Answer the questions in the spaces provided. If you run out of room for an answer, continue on the back of the page.

Name: Solutions

Student ID: ____________________

1. Write the definition for a set of vectors to be linearly independent.

A set $S = \{v_1, \ldots, v_n\}$ of vectors is linearly independent iff when $a_1 v_1 + \cdots + a_n v_n = 0$ for $a_1, \ldots, a_n \in F$, then we must have $a_1 = a_2 = \cdots = 0$.

2. Let V be a vector space and $W \subset V$ a subset. What must be true about W for it to be a subspace?

W is a subspace iff

1) $0 \in V$
2) If $v, w \in W$ then $v + w \in W$
3) If $v \in W$, $a \in F$ then $av \in W$.

3. Which of the following are vector spaces? (Yes/no is enough.)

   1. $\mathbb{R}$ over $\mathbb{C}$ with the standard interpretation of addition and multiplication.

      No

   2. $\mathbb{C}$ over $\mathbb{R}$ with the standard interpretation of addition and multiplication.

      Yes

   3. $\mathbb{C}$ over $\mathbb{Q}$ with the standard interpretation of addition and multiplication.

      Yes

4. Let V be a vector space over F. Let $a \in F$. Prove that $a0 = 0$. Justify each step using vector space axioms and field axioms.

We have $a0 = a(0+0)$ by definition of the additive identity. Then, $a(0+0) = a0 + a0$ by distributivity of scalar multiplication. Since every vector has an additive inverse, there is $-a0$. Then, $a0 + -a0 = a0 + a0 + -a0$. By definition of additive inverse, $a0 + -a0 = 0$. So, we have $0 = a0$ as desired.

5. Let V be a vector space and S a set of vectors. Show that $\operatorname{span} S$ is the smallest subspace containing S. That is, prove:

   (a) $\operatorname{span} S$ is a subspace of V, and

   (b) if $W \subseteq V$ is a subspace of V containing S, then $\operatorname{span}(S) \subseteq W$.

a) We will use the subspace axioms to show span$S$ is a subspace of V.

We know $0 \in \operatorname{span} S$ since if $S = \{s_1, \ldots, s_n\}$ then $0 = 0s_1 + \cdots + 0s_n \in \operatorname{span} S$.

Let $v, w \in \operatorname{span} S$. Then there are $a_1, \ldots, a_n, b_1, \ldots, b_n \in F$ with $v = a_1 s_1 + \cdots + a_n s_n$ and $w = b_1 s_1 + \cdots + b_n s_n$ by definition of span. Then,
$v + w = (a_1 + b_1) s_1 + \cdots + (a_n + b_n) s_n \in \operatorname{span} S$.

Let $c \in F$ and $v \in \operatorname{span}(S)$. Then we can write $v = a_1 s_1 + \cdots + a_n s_n$, and $cv = (ca_1) s_1 + \cdots + (ca_n) s_n \in \operatorname{span} S$.

6. Let V be a vector space of dimension n. Assume that $V_1, \dots, V_k$ are (each different, nonzero) subspaces of V so that

$$V_1 \subseteq V_2 \subseteq \cdots \subseteq V_k.$$

Show that $k \leq n$.

Lemma: If $V_1 \subseteq V_2$ and $V_1 \neq V_2$, then $\dim V_1 < \dim V_2$.

proof: Let $v_1, \dots, v_n$ be a basis for V_1. $v_1, \dots, v_n$ is still linearly independent in V_2 but isnt spanning since span $v_1, \dots, v_n = V_1 \subsetneq V_2$. So, we nontrivially extend $v_1, \dots, v_n$ to a basis $v_1, \dots, v_n, v_{n+1}, \dots, v_m$ for V_2. But, $\dim V_2 = m > n = \dim V_1$.

proof:

Since V_1 is nonzero, $\dim V_1 \geq 1$. Then, $1 \leq \dim V_1 < \dim V_2 < \dots < \dim V_k \leq n$ so $k \leq n$.