1 Section 1: Monday, June 23

1.1 Let S be a set of natural numbers. Consider the statement "If any number in S is odd, then 2 is not in S."

Note that this statement is of the form $A \Rightarrow B$ where

A: Any number in S is odd.

B: 2 is not in S.

(a) Write the contrapositive of this statement.

This is (not B) $\Rightarrow$ (not A).

not B: 2 is in S.

not A: There is a number in S which is not odd (even).

Contrapositive: If 2 is in S, then there is a number in S which is even.

(b) Write the converse of this statement.

This is $B \Rightarrow A$.

Converse: If 2 is not in S, then any number in S is odd.

(c) Write the negation of this statement.

This is (not A) or B.

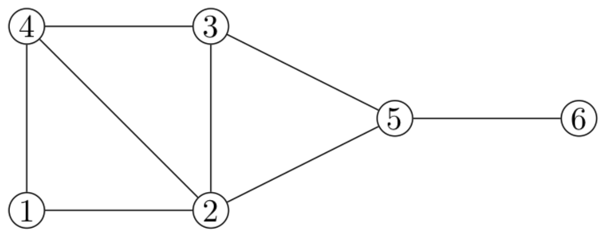
Negation: Either there is an even number in S or 2 is not in S.

1.2 List all subsets of the set $\{0, 1, 2\}$.

$\emptyset$, $\{0\}$, $\{1\}$, $\{2\}$, $\{0,1\}$, $\{0,2\}$, $\{1,2\}$, $\{0,1,2\}$

2 Section 2: Tuesday, June 24

2.1 Consider the following graph.



Write the adjacency matrix for this graph.

The adjacency matrix for this graph is the matrix A where

$$A_{i,j} = \begin{cases} 1 & \text{if there is an edge from vertex } i \text{ to vertex } j \\ 0 & \text{otherwise.} \end{cases}$$

This is the matrix:

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

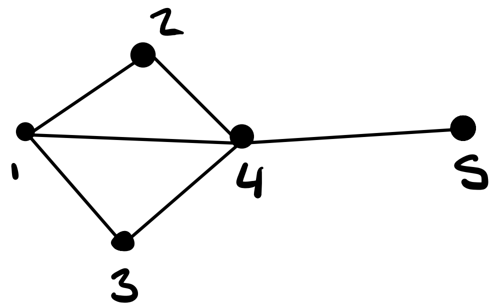
2.2 How would the adjacency matrix change if we add self-loops at each vertex?

A self-loop is an edge from a vertex to itself. This corresponds to the entries of the form $A_{i,i}$ being 1 for vertex i. If all vertices have self-loops, the values on the diagonal all become 1. Nothing else is changed.

2.3 Let G be any graph. Let A be its adjacency matrix. What do the entries of A^n represent? Prove your answer.

The value of $(A^n)_{i,j}$ is equal to the number of paths of length n from vertex i to vertex j.

We will go through an example to demonstrate intuition



$$A^2 = \begin{pmatrix} 0&1&1&1&0\\ 1&0&0&1&0\\ 1&0&0&1&0\\ 1&1&1&0&1\\ 0&0&0&1&0 \end{pmatrix}^2 = \begin{pmatrix} 0&1&1&1&0\\ 1&0&0&1&0\\ 1&0&0&1&0\\ 1&1&1&0&1\\ 0&0&0&1&0 \end{pmatrix}\begin{pmatrix} 0&1&1&1&0\\ 1&0&0&1&0\\ 1&0&0&1&0\\ 1&1&1&0&1\\ 0&0&0&1&0 \end{pmatrix} = \begin{pmatrix} 3&1&1&2&0\\ 1&2&2&1&1\\ 1&2&2&1&1\\ 2&1&1&3&0\\ 0&1&1&0&1 \end{pmatrix}$$

Note that $A^2_{i,j}$ is the number of times there is an edge from vertex i to some vertex k and from vertex k to vertex j. These form paths of length 2.

proof: We will prove this by induction.

Base Case: Note there is a path of length 1 from vertex i to vertex j if and only if there is an edge from vertex i to vertex j iff $(A^1)_{i,j} = 1$. (In this case, there can only be a single path of length 1 between two vertices.)

Induction Hypothesis: Fix n. Assume that $(A^{n-1})_{i,j}$ is the number of paths of length $n-1$ from vertex i to vertex j.

Induction Step: Note that any path of length $n \geq 2$ from vertex i to vertex j can be uniquely split into a path consisting of the first $n-1$ edges and a path consisting of the last edge.

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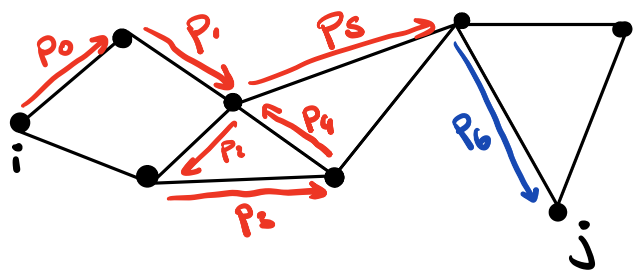

If our path is $p_0, \dots, p_6$ then we can split this into $p_0, \dots, p_5$ and p_6.

Since there is one way to do this for each path, there are exactly as many paths of length n from vertex i to vertex j as there are paths of length $n-1$ from vertex i to some other vertex adjacent to vertex j.

So, we can write

$$\# \text{ paths length } n \text{ from } i \text{ to } j = \sum_{\substack{\text{vertices} \\ k \text{ adjacent to } j}} \# \text{ paths of length } n-1 \text{ from } i \text{ to } k.$$

By our induction hypothesis, $\#$ paths length $n-1$ from i to k $= (A^{n-1})_{i,k}$.

Also note that

$$(A^n)_{i,j} = (A^{n-1})_{\text{row } i} \cdot A_{\text{column } j} = \sum_{\substack{\text{all} \\ \text{vertices } k}} (\# \text{ paths length } n-1 \text{ from } i \text{ to } k) \cdot A_{k,j}$$

This is 0 if there is no edge from k to j

$$= \sum_{\substack{\text{vertices} \\ k \text{ adjacent to } j}} \# \text{ paths of length } n-1 \text{ from } i \text{ to } k = \# \text{ paths length } n \text{ from } i \text{ to } j.$$

So, $(A^n)_{i,j} = \#$ paths length n from i to j. $\square$

3.1 Let V and W be vector spaces over the field $\mathbb{F}$. Prove that the product $V \times W$ is also a vector space with the same field $\mathbb{F}$.

proof: We will prove this by showing $V \times W$ satisfies the vector space axioms.

Closure under addition: Let $(v_1, w_1), (v_2, w_2) \in V \times W$. Since $v_1, v_2 \in V$ and V is a vector space, $v_1 + v_2 \in V$. Similarly, $w_1 + w_2 \in W$. So,

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) \in V \times W.$$

Addition is commutative: Let $(v_1, w_1), (v_2, w_2) \in V \times W$. Since $v_1, v_2 \in V$ and V is a vector space, $v_1 + v_2 = v_2 + v_1$. Similarly $w_1 + w_2 = w_2 + w_1$. So,

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) = (v_2 + v_1, w_2 + w_1) = (v_2, w_2) + (v_1, w_1).$$

Addition is associative: Let $(v_1, w_1), (v_2, w_2), (v_3, w_3) \in V \times W$. Since $v_1, v_2, v_3 \in V$, and V is a vector space, $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$. Similarly $(w_1 + w_2) + w_3 = w_1 + (w_2 + w_3)$.

So, $\big((v_1, w_1) + (v_2, w_2)\big) + (v_3, w_3) = (v_1 + v_2, w_1 + w_2) + (v_3, w_3) = \big((v_1 + v_2) + v_3, (w_1 + w_2) + w_3\big) = \big(v_1 + (v_2 + v_3), w_1 + (w_2 + w_3)\big)$

$= (v_1, w_1) + (v_2 + v_3, w_2 + w_3) = (v_1, w_1) + \big((v_2, w_2) + (v_3, w_3)\big)$.

Additive Identity: Note that since V is a vector space, it has an additive identity O_V. Similarly, O_W is the additive identity for W. Let $(v, w) \in V \times W$. Then, since O_V is the additive identity for V,

lowercase

$O_V + v = v$. Similarly $O_W + w = w$. Then,

capital

$(O_V, O_W) \in V \times W$ is so that for any $(v, w) \in V \times W$ we have

$(O_V, O_W) + (v, w) = (O_V + v, O_W + w) = (v, w)$. So, (O_V, O_W) is the additive identity for $V \times W$.

Additive inverse: Let $(v,w) \in V \times W$. Since $v \in V$ and V is a vector space, there is a vector $-v \in V$ so $v + -v = 0_V$. Similarly $-w \in W$. Then, $(-v,-w) \in V \times W$ is so $(v,w) + (-v,-w) = (v+-v,\ w+-w) = (0_V, 0_W)$.

Closure under scalar multiplication: Let $c \in \mathbb{F}$, $(v,w) \in V \times W$. Since V is a vector space over $\mathbb{F}$, $cv \in V$. Similarly, $cw \in W$. So, $c(v,w) = (cv, cw) \in V \times W$.

Scalar multiplication is associative: Let $c,d \in \mathbb{F}$, $(v,w) \in V \times W$. Since V is a vector space, $(cd)v = c(dv)$. Similarly $(cd)w = c(dw)$. So, $(cd)(v,w) = ((cd)v, (cd)w) = (c(dv), c(dw))$
$= c(dv, dw) = c(d(v,w))$.

Distributive properties: Let $c \in \mathbb{F}$, $(v_1,w_1)(v_2,w_2) \in V \times W$. Since $v_1, v_2 \in V$ and V is a vector space over $\mathbb{F}$, we have $c(v_1+v_2) = cv_1 + cv_2$. Similarly, $c(w_1+w_2) = cw_1 + cw_2$.
So, $c((v_1,w_1) + (v_2,w_2)) = c(v_1+v_2,\ w_1+w_2) = (c(v_1+v_2),\ c(w_1+w_2)) = (cv_1 + cv_2,\ cw_1 + cw_2)$
$= (cv_1, cw_1) + (cv_2, cw_2) = c(v_1,w_1) + c(v_2,w_2)$.

Let $c,d \in \mathbb{F}$ and $(v,w) \in V \times W$. Since $v \in V$ and V is a vector space over $\mathbb{F}$, we have $(c+d)v = cv + dv$. Similarly, $(c+d)w = cw + dw$. So, $(c+d)(v,w) =$
$((c+d)v, (c+d)w) = (cv+dv,\ cw+dw) = (cv, cw) + (dv, dw) = c(v,w) + d(v,w)$. $\square$