Abstract Linear Algebra - Problem Set 5
Instructor: Katalin Berlow

The homework is out of 10 points total.

1. (6 points) Let $T : V \to W$ be a linear map between finite dimensional vector spaces. Prove that the following are equivalent.
   I. T is bijective (both injective and surjective).
   II. There is a another linear map $T' : W \to V$ where $T \circ T'$ is the identity map on W and $T' \circ T$ is the identity map on V. We call this T' the *inverse* of T and write it as T^{-1}.
   III. T is injective and $\dim V \geq \dim W$.
   IV. T is surjective and $\dim V \leq \dim W$.
2. (2 points) Let $T : V \to W$ be a linear map between finite dimensional vector spaces. Prove that the following are equivalent.
   I. T is bijective.
   II. For every basis $v_1, \ldots, v_n$ of V, we have that $T(v_1), \ldots, T(v_n)$ is a basis for W.
   III. For some basis $v_1, \ldots, v_n$ of V, we have that $T(v_1), \ldots, T(v_n)$ is a basis for W.
3. (1 point) Let $T : V \to W$ and $S : W \to U$ be bijective linear maps. Prove that
$$(S \circ T)^{-1} = T^{-1} \circ S^{-1}.$$
4. (1 point) Let $T : V \to V$ be a linear map. Prove that if $T^n = 0$ for some $n \in \mathbb{N}$, then T is not invertible.

Extra Credit:

1. Let $T : V \to V$ and $S : V \to V$ be linear maps on a vector space V.
   (a) (1 point) Assume that V is finite dimensional. Show that S and T are both bijective if and only if $S \circ T$ is bijective.
   (b) (2 points) Find a counterexample to this when V is infinite dimensional.

1. (6 points) Let $T : V \to W$ be a linear map between finite dimensional vector spaces. Prove that the following are equivalent.

I. T is bijective (both injective and surjective).

II. There is a another linear map $T' : W \to V$ where $T \circ T'$ is the identity map on W and $T' \circ T$ is the identity map on V. We call this T' the *inverse* of T and write it as T^{-1}.

III. T is injective and $\dim V \geq \dim W$.

IV. T is surjective and $\dim V \leq \dim W$.

$\boxed{\text{I} \Rightarrow \text{III}}$ It was proven in lecture that if $T: V \to W$ is surjective, then $\dim V \geq \dim W$. So, if T is bijective (I), then T is injective and surjective and so, we have $\dim V \geq \dim W$, giving us III.

$\boxed{\text{I} \Rightarrow \text{IV}}$ It was proven in lecture that if $T: V \to W$ is injective, then $\dim V \leq \dim W$. So, if T is bijective (I), then T is injective and surjective and so, we have $\dim V \leq \dim W$, giving us IV. .

$\boxed{\text{III} \Rightarrow \text{II}}$ Assume T is injective and $\dim V \geq \dim W$. Since T is injective, $T(v_1), \ldots, T(v_n)$ are linearly independent for any basis $v_1, \ldots, v_n$ of V. Since $T(v_1), \ldots, T(v_n)$ are linearly independent and there are $n = \dim V \geq \dim W$ many of them, they form a basis. So, mapping $S(T(v_i))$ to v_i gives us a well-defined map S, which is an inverse for T.

$\boxed{\text{IV} \Rightarrow \text{II}}$ Assume T is surjective and $\dim V \leq \dim W$. Since T is surjective, if $v_1, \ldots, v_n$ is a basis for V, $T(v_1), \ldots, T(v_n)$ is spanning in W. But, since $\dim V \leq \dim W$, $T(v_1), \ldots, T(v_n)$ is a Basis. So, defining S

by $S(T(v_i)) = v_i$ gives us a well-defined inverse for T.

$\boxed{\mathrm{I} \Rightarrow \mathrm{II}}$ If T is bijective, for each $w \in W$ there is exactly one $v \in V$ with $T(v) = w$. Sending $T^{-1}(w) = v$ then gives us a well-defined inverse. This is also linear since for $w_1, w_2 \in W$, we can write $w_1 = T(v_1)$ and $w_2 = T(v_2)$. So, $T^{-1}(w_1 + w_2) = T^{-1}(T(v_1) + T(v_2)) = T^{-1}(T(v_1 + v_2))$ since T is linear, $= T^{-1} \circ T(v_1 + v_2) = v_1 + v_2 = T^{-1}(w_1) + T^{-1}(w_2)$. To prove that T is homogeneous, note that if $w = T(v)$, then $T^{-1}(cw) = T^{-1}(c\,T(v)) = T^{-1} \circ T(cv)$ $= cv = c\,T^{-1}(v)$. So, T^{-1} is linear.

$\boxed{\mathrm{II} \to \mathrm{I}}$ Assume T has inverse T^{-1}. Then, T^{-1} is both a left inverse and a right inverse. We've shown in class that having a left inverse is equivalent to being injective and having a right inverse is equivalent to being surjective. So, T is bijective.

We've proven:

$$\begin{tikzcd} \boxed{\mathrm{I}} \arrow[r, leftrightarrow] \arrow[d] \arrow[dr] & \boxed{\mathrm{II}} \\ \boxed{\mathrm{III}} \arrow[ur] & \boxed{\mathrm{IV}} \arrow[u] \end{tikzcd}$$

Since we can get from any statement to any other, they are all equivalent.

(Note $\mathrm{II} \Rightarrow \mathrm{I}$ was unnecessary.) $\square$

2. (2 points) Let $T : V \to W$ be a linear map between finite dimensional vector spaces. Prove that the following are equivalent.

I. T is bijective.

II. For every basis $v_1, \ldots, v_n$ of V, we have that $T(v_1), \ldots, T(v_n)$ is a basis for W.

III. For some basis $v_1, \ldots, v_n$ of V, we have that $T(v_1), \ldots, T(v_n)$ is a basis for W.

$\boxed{\mathrm{I} \Rightarrow \mathrm{II}}$ Assume T is bijective.

Let $v_1, \ldots, v_n$ be any basis for V. Then $T(v_1), \ldots, T(v_n)$ is linearly independent since T is injective. Let $w \in W$. Since $W = \operatorname{ran}(T)$, there is a $v \in V$ with $T(v) = w$. Write $v = a_1 v_1 + \cdots + a_n v_n$ since $v_1, \ldots, v_n$ is a basis for V. Then, $w = T(v) = a_1 T(v_1) + \cdots + a_n T(v_n)$ and so $w \in \operatorname{span}(T(v_1), \ldots, T(v_n))$. Since w was arbitrary, $T(v_1), \ldots, T(v_n)$ are spanning and thus a basis.

$\boxed{\mathrm{II} \Rightarrow \mathrm{III}}$ This is clear.

$\boxed{\mathrm{III} \Rightarrow \mathrm{I}}$ Assume T is not bijective. Let $v_1, \ldots, v_n$ be a basis for V. Then, we will show $T(v_1), \ldots, T(v_n)$ is not a basis for W, thus showing not III.

Case 1: T is not injective. Then, $\operatorname{Null}(T) \neq \{0\}$. Let $v \in \operatorname{Null}(T)$ be nonzero. Then, we write $v = a_1 v_1 + \cdots + a_n v_n$. Then, $T(v) = a_1 T(v_1) + \cdots a_n T(v_n) = 0$ so $T(v_1), \ldots T(v_n)$ are not linearly independent.

Case 2: T is not surjective. Then there is $w \in W$ with $w \notin \operatorname{span}(T)$. But, then for any $a_1 v_1 + \cdots + a_n v_n \in V$, we have $T(a_1 v_1 + \cdots + a_n v_n) = a_1 T(v_1) + \cdots + a_n T(v_n) \neq w$. So, $w \notin \operatorname{span}(T(v_1), \ldots, T(v_n))$. So, $T(v_1), \ldots, T(v_n)$ is not spanning. $\square$

3. (1 point) Let $T : V \to W$ and $S : W \to U$ be bijective linear maps. Prove that

$$(S \circ T)^{-1} = T^{-1} \circ S^{-1}.$$

proof: Note that: $(S \circ T) \circ (T^{-1} \circ S^{-1}) = S \circ (T \circ T^{-1}) \circ S^{-1} = S \circ S^{-1} = id$, and $(T^{-1} \circ S^{-1}) \circ (S \circ T) = T^{-1} \circ (S^{-1} \circ S) \circ T = T^{-1} \circ T = id$. So, $T^{-1} \circ S^{-1}$ is an inverse for $S \circ T$.

4. (1 point) Let $T : V \to V$ be a linear map. Prove that if $T^n = 0$ for some $n \in \mathbb{N}$, then T is not invertible.

proof. If T is bijective, then $\text{null}(T^2) = 0$ since if $x \neq 0$ so $T^2(x) = 0$, then $T(T(x)) = 0$. If $T(x) = 0$, then T is not injective since $x \neq 0$. If $T(x) \neq 0$, then the same argument holds since $T(\underbrace{T(x)}^{\neq 0}) = 0$. By iterating this, we see that if T is bijective, $\text{Null}(T^n) = 0$ for any $n \in \mathbb{N}$. So, we cannot have $T^n = 0$ (when $V \neq \{0\}$).

1. Let $T : V \to V$ and $S : V \to V$ be linear maps on a vector space V.

   (a) (1 point) Assume that V is finite dimensional. Show that S and T are both bijective if and only if $S \circ T$ is bijective.

   (b) (2 points) Find a counterexample to this when V is infinite dimensional.

a) If S and T are bijective, so is $S \circ T$ since the composition of two bijections is a bijection.

Assume T is not injective. Then let $a \neq b$ be so $T(a) = T(b)$. Then $S \circ T(a) = S \circ T(b)$, so $S \circ T$ is not bijective. So, if $S \circ T$ is bijective,

T is injective. But, being bijective is equivalent to being injective and if $T: V \to W$, $\dim V \geq \dim W$. Since $\dim V = \dim V$, T is bijective

If S is not surjective, there is some $b \in V$ with $b \neq S(v)$ for any $v \in V$. Then $b \neq S \circ T(v)$ for any $v \in V$, so $S \circ T$ is not bijective So, if $S \circ T$ is bijective, S is surjective. Since also, $\dim V = \dim V$, S is bijective

b) Let $V = \mathbb{R}^{\mathbb{N}}$. Then, let $F: V \to V$ be so $F(x_0, x_1, \ldots) = (0, x_0, x_1, \ldots)$ and $B: V \to V$ be so $B(x_0, x_1, \ldots) = (x_1, x_2, \ldots)$. B is not injective and F is not surjective but $B \circ F = id_V$.