Abstract Linear Algebra - Problem Set 6
Instructor: Katalin Berlow

The homework is out of 10 points total.

1. Let $T : V \to V$ be a linear operator on a finite-dimensional vector space V, and suppose $U, W \subseteq V$ are T-invariant subspaces. That is,

$$T(U) \subseteq U \quad \text{and} \quad T(W) \subseteq W.$$

   (a) (1.5 points) Prove that $U \cap W$ is a T-invariant subspace.

   (b) (1.5 points) Prove that $U + W$ is a T-invariant subspace.

   (c) (1 point) Suppose further that $V = U \oplus W$. Show there is a basis so that the matrix of T in this basis has the form

$$[T]_{\mathcal{B}} = \begin{bmatrix} [T_U] & 0 \\ 0 & [T_W] \end{bmatrix},$$

   where $[T_U]$ and $[T_W]$ are the matrices of $T|_U$ and $T|_W$, respectively.

2. (1 point) Let S and T be linear maps from a vector space V to itself. Assume that $S \circ T = T \circ S$. Show that the range(S) is invariant under T.

3. (1 point) Find a 2×2 matrix over $\mathbb{F}_2$ with no eigenvalues.

4. (2 points) Suppose that T is a linear map from V to itself. Assume that $T^2 = T$. Show that all eigenvalues of T are 0 or 1.

5. (2 points) We call an $n \times n$ matrix with nonnegative integer entries a *magic square* if all rows and all columns sum to the same number, called the *magic constant*. Show that for any magic square, the magic constant is an eigenvalue.

Bonus:

1. (2 points) We call an $n \times n$ matrix with nonnegative integer entries a *permutation square* if all rows and all columns contain each of the numbers $\{1, \ldots, n\}$. Show that all $n \times n$ permutation matrices have the same largest eigenvalue, and find this eigenvalue.