Abstract Linear Algebra - Problem Set 5
Instructor: Katalin Berlow

The homework is out of 10 points total.

1. (6 points) Let $T : V \to W$ be a linear map between finite dimensional vector spaces. Prove that the following are equivalent.
   I. T is bijective (both injective and surjective).
   II. There is a another linear map $T' : W \to V$ where $T \circ T'$ is the identity map on W and $T' \circ T$ is the identity map on V. We call this T' the *inverse* of T and write it as T^{-1}.
   III. T is injective and $\dim V \geq \dim W$.
   IV. T is surjective and $\dim V \leq \dim W$.
2. (2 points) Let $T : V \to W$ be a linear map between finite dimensional vector spaces. Prove that the following are equivalent.
   I. T is bijective.
   II. For every basis $v_1, \ldots, v_n$ of V, we have that $T(v_1), \ldots, T(v_n)$ is a basis for W.
   III. For some basis $v_1, \ldots, v_n$ of V, we have that $T(v_1), \ldots, T(v_n)$ is a basis for W.
3. (1 point) Let $T : V \to W$ and $S : W \to U$ be bijective linear maps. Prove that
$$(S \circ T)^{-1} = T^{-1} \circ S^{-1}.$$
4. (1 point) Let $T : V \to V$ be a linear map. Prove that if $T^n = 0$ for some $n \in \mathbb{N}$, then T is not invertible.

Extra Credit:

1. Let $T : V \to V$ and $S : V \to V$ be linear maps on a vector space V.
   (a) (1 point) Assume that V is finite dimensional. Show that S and T are both bijective if and only if $S \circ T$ is bijective.
   (b) (2 points) Find a counterexample to this when V is infinite dimensional.