Abstract Linear Algebra - Problem Set 4
Instructor: Katalin Berlow

The homework is out of 10 points total.

1. (2 points) Let V and W be vector spaces. Let $L(V,W)$ denote the set of all linear transformations from V to W.
   (a) Show that $L(V,W)$ is a subspace of W^V.
   (b) If $\dim V = n$ and $\dim W = n$, what is the dimension of $L(V,W)$? Prove your answer.
2. (2 points) Let $x \in \mathbb{R}$ be a fixed real number. Define the map $\phi_x : L(\mathbb{R},\mathbb{R}) \to \mathbb{R}$ by $\phi(f) = f(x)$. Show that ϕ is a linear map.
3. (4 points) Let $f : V \to W$ be a linear map between vector spaces V and W.
   (a) Show that if f is injective, then $\dim V \leq \dim W$.
   (b) Show that if f is surjective, then $\dim V \geq \dim W$.
4. (2 points) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with $T \circ T = 0$.
   (a) Show that $\operatorname{im}(T) \subseteq \ker(T)$.
   (b) Show that $\dim(\operatorname{im}(T)) \leq \frac{1}{2}n$.

1. (2 points) Let V and W be vector spaces. Let $L(V, W)$ denote the set of all linear transformations from V to W.

   (a) Show that $L(V, W)$ is a subspace of W^V.

We will show that $L(V,W)$ satisfies the subspace axioms.

1) The zero function f_0 is linear: $f_0(a+b) = 0 = 0+0 = f_0(a)+f_0(b)$ and $f_0(ca) = 0 = c0 = cf_0(a)$ for any $a,b \in V$, and $c \in F$. So, $f_0 \in L(V,W)$.

2) Let $f,g \in L(V,W)$. Then, $f+g$ is linear. For $v,w \in V$ and $c \in F$, we have $(f+g)(v+w) = f(v+w) + g(v+w) = f(v)+f(w)+g(v)+g(w)$ since f and g are linear. Then, $f(v)+f(w)+g(v)+g(w) = f(v)+g(v)+f(w)+g(w) = (f+g)(v)+(f+g)(w)$.
Also, $(f+g)(cv) = f(cv)+g(cv) = cf(v)+cg(v) = (cf+cg)(v) = c(f+g)(v)$. So, we have $f+g \in L(V,W)$.

3) We will show that if $f \in L(V,W)$ and $c \in F$, then $cf \in L(V,W)$.
To do this, we will show cf is linear. Let $v,w \in V$, $a \in F$. Then,
$(cf)(v+w) = c(f(v+w)) = c(f(v)+f(w)) = cf(v)+cf(w)$. Also, $(cf)(av) = c(f(av))$ $= caf(v)$ since f is linear. $caf(v) = a(cf)(v)$. So, $cf \in L(V,W)$.

Thus, $L(V,W)$ is a subspace of W^V.

(b) If $\dim V = n$ and $\dim W = m$ what is the dimension of $L(V, W)$? Prove your answer.

Claim: $\dim L(V,W) = nm$.

proof: We will construct a basis of size nm for $L(V,W)$.

Let $v_1, \dots, v_n$ and $w_1, \dots, w_m$ be bases for V and W. Define linear maps $T_{i,j} : V \to W$ by $T_{i,j}(v_i) = w_j$ and $T_{i,j}(v_k) = 0$ for $k \neq i$.

Then $\{T_{i,j} : i \leq n, j \leq m\}$ is a basis for $L(V,W)$.

These are linearly independent: Suppose $a_{1,1}T_{1,1} + a_{1,2}T_{1,2} + \dots + a_{n,m}T_{n,m} = 0$. Then, for each $i \leq n$, $(a_{1,1}T_{1,1} + \dots + a_{n,m}T_{n,m})(v_i) = a_{i,1}T_{i,1}(v_i), \dots a_{i,m}T_{i,m}(v_i)$ since $T_{k,j}(v_i) = 0$ if $k \neq i$. This is equal to $a_{i,1}w_1 + \dots + a_{i,m}w_m = 0$ and so, $a_{i,1} = \dots = a_{i,m} = 0$ since $w_1, \dots, w_m$ are linearly independent. Since this holds for each $i \leq n$, the coefficients are all zero.

These are spanning: Let $S \in L(V,W)$. As shown in lecture, we can uniquely define S by where basis vectors are sent. Since $w_1, \dots, w_m$ is a basis for W, we can write $S(v_i) = a_{i,1}w_1 + \dots + a_{i,m}w_m$ for each $i \leq n$. Then, $S(v_i) = a_{i,1}T_{i,1}(v_i) + \dots + a_{i,1}T(v_m)$ for each i. So, we have that for any basis vector v_k, $\sum_{i,j \leq n,m} a_{i,j}T_{i,j}(v_k) = \sum_{j \leq m} a_{k,j}T_{k,j}(v_k) = S(v_k)$.

Since a linear map is defined uniquely by where basis vectors are sent,

$$S = \sum_{i,j \leq n,m} a_{i,j} T_{i,j}.$$

So, $\dim(L(V,W)) = nm$. $\square$

2. (2 points) Let $x \in \mathbb{R}$ be a fixed real number. Define the map $\phi_x : L(\mathbb{R}, \mathbb{R}) \to \mathbb{R}$ by $\phi(f) = f(x)$. Show that ϕ is a linear map.

We will show Φ_x is linear for any $x \in \mathbb{R}$.

Let $S, T \in L(\mathbb{R}, \mathbb{R})$. Then $\Phi_x(S+T) = (S+T)(x) = S(x) + T(x) = \Phi_x(S) + \Phi_x(T)$.

Let $a \in \mathbb{R}$, $S \in L(\mathbb{R}, \mathbb{R})$. Then $\Phi_x(aS) = (aS)(x) = a\,S(x) = a\Phi_x(S)$.

So, Φ_x is linear. $\square$

3. (4 points) Let $f : V \to W$ be a linear map between vector spaces V and W.

(a) Show that if f is injective, then $\dim V \leq \dim W$.

proof: Let $f: V \to W$ be injective. Then, let $v_1, \dots, v_n$ be a basis for V. We then have that $f(v_1), \dots, f(v_n)$ is linearly independent as shown in lecture since f is injective. So, since there are $n = \dim V$ many linearly independent vectors in W, $\dim W \geq n = \dim V$. $\square$

(b) Show that if f is surjective, then $\dim V \geq \dim W$.

proof: Let $f: V \to W$ be surjective Let $w_1, \dots, w_m$ be a basis for W. Then, since f is surjective, there are $v_1, \dots, v_m \in V$ with $f(v_1) = w_1, \dots, f(v_m) = w_m$ As shown in lecture, since $f(v_1), \dots, f(v_m)$ are linearly independent, so are $v_1, \dots, v_m$. So $\dim V \geq \dim W$.

4. (2 points) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with $T \circ T = 0$.

(a) Show that $\mathrm{im}(T) \subseteq \ker(T)$.

Let $v \in \mathrm{im}(T)$. Then, there is $w \in \mathbb{R}^n$ where $T(w) = v$. But, $T \circ T(w) = 0$ and so $T(v) = 0$ and $v \in \mathrm{Null}(T)$.

(b) Show that $\dim(\mathrm{im}(T)) \leq \frac{1}{2}n$.

By rank-nullity, we have $\dim(\mathbb{R}^n) = \dim \mathrm{Null}(T) + \dim \mathrm{ran}(T)$, but $\mathrm{ran}(T) \subseteq \mathrm{Null}(T)$ so $\dim \mathrm{ran}(T) \leq \dim \mathrm{Null}(T)$. So, $2 \dim \mathrm{ran}(T) \leq \dim \mathrm{ran}(T) + \dim \mathrm{Null}(T) = \dim \mathbb{R}^n = n$. Thus, $\dim(\mathrm{ran}(T)) \leq \frac{1}{2} n$.

1. (3 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a function which is additive: for any $x, y \in \mathbb{R}$ we have $f(x+y) = f(x) + f(y)$. Assume also that $f(1) = 1$. Does f have to be the identity function? (The function where $f(x) = x$.) Prove or disprove.

We can view $\mathbb{R}$ as a vector space over $\mathbb{Q}$. Let B be a basis for this space. Let $b_1 \neq b_2$ be two irrational basis vectors. define $f: (1) = 1$, $f(b_1) = b_2$, $f(b_2) = b_1$, and $f(b) = b$ for all other $b \in B$. This defines a $(\mathbb{Q})$-linear map since we've specified where the basis vectors are sent. So it is additive, $f(1) = 1$, and by construction, $f \neq id_{\mathbb{R}}$.