}(x)) = \pm \sum_{\beta=\beta_1+\beta_2} \sum_{ab} g^{ab} I^M_{g_1,m_1+1,\beta_1}(\rho_1^*(x) \times \Delta_a)\, I^M_{g_2,m_2+1,\beta_2}(\rho_2^*(x) \times \Delta_b).$$

Here $\pm$ depends only on the degree of the factors [39].

*Proof of Theorem* 23.1.6. By the evaluation at last marked points, we have

$$E\quad L : \bigcup_{\beta_1+\beta_2=\beta} C\mathcal{M}_{g_1,m_1+1}(M,J,\beta_1) \times C\mathcal{M}_{g_2,m_2+1}(M,J,\beta_2) \to M^2 \tag{23.7}$$

Let $\Delta \subseteq M^2$ be the diagonal. We find that the inverse, image $E\quad L^{-1}(\Delta)$ can be identified to a union of strata of $C\mathcal{M}_{g,m}(M,J,0)$ and that the following diagram commutes.

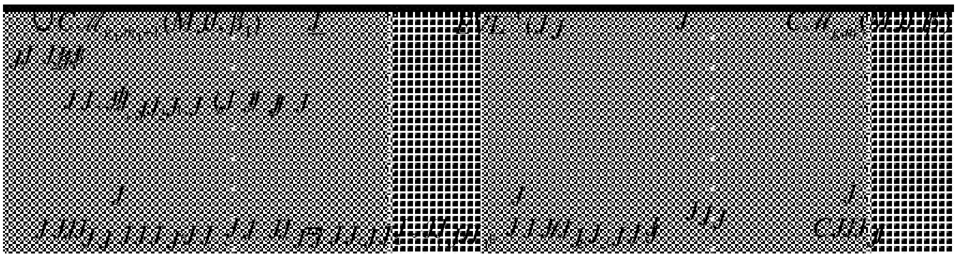


Diagram 23.8.

We remark that we may assume that these maps are extended to the products of their Kuranishi neighborhoods. We can choose our Kuranishi structure so that $E\quad L : U_\sigma \times U_{\sigma'} \to M^2$ is of maximal rank everywhere. Then the Kuranishi structure of $C\mathcal{M}_{g_1,m_1+1}(M,J,\beta_1) \times C\mathcal{M}_{g_2,m_2+1}(M,J,\beta_2)$ induces a Kuranishi structure on $E\quad L^{-1}(\Delta)$. We can then extend this Kuranishi structure to $C\mathcal{M}_{g_2,m_2+1}(M,J,\beta)$ and use it to define Gromov–Witten invariant. This argument we use to construct the Kuranishi structure , etc. compatible to the Diagram 23.8, is the same as we discussed in detail in Section 19, namely using induction on energy to make Kuranishi structure compatible to all the gluing maps. Hence we do not repeat it here.

We next remark that $\sum_{a,b} g^{ab}\Delta_a\Delta_b$ is Poincaré dual to the diagonal. Hence we have

$$\pi_*[E\quad L^{-1}(\Delta)] \backslash (\rho_1^*(x) \times \rho_2^*(x)) = \pm \sum_{\beta=\beta_1+\beta_2} \sum_{ab} g^{ab} I^M_{g_1,m_1+1,\beta_1}(\rho_1^*(x) \times \Delta_a)\, I^M_{g_2,m_2+1,\beta_2}(\rho_2^*(x) \times \Delta_b)$$

Theorem 23.1.6 then follows from the commutavity of Diagram 23.8. □

Let $\psi : C\mathscr{M}_{g-1,n+2} \to C\mathscr{M}_{g,n}$ be the map corresponding to gluing together last two marked points.

THEOREM 23.1.7.

$$\psi^*(I^M_{g,m,\beta}(x)) = \pm \sum_{ab} g^{ab} I^M_{g-1,m+1,\beta}(x \times \Delta_a \times \Delta_b).$$

The proof is the same as Theorem 23.1.6.

We thus completed the proof of the axioms of Kontsevich–Manin [39] except the Motivik axiom.

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The results and outline of the proof of this paper was written and distributed in our paper [31] in February 1996. We refer to [31] for more remarks on the relation of our method to the study of other equations, etc.

After [31] had been distributed, we were informed by J. Li and G. Tian that they proved Corollary 1.4 independently, in the case when $M$ is a projective variety, in their paper [41].† In that case, they proved also Motivic Axiom. They also informed us that their method could be used also in symplectic case, and that G. Liu and G. Tian were doing Arnold Conjecture using their method. We learn also of the existence of the papers [5, 6] by Behrend-Fantechi and Behrend which proves Corollary 1.4 and Motivic Axiom in the case where $M$ is a projective variety. Y. Ruan also sent his paper [57] defining a notion similar to our Kuranishi structure in the case when the group is trivial. He calls it virtual neighborhood. Based on Furuta's work [32] on monopole equation, Ruan used his virtual neighborhood to study moduli space of the solutions of monopole equation. He also asserted that he also proved Theorem 1.1 and Corollary 1.4 independently by a method different from ours.‡

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‡After this paper had been completed and distributed, we received a paper by Siebert "Gromov–Witten invariant

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## APPENDIX: ANOTHER NORMALIZATION

In Section 15, we used a kind of "center of mass' argument in order to kill ambiguity coming from infinitesimal automorphisms of $\Sigma_{\tau_i}$. Here we give another way of killing ambiguity.

Let $\tau = (\Sigma_\tau, h_\tau)$ be a stable map. (Here $\Sigma_\tau$ denotes a semi-stable curve possibily with marked point.) Recall that our pair $(\Sigma, h)$ of a semi-stable curve $\Sigma$ and a map $h : \Sigma \to M$ is said to be close to $\tau$, if there exists $\zeta = (\xi, a) \in {}_{\mathrm{deform},\tau} \times {}_{\mathrm{resolve},\tau}$ $u \in C^{\infty}(h_{\tau}^{\infty} TM)$ and a biholomorphic map $\vartheta : \Sigma \to \Sigma_{\tau,\zeta}$ preserving marked points, such that $h \circ \vartheta^{-1}$ and $(h_{\tau})_{\mathrm{approx},\zeta,u}$ are close in smooth topology on each irreducible components of $\Sigma_{\tau,\zeta}$. If $\Sigma_{\tau}$ contains unstable components, $Aut(\Sigma_{\tau})$ is not finite and there are uncountably many choices of a biholomorphic, map $\vartheta$.

To make sense of the equation $\bar{\partial} h \in \Sigma_{\tau}$, we have to fix a representative $(\Sigma_{\tau}, h_{\tau})$ of $\tau$ and normalize the identification map $\vartheta$. (We choose and fix a represntative of $\tau$ once for all.)

There are six possibilities of unstable components.

(i) $g = 0$ and two singular points without marked points.
(ii) $g = 0$ one singular point and one marked point.
(iii) $g = 0$ and one singular point without marked points.
(iv) $g = 0$ and one or two marked points.
(v) $g = 0$ without singular or marked points.
(vi) $g = 1$ without singular or marked points.

(Among these six cases, (iv)–(vi) are the cases when $\Sigma_{\tau}$ consists of one irreducible component. In the case when two singular points are identified in case (i), $\Sigma_{\tau}$ also consists of one irreducible component. In other cases, $\Sigma_{\tau}$ consists at least of two components.)

For each unstable component $\Sigma_{\tau,v}$, we put new marked points away from neck regions so that $\Sigma_{\tau,v}$ becomes a stable component and the number of new marked points is as small as possible. We also require the following condition. Suppose that there exists $\varphi \in Aut(\tau)$ and unstable components $\Sigma_{\tau,v_1}$, $\Sigma_{\tau,v_2}$ so that $\varphi(\Sigma_{\tau,v_1}) = \Sigma_{\tau,v_2}$. If $h$ is not a multiple covered map on these components, such a $\varphi$ is unique. In this case we require that $\varphi$ maps new marked point on $\Sigma_{\tau,v_1}$ to new marked point on $\Sigma_{\tau,v_2}$, and $h$ is an embedding near these marked points. If $h$ is a multiple covered map, we take all $\Sigma_{\tau,v_j}$ $(j = 1, \ldots, \ell)$ such that $\varphi_j(\Sigma_{\tau,v_1}) = \Sigma_{\tau,v_j}$ for some $\varphi_j \in Aut(\tau)$ for $j = 2, \ldots, \ell$. Put minimal number of marked points on $\Sigma_{\tau,v_1}$ so that it becomes stable. We choose marked points so that $h$ is an immersion near these points. These points are mapped by $\varphi_j$ to $\Sigma_{\tau,v_j}$ and we regard them as new marked points on $\Sigma_{\tau,v_j}$ $(j = 2, \ldots, \ell)$. (This construction is not invariant under $Aut(\tau)$. We however, restore $Aut(\tau)$-action on a neighborhood of $\tau$, later.)

For each new marked point $p \in \Sigma_{\tau,v}$, take an embedded $(2n-2)$-dimensional disk $\mathscr{D}_p$ in $M$, which is transversal to $h(\Sigma_{\tau,v})$ at $p$. We assume that $\mathscr{D}_{\varphi(p)} = \mathscr{D}_p$ when $p$ and $\varphi(p)$ are marked points, where $\varphi \in Aut(\tau)$.

Recall that $\Sigma_{\tau,\xi}$ is obtained by gluing

$$\Sigma_{\tau,v}(v) = \Sigma_{\tau,v} - \bigcup_{\substack{s:\,\text{singular}\\ \text{points of } \Sigma_{\tau,\xi}}} \mathring{D}_{|a_x|^{1/2}}(x)$$

along boundaries. Hence, if $a = (a_x) \in {}_{\mathrm{resolve}.\tau}$ is small and $p \in \Sigma_{\tau,\xi}(v)$, the following condition for $\vartheta : \Sigma \to \Sigma_{\tau,\xi}$ makes sense:

$$h \circ \vartheta^{-1}|_{\Sigma_{\tau,\xi}(v)}(p) \in \mathscr{D}_p. \tag{A.1}$$

(In other words, we consider a subspace $C^{\infty}(h_{\tau}^{*} TM)$ consisting of $u \in C^{\infty}(h_{\tau}^{*} TM)$ tangent to $\mathscr{D}_p$ at $p$, if $\mathscr{D}_p$ is an image of a flat disk in $T_pM$ by the exponential map.)

We can establish results in Sections 12, 13, 14 using (A1) in place of (15.8).

The action of $Aut(\tau)$ is restored in the following way. Firstliy, we consider $\varphi \in Aut(\tau)$ such that $\varphi$ is identity except on one component $\Sigma_{\tau,v}$. If $h$ is close to $h_{\tau}$, so is $h \circ \vartheta^{-1} \circ \varphi \circ \vartheta$. Remember that $h_{\tau}$ is an immersion on some neighborhood $U(p_j)$ of new marked point $p_j$. Hence, for each $p_j$, there is a unique point $p_j' \in U(p_j)$ such that $h \circ \vartheta^{-1} \circ \varphi(p_j') \in \mathscr{D}_p$. Note that $\Sigma_\tau$ becomes a stable curve after adding "new marked points" on unstable components. We denote it by $\hat{\Sigma}_\tau$. Since $\Sigma_{\tau,\xi} = \bigcup \Sigma_{\tau,\xi}(v)$ and $p'_j$ are close to $p_j$. $\Sigma_{\tau,\xi}$ with extra marked points $p'_j$ is close to $\Sigma_{\tau,\xi}$ with extra marked points $p_j$ in the sense of stable curves. Hence, there exist $\xi' \in {}_{\mathrm{deform}:\tau} \times {}_{\mathrm{resolve},\tau}$ and a biholomorphic map $\phi : \Sigma_{\tau,\xi} \to \Sigma_{\tau,\xi'}$ such that $\phi(p'_j) = p_j$. (Here $p_j \in \Sigma_{\tau,v}$ is considered also as points in $\Sigma_{\tau,\xi'}(v) \subset \Sigma_{\tau,\xi'}$.) We define an action of $\varphi \in Aut(\tau)$ by sending $(h, \vartheta)$ to $(h \circ \vartheta^{-1} \circ \varphi \circ \vartheta, \phi \circ \vartheta)$. (Here we forget extra marked points.)

As for other $\varphi \in Aut(\tau)$, there are $\varphi_j \in Aut(\tau), j = 1, \dots, k$, which are identities except on one component for each $j$, such that $\varphi_1 \circ \cdots \circ \varphi_k \circ \varphi$ interchanges new marked points on different components as prescribed before. Then it is obvious that our construction is in variant under the action of $\varphi_1 \circ \cdots \circ \varphi_k \circ \varphi$. Hence we get the action of $Aut(\tau)$ on the space of collections $(\xi, h, \vartheta)$, where $\xi \in {}_{\mathrm{deform},\tau} \times {}_{\mathrm{resolve},\tau}$, $h : \Sigma \to M$, and $\vartheta : \Sigma \to \Sigma_{\tau,\xi}$ is a biholomorphic map satisfying $h \circ \vartheta^{-1}(p_j) \in \mathscr{D}_{p_j}$.

For $h : \Sigma \to M$ close to $\tau$, we consider all $\vartheta : \Sigma \to \Sigma_{\tau,\xi}$ such that $h \circ \vartheta^{-1}(p_j) \in \mathscr{D}_{p_j}$, which are finitely many. Then, for $s \in E_\tau$, we take the average of $Emb_{(\xi,\vartheta),\tau}(s)$ and denote it by $s(h)$. Then the equation

$$\overline{\partial} h \in E_\tau$$

means that $\overline{\partial} h = s(h)$ for some $s \in E_\tau$. Note that this equation is invariant under the action of $Aut(\tau)$ described above. We write

$$\hat{}_{\mathrm{map},\tau} = \left\{ u \in C^\infty(h_\tau^* TM) \;\middle|\; \begin{array}{l} D_{h_\tau} \overline{\partial}_{\Sigma_\tau} u = 0 \\ u(p_j) \text{ is tangent to } \mathscr{D}_{p_j} \end{array} \right\}.$$

Then $Aut(\tau)$ acts on ${}_{\mathrm{deform},\tau} \times {}_{\mathrm{resolve},\tau} \times \hat{}_{\mathrm{map},\tau}$ in a similar way as above.

This gives a local chart of a neighborhood of $\tau$. (cf. Sections 12 and 13.)