Abstract Linear Algebra - Problem Set 4
Instructor: Katalin Berlow

The homework is out of 10 points total.

1. (2 points) Let V and W be vector spaces. Let $L(V, W)$ denote the set of all linear transformations from V to W.
   (a) Show that $L(V, W)$ is a subspace of W^V.
   (b) If $\dim V = n$ and $\dim W = m$, what is the dimension of $L(V, W)$? Prove your answer.
2. (2 points) Let $x \in \mathbb{R}$ be a fixed real number. Define the map $\phi_x : L(\mathbb{R}, \mathbb{R}) \to \mathbb{R}$ by $\phi(f) = f(x)$. Show that ϕ is a linear map.
3. (4 points) Let $f : V \to W$ be a linear map between vector spaces V and W.
   (a) Show that if f is injective, then $\dim V \leq \dim W$.
   (b) Show that if f is surjective, then $\dim V \geq \dim W$.
4. (2 points) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with $T \circ T = 0$.
   (a) Show that $\text{range}(T) \subseteq \text{null}(T)$.
   (b) Show that $\dim(\text{range}(T)) \leq \frac{1}{2}n$.

Bonus Question

1. (3 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a function which is additive: for any $x, y \in \mathbb{R}$ we have $f(x + y) = f(x) + f(y)$. Assume also that $f(1) = 1$. Does f have to be the identity function? (The function where $f(x) = x$.) Prove or disprove.