**Abstract Linear Algebra - Problem Set 3**

Instructor: Katalin Berlow

The homework is out of 10 points total.

1. (a) (1 point) Consider the vector space $\mathbb{R}$ over the field $\mathbb{Q}$. Show that the vectors 1, $\sqrt{2}$, and $\sqrt{3}$ are linearly independent.

   Hint: You may assume $\sqrt{6}$ is irrational.

   (b) (1 point) For $n, m \in \mathbb{N}$, are the vectors 1, $\sqrt{n}$, and $\sqrt{m}$ always linearly independent? If not, when are they linearly independent?

   (c) (0.5 point) How many linearly independent vectors can there be in $\mathbb{R}$ over $\mathbb{Q}$?

2. (2 points) Suppose $v_1, \ldots v_n$ are linearly independent in $V$ and $w \in V$. Show that $v_1, \ldots, v_n, w$ are linearly independent if and only if $w \notin \text{span}\{v_1, \ldots, v_n\}$.

3. (a) (2 points) Let $(\mathbb{F}_2)^{\mathbb{N}}$ denote the vector space of all infinite sequences of elements in $\mathbb{F}_2$ over the field $\mathbb{F}_2$. Show that if $S$ is a set of finitely many linearly independent vectors, then we can extend $S$ to a larger set $T$ of linearly independent vectors so that $S \subseteq T$.

   Hint: Consider the span of $S$.

   (b) (1 point) Does this imply that $(\mathbb{F}_2)^{\mathbb{N}}$ is infinite dimensional? Prove your answer.

4. (2.5 points) Let V be a 6 dimensional vector space. Let $U, W \subseteq V$ each be subspaces with dimension 4. What is the maximum dimesion $U \cap W$ can be? What is the minimum dimension? Prove your answers.

**Extra Credit:**

4. (3 points) We call a polynomial *prime-ish* if all of its exponents are prime. For example, $12x^7 + \frac{4}{7}x^3$ is prime-ish but $x^2 + 3x$ is not. Show that any polynomial with real valued coefficients has a prime-ish multiple.

   Hint: This is a linear algebra class: this will use the fact that polynomials form a vector space.

(a) (1 point) Consider the vector space $\mathbb{R}$ over the field $\mathbb{Q}$. Show that the vectors 1, $\sqrt{2}$, and $\sqrt{3}$ are linearly independent.

Hint: You may assume $\sqrt{6}$ is irrational.

<u>Lemma:</u> Let $a \in \mathbb{Q}$ and $b \in \mathbb{R} \setminus \mathbb{Q}$. If $ab \in \mathbb{Q}$ then $a=0$.

proof: Let $a \neq 0$. If $ab \in \mathbb{Q}$, then $a^{-1}ab \in \mathbb{Q}$ since $\mathbb{Q}$ is closed under multiplication and inverses. But, $a^{-1}ab = b \in \mathbb{Q}$, which is a contradiction. $\square$

Assume there are scalars $a, b, c \in \mathbb{Q}$ so $a1 + b\sqrt{2} + c\sqrt{3} = 0$. Then, we have $a = -b\sqrt{2} - c\sqrt{3}$. So, by squaring: $a^2 = (-b\sqrt{2} - c\sqrt{3}) = 2b + 3c + bc\sqrt{6}$. So, $bc\sqrt{6} = a^2 - 2b - 3c \in \mathbb{Q}$ since $a,b,c \in \mathbb{Q}$ and $\mathbb{Q}$ is closed under adittion and multiplication

We know $\sqrt{6}$ is irrational so $bc = 0$, by the lemma above. So, $b$ or $c$ is zero. This either gives us $a \cdot 1 + b\sqrt{2} = 0$ or $a \cdot 1 + c\sqrt{3} = 0$, either way, we would need $a=b=c=0$ since $\sqrt{2}, \sqrt{3}$ are both irrational. So, $1, \sqrt{2}, \sqrt{3}$ is linearly independent. $\square$

b) (1 point) For $n, m \in \mathbb{N}$, are the vectors 1, $\sqrt{n}$, and $\sqrt{m}$ always linearly independent? If not, when are they linearly independent?

No, they are not always linearly independent: $1, \sqrt{4}, \sqrt{16}$ are dependent:

$1 + -\frac{1}{2}\sqrt{4} + -\frac{1}{4}\sqrt{16} = 0.$

<u>Claim:</u> If $n, m$ are prime then $1, \sqrt{n}, \sqrt{m}$ are independent.

proof: Replace 2 by $n$ and 3 by $m$ in the proof of a. The same proof goes through. $\square$

(c) (0.5 point) How many linearly independent vectors can there be in $\mathbb{R}$ over $\mathbb{Q}$?

The set $\{\sqrt{n} \in \mathbb{R} : n \text{ is prime}\}$ is linearly independent.

This can be proven using induction and a technique similar to (a) but this is difficult – thus the extra credit.

2. (2 points) Suppose $v_1, \ldots v_n$ are linearly independent in $V$ and $w \in V$. Show that $v_1, \ldots, v_n, w$ are linearly independent if and only if $w \notin \text{span}\{v_1, \ldots, v_n\}$.

proof: ($\Rightarrow$) If $w \in \text{span}\{v_1, \ldots, v_n\}$, then by $v_1, \ldots, v_n, w$ are linearly dependent by one of the equivalent definitions.

($\Leftarrow$) Assume $v_1, \ldots, v_n, w$ are linearly dependent. Then there are scalars $a_1, \ldots, a_{n+1} \in F$ not all zero, so $a_1v_1 + \cdots + a_nv_n + a_{n+1}w = 0$. If $a_{n+1} = 0$, then $a_1v_1 + \cdots + a_nv_n = 0$, contradicting linear independence of $v_1, \ldots, v_n$. So, assume $a_{n+1} \neq 0$. Then $w = \left(-\frac{a_1}{a_{n+1}}\right)v_1 + \cdots + \left(-\frac{a_n}{a_{n+1}}\right)v_n$ as desired. □

3. (a) (2 points) Let $(\mathbb{F}_2)^{\mathbb{N}}$ denote the vector space of all infinite sequences of elements in $\mathbb{F}_2$ over the field $\mathbb{F}_2$. Show that if $S$ is a set of finitely many linearly independent vectors, then we can extend $S$ to a larger set $T$ of linearly independent vectors so that $S \subseteq T$. Hint: Consider the span of $S$.

proof: Let $S \subseteq \mathbb{F}_2^{\mathbb{N}}$ be a finite set of linearly independent vectors. Then, span$(S)$ is finite: since there are finitely many scalars, there are finitely many possible linear combinations of elements of $S$. So, since $\mathbb{F}_2^{\mathbb{N}}$ is infinite, $\mathbb{F}_2^{\mathbb{N}} \setminus \text{span}(S) \neq \emptyset$. Choose $v \in \mathbb{F}_2^{\mathbb{N}} \setminus \text{span}(S)$. By the previous problem, $S \cup \{v\}$ is linearly independent since $v \notin \text{span } S$. This is a linearly independent set strictly extending $S$. □

(b) (1 point) Does this imply that $(\mathbb{F}_2)^{\mathbb{N}}$ is infinite dimensional? Prove your answer.

Yes: If $\mathbb{F}_2^{\mathbb{N}}$ were finite dimensional, it would have a finite basis $v_1, \ldots, v_n$. By part (a), we can extend this linearly independent set to a larger one, contradicting the fact that $v_1, \ldots, v_n$ is a basis. $\square$

4. (2.5 points) Let V be a 6 dimensional vector space. Let $U, W \subseteq V$ each be subspaces with dimension 4. What is the maximum dimesion $U \cap W$ can be? What is the minimum dimension? Prove your answers.

$\dim U \cap W \leq 4$ since $U \cap W \subseteq U$ and $\dim U = 4$.

Also, we know $\dim V \geq \dim(U+W) = \dim U + \dim W - \dim U \cap W$, so,

$\dim U \cap W \geq \dim U + \dim W - \dim V = 4+4-6 = 2$. So, $\dim U \cap W \geq 2$.

To see that these are strict, note that if $V = \mathbb{R}^6$,

$U = \{(x_1, x_2, x_3, x_4, 0, 0) \in \mathbb{R}^6 : x_1, \ldots, x_4 \in \mathbb{R}\}$, and $W_1 = \{(0, 0, x_3, x_4, x_5, x_6) \in \mathbb{R}^6 : x_4, \ldots, x_6 \in \mathbb{R}\}$,

Then, $U \cap W_1 = \{(0, 0, x_3, x_4, 0, 0) \in \mathbb{R}^6 : x_3, x_4 \in \mathbb{R}\}$ and so $\dim U \cap W = 2$.

To see $\dim U \cap W$ can equal 3, let $W_2 = \{(0, x_2, x_3, x_4, x_5, 0) \in \mathbb{R}^6 : x_2, \ldots, x_5 \in \mathbb{R}\}$. Then, $\dim U \cap W = 3$. For $\dim U \cap W = 4$, let $W = U$. So, we have $\dim U \cap W$ can be 2, 3, or 4. $\square$

4. (3 points) We call a polynomial *prime-ish* if all of its exponents are prime. For example, $12x^7 + \frac{4}{7}x^3$ is prime-ish but $x^2 + 3x$ is not. Show that any polynomial with real valued coefficients has a prime-ish multiple.

Hint: This is a linear algebra class: this will use the fact that polynomials form a vector space.

Let $q$ be a polynomial of degree $d$. Let $S = \{x^{p_1}, x^{p_2}, \ldots, x^{p_{d+1}}\}$ be a set of $d+1$ many prime-ish monomials: Let $p_1, p_2, \ldots, p_{d+1}$ be the first $d+1$-many primes. For those $x^{p_i} \in S$ with $p \geq d$, we can perform polynomial long-division to write $x^{p_i} = q \cdot f_i + r_i$ where $r_i$ is now degree $< d$. Replace $x^{p_i}$ by $r_i$ in $S$. Now, $S$ is a set of $d+1$ many vectors in $P_{d-1}(\mathbb{R})$. Since $\dim P_{d-1}(\mathbb{R}) = d$, there must be a linear dependency in $S$. Write $S = \{r_1, \ldots r_{d+1}\}$. Then, we can find $a_1, \ldots, a_{d+1} \in \mathbb{R}$ so $a_1 r_1 + \ldots + a_{d+1} r_{d+1} = 0$.

But, we have $x^{p_i} = q f_i + r_i$ so $r_i = x^{p_i} - q f_i$. Plugging this in, we get $a_1(x^{p_1} - q f_1) + \ldots + a_{d+1}(x^{p_{d+1}} - q f_{d+1}) = 0$. We can expand and regroup to get $a_1 x^{p_1} + \ldots + a_{d+1} x^{p_{d+1}} = (a_1 f_1 + \ldots + a_{d+1} f_{d+1}) q$. The left hand side is prime-ish, and the right hand side is a multiple of $q$. $\square$