Continuity and Differentiability

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Abstract

The difference between continuity and differentiability is a critical issue. If you haven't thought it carefully before, I would suggest you try to think some examples to convince yourself that they are not really quite the same concept. These are some notes what I will try to cover these days. Some proofs here will need some theorem in Chp. 4. However, don't be afraid of this since if you have examples in your mind to cover the most essential idea, then it will be fine. A proof is no more than trying to translate the key ideas into mathematical language.

1 Continuity and Differntiability on an Interval

Example 1. Is $\sqrt{x}$ continuous at 0? How do we define continuity for functions whose domain is not the whole real numbers?

Definition (interval)**.** *A subset I of $\mathbb{R}$ is called an interval if for every $x, y \in I$, then $sx + (1-s)y \in I$, where $0 \leq s \leq 1$.*

Example 2. $[0,1]$, $(2,3)$, $\mathbb{R}$ are intervals. $[0,1] \cup (2,3)$ is not an interval.

Definition. *A subset I of $\mathbb{R}$ is called a closed interval if it is of one of the following forms: $[a,b]$, $[a,\infty)$, $(-\infty,b]$, $(-\infty,\infty)$. That is, a closed interval is an interval which includes its "endpoints."*

Example 3. $[0,1]$, $\mathbb{R}$ are closed intervals. $(2,3)$ is not a closed interval.

Definition. *A subset I of $\mathbb{R}$ is called an open interval if it is of one of the following forms: (a,b), (a,∞), $(-\infty,b)$, $(-\infty,\infty)$. That is, an open interval is an interval which excludes its "endpoints."*

Example 4. $(2,3)$, $\mathbb{R}$ are open intervals. $[1,2]$ is not an open interval.

Example 5. Does there exist any interval neither open nor closed?

Definition (Continuity)**.** *A function $f(x)$ defined on an interval I is continuous at a if*

- *$f(a)$ exists.*
- *$\lim_{x\to a} f(x)$ exists.*
- *$\lim_{x\to a} f(x) = f(a)$.*

When a is an endpoint of I, then the definition above should replace $\lim_{x\to a} f(x)$ by one-sided limit $\lim_{x\to a^+} f(x)$ or $\lim_{x\to a^-} f(x)$.

Example 6. Under this definition, $\sqrt{x}$ is continuous at 0.

Definition (Differentiability)**.** *A function $f(x)$ defined on an* ***open*** *interval I is differentiable at a if $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a)$ exists.*

Example 7. $x\sqrt{x}$ is continuous at 0 but NOT differentiable at 0. Even if its derivative is $\frac{3}{2}\sqrt{x}$, whose domain contains 0, we still do not say $x\sqrt{x}$ is differentiable at 0.

Definition (Semi-differentiability)**.** *A function $f(x)$ defined on an interval I is semi-differentiable at a if either $\lim_{x\to a^-} \frac{f(x)-f(a)}{x-a} = f'_-(a)$ or $\lim_{x\to a^+} \frac{f(x)-f(a)}{x-a} = f'_+(a)$ exists.*

Remark 8. Use $\delta - \epsilon$ argument, we can show $f'(a)$ exists if and only if both $f'_-(a)$ and $f'_+(a)$ exist and they have the same value.

Remark 9. Don't use the word "semicontinuous." It is reserved for other situation.

Remark 10. The reason why we can consider only continuity at the endpoints of interval but not differentiability is due to the following proposition.

Proposition 11. *Consider $f(x)$ is defined on both intervals with one common endpoint b, e.g. $[a, b]$ and $[b, c]$. If $f(x)$ is continuous at b when $f(x)$ is considered only defined on each interval, then $f(x)$ is continuous at b. However, if $f(x)$ is semi-differentiable at b when $f(x)$ is considered only defined on each interval, then $f(x)$ is NOT necessarily differentiable at b.*

Proof. $f(x)$ is continuous at b when considered define on $[a, b]$ iff $\lim_{x\to b^-} f(x) = f(b)$. Similarly, $f(x)$ is continuous at b when considered define on $[b, c]$ iff $\lim_{x\to b^+} f(x) = f(b)$. And the fact that $\lim_{x\to b} f(x) = f(b)$ iff $\lim_{x\to b^-} f(x) = \lim_{x\to b^+} f(x) = f(b)$ gives us the desired result.
However, $f(x)$ is semi-differentiable at b when considered define on $[a, b]$ iff $f'_-(b)$ exists. Similarly, $f(x)$ is semi-differentiable at b when considered define on $[b, c]$ iff $f'_+(b)$ exists. And in order to show $f'(b)$ exists, we need $f'_-(b) = f'_+(b)$, but it is not necessarily true. For example, see [1] Sec 2.8, exercise 56.

□

2 Class C^k and Classification

Example 12. $[\![x]\!]$ is a discontinuous function with "jump discontinuity."

Example 13. $\sin(\frac{1}{x})$ is a discontinuous function without jump discontinuity.

Example 14. $f(x) = \begin{cases} x & \text{if } x \text{ is irrational} \\ 0 & \text{if } x \text{ is rational} \end{cases}$ is function only continuous at a point 0 and discontinuous everywhere outside 0.

Example 15. $|x|$ is a continuous function but not differentiable at 0.

Example 16. $f(x) = \begin{cases} x^2\sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is a continuous, differentiable function. But its derivative is NOT continuous at 0.

Proof. Because $x^2, \sin(x), \frac{1}{x}$ are continuous, and product or composite of continuous functions is still continous, $f(x)$ is continuous ouside 0. And $\lim_{x\to 0} x^2\sin(\frac{1}{x}) = 0 = f(0)$, hence $f(x)$ is continuous everywhere. It is differentiable since for $x \neq 0$, $(x^2\sin(\frac{1}{x}))' = 2x\sin(\frac{1}{x}) - \cos(\frac{1}{x})$. And it is differentiable at 0 since $\lim_{x\to 0}\frac{x^2\sin(\frac{1}{x})-0}{x} = \lim_{x\to 0} x\sin(\frac{1}{x}) = 0$. However, $\lim_{x\to 0}\left(2x\sin(\frac{1}{x}) - \cos(\frac{1}{x})\right)$ DNE. □

Remark 17. (Hard! Just need to know this fact.) If $f(x)$ is differentiable (in general sense), that is, we allow $f'(x)$ to be $\pm\infty$, and $\lim_{x\to a} f'(x)$ exists (in general sense), then $\lim_{x\to a} f'(x) = f'(a)$, that is, the derivative is continuous at a (in general sense)! This remark tells you **the derivative doesn't allow jump discontinuity**.

Proof. Use $\delta - \epsilon$ argument and MVT. □

So, the classification of all the functions are as follows:

$$\begin{aligned} \{\text{all functions}\} &\supset C^0 := \{\text{continuous functions}\} \supset \{\text{differentiable functions}\} \\ &\supset C^1 := \{\text{differentiable functions with its derivative continuous}\} \\ &\supset \{2^{\text{nd}} \text{ order differentiable functions}\} \\ &\supset C^2 := \{2^{\text{nd}} \text{ order differentiable functions with its second derivative continuous}\} \\ &\supset ... \end{aligned}$$

The examples above give us the reason why each set is a proper subset of the bigger one.

3 Secant Line and Mean Value Theorem

Proposition 18. *Consider the following three points $a-h$, a, $a+h$. Call L_1 be the secant line passing through $(a-h, f(a-h))$ and $(a, f(a))$. Call L_2 be the secant line passing through $(a+h, f(a+h))$ and $(a, f(a))$. And let L_3 be the secant line passing through $(a-h, f(a-h))$ and $(a+h, f(a+h))$. Then the average of the slopes of L_1 and L_2 are the slope of L_3.*

Proof. Slope of L_1 is $\frac{f(a-h)-f(a)}{-h}$; slope of L_2 is $\frac{f(a+h)-f(a)}{h}$; slope of L_3 is $\frac{f(a+h)-f(a-h)}{(a+h)-(a-h)}$. And the average will be

$$\frac{\frac{f(a-h)-f(a)}{-h}+\frac{f(a+h)-f(a)}{h}}{2}=\frac{\frac{-f(a-h)+f(a)}{h}+\frac{f(a+h)-f(a)}{h}}{2}=\frac{f(a+h)-f(a-h)}{2h}$$

□

Proposition 19. $f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a-h)}{(a+h)-(a-h)}=\lim_{h\to 0}\frac{f(a+h)-f(a-h)}{2h}$.

Proof. By definition, $f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{f(a-h)-f(a)}{-h}$. Hence, by the proposition above,

$$\begin{aligned} f'(a) &= \frac{1}{2}\lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{h}+\frac{f(a-h)-f(a)}{-h}\right) \\ &= \lim_{h\to 0}\left(\frac{\frac{f(a+h)-f(a)}{h}+\frac{f(a-h)-f(a)}{-h}}{2}\right) \\ &= \lim_{h\to 0}\frac{f(a+h)-f(a-h)}{(a+h)-(a-h)} \end{aligned}$$

□

Theorem 20 (MVT)**.** *Suppose f is a continuous function on the closed interval $[a,b]$ and differentiable on (a,b). Then, there is a number c in (a,b) such that $f'(c)=\frac{f(b)-f(a)}{b-a}$.*

Proof. [1], Sec 4.2. □

Remark 21. The proposition above gives you the sense why MVT should be correct. To compute the derivative, you can take the secant line only passing through neighborhood points and take the limit. And MVT tells you more: when you consider such secant line passing through two points, it is actually the deravitive at some points between them.

4 Vertical Tangent Line/ Vertical Cusp

Definition (Vertical Tangent Line)**.** *We say a function f has a vertical tangent line at a if f is continuous at a and $\lim_{x\to a}|f'(x)|=\infty$.*

Example 22. $\sqrt[3]{x}=x^{\frac{1}{3}}$ has a vertical tangent line at 0.

Example 23. $f(x)=\begin{cases}\sqrt{x} & \text{if } x\geq 0\\ \sqrt{-x} & \text{if } x<0\end{cases}$ has a vertical tangent line/vertical cusp at 0.

We can show the following is equivalent to the definition of vertical tangent line: we say a function f has a vertical tangent line at a if f is continuous at a and $\lim_{x\to a}|\frac{f(x)-f(a)}{x-a}|=\infty$. This definition may give you more sense why the tangent line is "vertical."

Proposition 24. *Suppose f is continuous at least in a neighborhood of a, and f is differentiable except at a. If both limits $\lim_{x\to a} |f'(x)|$, $\lim_{x\to a} |\frac{f(x)-f(a)}{x-a}|$ exist in general sense, then they get the same value.*

Proof. Try to argue like what we did in Remark 17. $\square$

References

[1] James Stewart, *Single Variable Calculus Early Transcendentals, Seventh Edition*