Abstract Linear Algebra - Problem Set 2
Instructor: Katalin Berlow

The homework is out of 10 points total.

1. (3 points) Let E denote the set of all points on earth (a sphere). We will call a continuous function $f : E \to \mathbb{R}$ a **temperature map**. Show that the set T of all temperature maps forms a vector space over the field $\mathbb{R}$.

   In this vector space, addition and scalar multiplication of two functions is pointwise: $(f+g)(x) = f(x) + g(x)$ and $(cf)(x) = c \cdot f(x)$.

   Hint: Recall that adding two continuous functions, or scaling a continuous function, gives a continuous function.

2. (2 points) Let V be a vector space over F with subspaces $W, U \subseteq F$. Recall the definition of a direct sum:

   $$W + U := \{v \in V : \exists w \in W,\ \exists u \in U,\ v = w + u\}.$$

   Show that $W + U$ is a subspace of V.

3. (5 points) Let $S = \{1, \ldots, 5\}$. We let $\mathcal{P}(S)$ denote the **powerset** of S, which is the set of all subsets of S. Given two subsets $A, B \subseteq S$, we let

   $$A \triangle B := (A \cup B) \setminus (A \cap B).$$

   This is the set of element which are in A xor (exclusive or) B. The set $A \triangle B$ is called the **symmetric difference** of A and B. We let $A + B := A \triangle B$.

   Recall the field on two elements $\mathbb{F}_2 := \{0, 1\}$. For a subset $A \subseteq S$ we define

   $$0 \cdot A = \varnothing \text{ and } 1 \cdot A = A.$$

   Show that $\mathcal{P}(S)$ is a vector space over $\mathbb{F}_2$ with addition and scalar multiplication defined as above.

Extra Credit:

4. (3 points) Show that the vector space $\mathcal{P}(S)$ defined in problem 3 is the same vector space as $(\mathbb{F}_2)^5$. This is the vector space over $\mathbb{F}_2$ whose elements are vectors of length 5 with entries in $\mathbb{F}_2$.

   That is, find a way to bijectively match elements of $\mathcal{P}(S)$ to elements of $\mathbb{F}_2$ in such a way that addition and scalar multiplication doesn't change. This is the same as finding an invertible linear map between these vector spaces.

1. (3 points) Let E denote the set of all points on earth (a sphere). We will call a continuous function $f : E \to \mathbb{R}$ a **temperature map**. Show that the set T of all temperature maps forms a vector space over the field $\mathbb{R}$.

In this vector space, addition and scalar multiplication of two functions is pointwise: $(f+g)(x) = f(x) + g(x)$ and $(cf)(x) = c \cdot f(x)$.

Hint: Recall that adding two continuous functions, or scaling a continuous function, gives a continuous function.

We will show that T is a vector space over $\mathbb{R}$.

Commutativity: Let $f, g \in T$. Then for $x \in E$, $(f+g)(x) = f(x) + g(x) = g(x) + f(x)$ because $f(x), g(x) \in \mathbb{R}$ and $+$ is commutative over $\mathbb{R}$. We then have $g(x) + f(x) = (g+f)(x)$ and so $f+g = g+f$.

Associativity: Let $f, g, h \in T$. Then, for $x \in E$, we have $((f+g)+h)(x) = (f(x)+g(x))+h(x)$ $= f(x) + (g(x)+h(x))$ since $f(x), g(x), h(x) \in \mathbb{R}$ and $+$ is associative in $\mathbb{R}$. We then have: $f(x) + (g(x)+h(x)) = (f+(g+h))(x)$ so $(f+g)+h = f+(g+h)$.

Also, let $a, b \in \mathbb{R}$, $f \in T$. Then $(ab)f(x) = a(bf(x))$ since $\cdot$ in $\mathbb{R}$ is associative. So, $(ab)f(x) = a(b(f(x)))$.

Additive identity: Define the function $f_0 \in T$ where for all $x \in E$, $f(x) = 0$. Then, for any $g \in T$, $x \in E$, $(f_0+g)(x) = f_0(x) + g(x) = 0 + g(x) = g(x)$, so $f_0 + g = g$ and so f_0 is the additive identity.

Additive inverse: Let $f \in T$ be arbitrary. Define $g \in T$ where $g(x) = -1 \cdot f(x)$. Then $(f+g)(x) = f(x) + g(x) = f(x) - f(x) = 0$. So, g is the additive inverse for f.

Multiplicitive identity: Note that $1 \in \mathbb{R}$ and for any $f \in T$, $1f(x) = f(x)$ so $1f = f$.

Distributive Properties: Let $a, b \in \mathbb{R}$, $f, g \in T$. Then, $(a(f+g))(x) = a((f+g)(x))$ by associativity of scalar multiplication. $a((f+g)(x)) = a(f(x)+g(x)) = af(x)+ag(x)$ by distributivity of $\cdot$ over $+$ in $\mathbb{R}$. $af(x)+ag(x) = (af+ag)(x)$ so $a(f+g) = af+ag$.

Let $a, b \in \mathbb{R}$, $f \in T$. Then, $(a+b)f(x) = af(x)+bf(x)$ since $\cdot$ is distributive over $+$ in $\mathbb{R}$. So, $(a+b)f = af+bf$.

Thus T is a vector space over $\mathbb{R}$. $\square$

2. (2 points) Let V be a vector space over F with subspaces $W, U \subseteq \cancel{F}$ V Recall the definition of a direct sum:

$$W + U := \{v \in V : \exists w \in W,\ \exists u \in U,\ v = w + u\}.$$

Show that $W + U$ is a subspace of V.

We will show $U+W$ satisfies the axioms of a vector space.

Additive identity: Note $0_V = 0_W \in W$ and $0_V = 0_U \in U$ since W and U are subspaces. So, $0_W + 0_U \in W+U$ by definition of the sum of vector spaces. But, $0_W + 0_U = 0_V$, So, $0_V \in W+U$.

Closure under $+$: Let $z_1, z_2 \in U+W$. By definition of $U+W$, z_1 and z_2 must be of the form $z_1 = w_1 + u_1$ and $z_2 = w_2 + u_2$ for $w_1, w_2 \in W$ and $u_1, u_2 \in U$. So, $z_1 + z_2 = (w_1+u_1) + (w_2+u_2) = (w_1+w_2) + (u_1+u_2)$ by comutativity and associativity of $+$ in V. Since W and U are closed under $+$, we have $w_1+w_2 \in W$ and $u_1+u_2 \in U$. So, $(w_1+w_2)+(u_1+u_2) = z_1+z_2 \in W+U$.

Closure under scalar multiplication: Let $a \in F$, $z \in W+U$. We can write $z = w+u$ for some $w \in W$ and $u \in U$ by definition of $W+U$. So, $az = a(w+u) = aw + au$ by distributivity of scalar multiplication in V. But, since W and V are closed under scalar multiplication, $aw \in W$ and $au \in U$ and so $aw + au = az \in W+U$.

So, we can conclude that $W+U$ is a subspace of V. $\square$

3. (5 points) Let $S = \{1, \ldots, 5\}$. We let $\mathcal{P}(S)$ denote the **powerset** of S, which is the set of all subsets of S. Given two subsets $A, B \subseteq S$, we let

$$A \triangle B := (A \cup B) \setminus (A \cap B).$$

This is the set of element which are in A xor (exclusive or) B. The set $A \triangle B$ is called the **symmetric difference** of A and B. We let $A + B := A \triangle B$.

Recall the field on two elements $\mathbb{F}_2 := \{0, 1\}$. For a subset $A \subseteq S$ we define

$$0 \cdot A = \varnothing \text{ and } 1 \cdot A = A.$$

Show that $\mathcal{P}(S)$ is a vector space over $\mathbb{F}_2$ with addition and scalar multiplication defined as above.

We will show that $P(S)$ is a vector space over $\mathbb{F}_2$

Lemma: $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

proof: $x \in A \triangle B$ iff $x \in (A \cup B) \setminus (A \cap B)$

iff $x \in A \cup B$ and $x \notin A \cap B$

iff $(x \in A$ or $x \in B)$ and not $(x \in A$ and $x \in B)$

iff $(x \in A$ or $x \in B)$ and $(x \notin A$ or $x \notin B)$

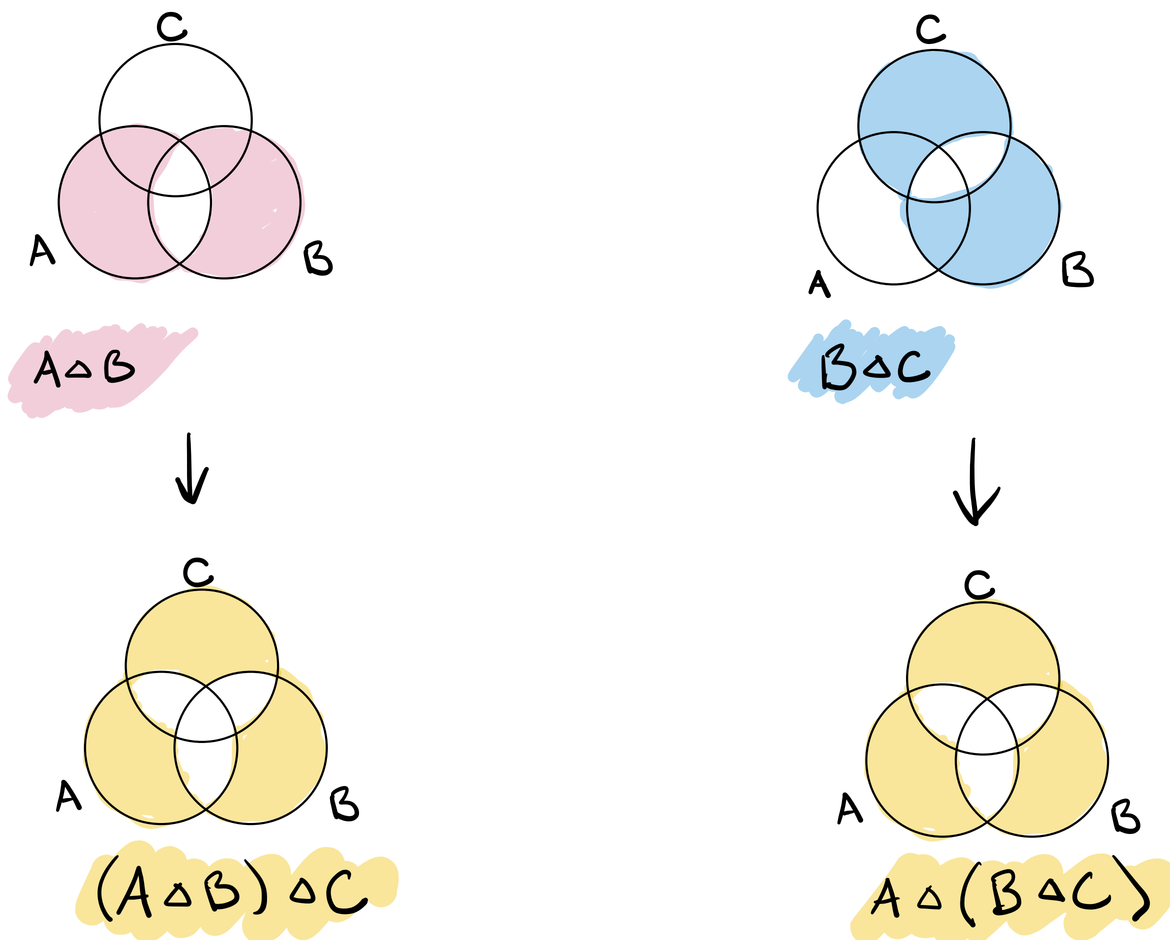
iff $(x \in A$ and $(x \notin A$ or $x \notin B))$ or $(x \in B$ and $(x \notin A$ or $x \notin B))$

iff $(x \in A$ and $x \notin B)$ or $(x \in B$ and $x \notin A)$ iff $x \in (A \setminus B) \cup (B \setminus A)$.

Commutativity: Let $A, B \in P(S)$. Then, $A+B = A \triangle B = (A \cup B) \setminus (A \cap B)$
$= (B \cup A) \setminus (B \cap A) = B \triangle A = B + A$. So, $A+B = B+A$.

Associativity: Let $A, B, C \in P(S)$. We wish to show $(A+B)+C = A+(B+C)$.

We will prove this via venn diagram.



As we can see, $(A \triangle B) \triangle C = A \triangle (B \triangle C)$.

Note: It is possible to prove this via set arithmetic, but it is painful.

Additive identity: Let $0 = \emptyset$. Then $\emptyset \triangle A = (A \cup \emptyset) \setminus (A \cap \emptyset) = A$ for any $A \in P(S)$

Additive inverse: Let $A \in \mathcal{P}(S)$. Then Let $-A = A$. Then $A \cup -A = A$ and $A \cap -A = A$ so $(A \cup -A) \setminus (A \cap -A) = A \setminus A = \emptyset$.

Multiplicative identity: By definition $1A = A$ for all $A \in \mathcal{P}(S)$.

Distributivity: Let $A, B \in \mathcal{P}(S)$. Note that:

$$1(A+B) = A+B = 1A + 1B \qquad (0+0)A = \emptyset = \emptyset + \emptyset = \emptyset A + \emptyset A$$

$$0(A+B) = \emptyset = 0A + 0B \qquad (0+1)A = 1A = A = \emptyset \triangle A = 0A + 1A$$

$$(1+1)A = 0A = \emptyset = A \triangle A = A + A = 1A + 1A.$$

And so, we have distributivity.

4. (3 points) Show that the vector space $\mathcal{P}(S)$ defined in problem 3 is the same vector space as $(\mathbb{F}_2)^5$. This is the vector space over $\mathbb{F}_2$ whose elements are vectors of length 5 with entries in $\mathbb{F}_2$.

That is, find a way to bijectively match elements of $\mathcal{P}(S)$ to elements of $\mathbb{F}_2$ in such a way that addition and scalar multiplication doesn't change. This is the same as finding an invertible linear map between these vector spaces.

We will define a bijective linear map from $(\mathbb{F}_2)^5$ to $\mathcal{P}(S)$.

Define $f : (\mathbb{F}_2)^5 \to \mathcal{P}(S)$ by $f(z) = A_z$ where $n \in A_z$ iff $z_n = 1$

for example if $z = (0,1,1,0,0)$ then $f(z) = \{2,3\}$.

This is bijective: If $z \neq y$ then there is an n with $z_n \neq y_n$, but then, $n \in f(z)$ but $n \notin f(y)$ or vice versa. so, f is injective.

If $A \subseteq S$, then defining z by $z_n = 1$ iff $n \in A$, we get $f(z) = A$.

so, f is bijective

f is linear: If $z, y \in (\mathbb{F}_2)^S$, then $n \in f(z) + f(y)$ iff $n \in f(z) \triangle f(y)$ iff $n \in f(z)$ exclusive or $n \in f(y)$. So, $n \in f(z) + f(y)$ iff either $z_n = 1$ exclusive or $y_n = 1$. This is true if and only if $z_n + y_n = 1$. So $n \in f(z+y)$.

Also, for $z \in (\mathbb{F}_2)^S$, $1f(z) = f(z) = f(1z)$ and $0f(z) = \emptyset = f(0) = f(0z)$. So, f is linear, and thus $(\mathbb{F}_2)^S$ is isomorphic to $\mathcal{P}(S)$ as vector spaces.