I know this time’s homework is hard ... So I’ll do as much detail as I can.

**Theorem 1** (Corollary 7). If \( f'(x) = g'(x) \) in \((a, b)\), then \( f - g \) is a constant function on \((a, b)\).

**Proof.** Call \( h(x) = f(x) - g(x) \). Then, we try to show \( h(x) \) is a constant function on \((a, b)\). First see \( h'(x) = f'(x) - g'(x) = 0 \). Given any \( x_1, x_2 \) with \( a < x_1 < x_2 < b \). By MVT, we have \( \frac{h(x_2) - h(x_1)}{x_2 - x_1} = h'(c) \) for some \( c \in (x_1, x_2) \). Hence, \( h(x_2) - h(x_1) = h'(c)(x_2 - x_1) = 0 \Rightarrow h(x_2) = h(x_1) \).

**Remark 2.** This is an important theorem because it just tells us the derivative can determine its original function up to a constant.

**Theorem 3.** Suppose \( f \) is continuous on \([a, b]\) and differentiable on \((a, b)\). Then \( f \) is (strictly) increasing on \([a, b]\) if and only if \( f' > 0 \) on \((a, b)\).

**Proof.** We try to show when \( b \geq x > y \geq a \), it implies \( f(x) > f(y) \). Consider \( \frac{f(x) - f(y)}{x - y} \), by MVT, there exists some \( c \in (y, x) \) such that \( \frac{f(x) - f(y)}{x - y} = f'(c) \), which is greater than 0. Therefore, as \( x - y > 0 \), we have \( f(x) - f(y) > 0 \).

**Remark 4.** The reverse statement is not true. For example, \( f(x) = x^3 \) is a strictly increasing function with its derivative 0 at \( x = 0 \). One can modify this statement by the following one: \( f \) is a non-decreasing function on \([a, b]\) if and only if \( f' \geq 0 \) on \((a, b)\).

23. \( f(1) = 10, \ f'(x) \geq 2 \). You may try to guess if \( f \) is a line with slope 2, then \( f(4) = f(1) + 2 \cdot (4 - 1) = 16 \). To complete the argument, we need to show \( f(4) \geq 16 \). If not, that is, \( f(4) < 16 \), then by MVT, there exists some \( c \) satisfying \( 1 < c < 4 \) such that \( f'(c) = \frac{f(4) - f(1)}{4 - 1} < \frac{16 - 10}{3} = 2 \), contradicting to the condition that \( f'(x) \geq 2 \) for \( 1 \leq x \leq 4 \).

27. (Sol 1) Consider \( f(x) = \sqrt{1 + x} \). Then, \( \frac{f(x) - f(0)}{x - 0} = \frac{\sqrt{1 + x} - 1}{x} \), by MVT, it also equals to \( f'(c) = \frac{1}{2\sqrt{1 + c}} \) for some \( c \) between 0 and \( x \). Therefore, \( \frac{\sqrt{1 + x} - 1}{x} = \frac{1}{2\sqrt{1 + x}} < \frac{1}{2} \) (because \( c > 0 \Rightarrow 1 + c > 1 \Rightarrow \sqrt{1 + c} > 1 \Rightarrow \frac{1}{\sqrt{1 + c}} < 1 \)). Hence, \( \frac{\sqrt{1 + x} - 1}{x} < \frac{1}{2} \Rightarrow 1 + x < 1 + \frac{1}{2} \). (Sol 2) Consider \( g(x) = 1 + \frac{1}{2} x - \sqrt{1 + x} \). \( g(0) = 0 \). \( g'(x) = \frac{1}{2} - \frac{1}{2\sqrt{1 + x}} > 0 \) for \( x > 0 \). So, \( g(x) \) is an increasing function on \([0, \infty)\) and this tells us \( g(x) > 0 \) if \( x > 0 \), i.e., \( 1 + \frac{1}{2} x > \sqrt{1 + x} \) if \( x > 0 \).

31. Clearly, \( f'(x) = g'(x) = -x^{-2} \) in their domains. We cannot use Corollary 7 because the domain is not an open interval. Actually, it is an union of two open intervals \((0, \infty), (-\infty, 0)\). So \( f(x) - g(x) \) is constant when restricted to one of them. In this situation, we usually call \( f(x) - g(x) \) is a locally constant function.

33. Call \( f(x) = \sin^{-1} \frac{x}{x + 1} - 2 \tan^{-1} \frac{x}{2} \). \( f'(x) = \frac{1}{\sqrt{1 - \left(\frac{x}{x + 1}\right)^2}} \cdot \left(\frac{2}{x + 1}\right) - 2 \cdot \frac{1}{2x} \cdot \frac{1}{(x + 1)\sqrt{x}} = \frac{1}{(x + 1)\sqrt{x}} - \frac{1}{(x + 1)\sqrt{x}} = 0 \). By Corollary 7, \( f(x) \)
35. Consider \( f(t) = g(t) - h(t) \). Suppose the two runners start a race at time \( t_1 \) and finish at time \( t_2 \). Then we have \( f(t_1) = g(t_1) - h(t_1) = 0 \) and \( f(t_2) = g(t_2) - h(t_2) = 0 \). By MVT, there exists some \( c \in (t_1, t_2) \) such that \( f'(c) = \frac{f(t_2) - f(t_1)}{t_2 - t_1} = 0 \). Therefore, \( f'(c) = g'(c) - h'(c) = 0 \Rightarrow g'(c) = h'(c) \). That is, at time \( c \), they have the same speed.

41. (a) \( C'(x) = \frac{4}{3}x^{1/3} + \frac{4}{3}x^{-2/3} = \frac{4}{3x^{2/3}}(x + 1) \). \( C'(x) = 0 \) for \( x = -1 \) and \( C'(x) \) DNE for \( x = 0 \). \( C''(x) > 0 \) for \( x > 0 \); \( C''(x) > 0 \) for \( x > -1 \); \( C''(x) < 0 \) for \( x < -1 \). Hence the function is increasing on the interval \((-1, 0) \cup (0, \infty) = (-1, \infty)\) and decreasing on the interval \((-\infty, -1)\).

(b) Local extremal point can only occur at 0 and \(-1\). But 0 does not satisfy the first derivative test, i.e., both sides have the same sign. So this function can only have \(-1\) as its local max.

(c) \( C''(x) = \frac{4}{3}x^{-2/3} - \frac{8}{9}x^{-5/3} = \frac{4}{9x^{5/3}}(x - 2) \). \( C''(x) = 0 \) for \( x = 2 \) and \( C''(x) \) DNE for \( x = 0 \). \( C'''(x) > 0 \) for \( x > 2 \); \( C'''(x) < 0 \) for \( 2 > x > 0 \); \( C'''(x) > 0 \) for \( x < 0 \). Hence the function is concave upward on the intervals \((2, \infty)\) and \((-\infty, 0)\). And it's concave downward on the interval \((0, 2)\). This means 0 and 2 are both inflection points.

61. (a) Concave upward and then concave downward; always increases.

(b) \( t \approx 8 \) hours.

(c) \((0, 8)\) concave upward; \((8, 18)\) concave downward.

(d) \((8, 350)\).

65. \( S(t) = Atp^{-k} \).
\[
S'(t) = Atp^{p-1}e^{-kt} - Aktpe^{-kt} = At^{p-1}(p - kt)e^{-kt}.
\]
\[
S''(t) = (p - 1)t^{p-2}(p - kt)e^{-kt} + At^{p-1}(-ke^{-kt} + At^{p-1}(p - kt)(-k)e^{-kt}
= At^{p-2}e^{-kt}[p - 1)(p - kt) - kt - (p - kt)kt]
= At^{p-2}e^{-kt}[k^2t^2 - 2pkt + p^2 - p]
\]

\( A = 0.01; \ p = 4; \ k = 0.07 \). Hence, \( S''(t) = At^2e^{-0.07t}[0.0049t^2 - 0.56t + 12] = 0 \) if and only if \( t = 0 \) or \( t = \frac{0.56 \pm \sqrt{0.56^2 - 4 \cdot 0.0049 \cdot 12}}{2 \cdot 0.0049} \approx 28.57, 85.71 \). \( S''(t) > 0 \) on the intervals \((85.71, \infty)\) and \((0, 28.57)\). \( S''(t) < 0 \) on the interval \((28.57, 85.71)\). \( t \approx 28.57 \), rate of increase is the greatest (local maximum); \( t \approx 85.71 \), rate of decrease is the greatest (local minimum).
67. \( f(-2) = 3; f'(-2) = 0; f(1) = 0; f'(1) = 0 \). We obtain:

\[
\begin{align*}
-8a + 4b - 2c + d &= 3 \\
12a - 4b + c &= 0 \\
a + b + c + d &= 0 \\
3a + 2b + c &= 0
\end{align*}
\]

\[\Rightarrow \begin{align*}
-8a + 4b - 2(3a - 2b) + d &= 3 \\
12a - 4b + (3a - 2b) &= 0 \\
a + b + (3a - 2b) + d &= 0 \\
3a + 2b + (3a - 2b) &= 0
\end{align*}\]

\[\Rightarrow \begin{align*}
-2a + 8b + d &= 3 \\
9a - 6b &= 0 \\
-2a - b + d &= 0
\end{align*}\]

Therefore, \( a = \frac{2}{3}b, c = -3a - 2b = -4b \). And we have \( 3 - 0 = (-2a + 8b + d) - (-2a - b + d) = 9b \Rightarrow b = \frac{1}{3} \Rightarrow a = \frac{2}{3}b = \frac{2}{9} \Rightarrow c = -4b = \frac{-4}{3} \Rightarrow d = 2a + b = \frac{7}{9} \).

71. \( y' = \frac{1+x^2-(1+x)2x}{(1+x^2)^2} = -\frac{x^2-2x+1}{(1+x^2)^2}. \) \( y'' = \frac{(1+x^2)^2(-2x-2)-2(1+x^2)(2x-x^2-2x+1)}{(1+x^2)^3} = \frac{(1+x^2)(-2x-2)+4x(x^2+2x-1)}{(1+x^2)^3} \frac{2x^3+6x^2-6x-2}{(1+x^2)^3} = \frac{2(x-1)(x^2+4x+1)}{(1+x^2)^3}. \) Hence, \( y'' = 0 \) if and only if \( x = 1 \) or \( x = \frac{-1\pm\sqrt{3}}{2} = -2 \pm \sqrt{3} \). So the inflection points are \((1, 1), (-2 \pm \sqrt{3}, \frac{1-2\pm\sqrt{3}}{4}) = (-2 \pm \sqrt{3}, \frac{1\pm\sqrt{3}}{4})\).

The line passing through \((-2 \pm \sqrt{3}, \frac{1\pm\sqrt{3}}{4})\) has the slope \( \frac{1\pm\sqrt{3}}{-2\pm\sqrt{3}(-2-\sqrt{3})} = \frac{1}{4}. \) And the line passing through \((1, 1)\) and \((-2 + \sqrt{3}, \frac{1-\sqrt{3}}{4})\) has the slope \( \frac{1\pm\sqrt{3}-1}{-2+\sqrt{3}-(-2-\sqrt{3})} = \frac{1}{4}. \) So these two lines must be the same line.

75. (a) It is sufficient to show \((fg)' > 0. \) \((fg)' = (f'g + fg)' = f''g + 2f'g' + fg''. \) Since \( f, g \) are positive, \( f > 0 \) and \( g > 0. \) Also because \( f, g \) are increasing, \( f' > 0 \) and \( g' > 0. \) Moreover, \( f, g \) are concave upward \( \Rightarrow f'' > 0 \) and \( g'' > 0. \) So, \((fg)'' > 0. \)

(b) If \( f \) and \( g \) are decreasing, then \( f' < 0 \) and \( g' < 0. \) But their product is still positive. So, \((fg)' \) is still positive.

(c) \( f = x^4, g_1 = \frac{1}{x^2}, g_2 = \frac{1}{x^2}, g_3 = \frac{1}{x^3} \) then \( fg_1, fg_2, fg_3 \) are concave upward, concave downward, linear, respectively. The argument doesn’t work because \( f'g' \) can be some large negative number such that \((fg)'' \) is no longer positive.

78. Since \( (e^x)' = e^x > 0, \) \( e^x \) is an increasing function.

(a) Consider \( f(x) = e^x - 1 - x. \) \( f(0) = 0. \) \( f'(x) = e^x - 1 > 0 \) for \( x > 0 \) because \( e^x \) is an increasing function. So, \( f \) is also an increasing function, that is, \( f(x) > f(0) = 0 \) for \( x > 0. \) Hence, \( e^x \geq 1 + x \) for \( x \geq 0. \)

(b) It suffices to prove (c).

(c) Claim: \( e^x > 1 + x + \cdots + \frac{x^n}{n!} \) for \( x > 0. \)

\( n = 1, \) it is proved in (a).

\( n = k, \) assume our claim is true.

\( n = k + 1, \) suppose \( g(x) = e^x - 1 - x - \cdots - \frac{x^{k+1}}{(k+1)!}. \) \( g(0) = 0. \) \( g'(x) = e^x - 1 - x - \cdots - \frac{x^k}{k!}, \) by assumption, it is positive. So \( g(x) \) is an increasing function, that
Theorem 5. Suppose $f$ is a differentiable function. $f$ is concave upward/downward if and only if $f'$ is increasing/decreasing.

The reason why we need this theorem is that we don’t know what concavity implies. What we only know is that $f'' > 0$ implies $f$ is concave upward. But the reverse statement is wrong. For example, $x^4$ is concave upward but its second derivative equals to 0 when $x = 0$.

To clarify the ideas, we have the following facts:

A. $f$ is differentiable. Then, $f$ is concave upward/downward if and only if $f'$ is increasing/decreasing.

B. $f$ is differentiable. Then, $f$ is (non-strictly) concave upward/downward if and only if $f'$ is (non-strictly) increasing/decreasing.

C. $f$ is twice differentiable. Then, $f$ is (non-strictly) concave upward/downward if and only if $f''$ is non-negative/non-positive. (this is simply followed from Remark 4)

Once you have these ideas, you can try to do this problem as follows:

If $c$ is an inflection point, then $f$ has different concavity in the two small intervals $(c-h, c)$ and $(c, c+h)$. Without loss of generality (abbreviated as WLOG afterwards), say $f$ is concave upward on $(c-h, c)$ and downward on $(c, c+h)$. Since $f$ is concave upward on $(c-h, c)$, it is (non-strictly) concave upward on $(c-h, c)$, by B., $f'$ is (non-strictly) increasing on $(c-h, c)$. Similarly, $f'$ is (non-strictly) decreasing on $(c, c+h)$.

So, $f'$ has a local maximum at $c$, and we already know $f''(c)$ exists, then by Fermat’s theorem, $f''(c) = 0$.

So the only thing I owe to you is how I prove the fact B, in fact, we only need the "only if" part:

WLOG, I only prove the case $f$ is (non-strictly) concave upward. Suppose $f$ is (non-strictly) concave upward. Consider $x < y < z$. By definition, we have $f(x) \geq f(y) + f'(y)(x-y)$, that is, $f'(y) \geq \frac{f(x) - f(y)}{x-y}$. Similarly, $f'(y) \leq \frac{f(z) - f(y)}{z-y}$. Hence, $\frac{f(x) - f(y)}{x-y} \leq \frac{f(z) - f(y)}{z-y}$. In general, if given $a < b < c < d$, then we have

$$\frac{f(b) - f(a)}{b-a} = \frac{f(a) - f(b)}{a-b} \leq \frac{f(c) - f(b)}{c-b} = \frac{f(b) - f(c)}{b-c} \leq \frac{f(d) - f(c)}{d-c}$$

Consider $f'(b) = \lim_{a \to b} \frac{f(b) - f(a)}{b-a}$ and $f'(c) = \lim_{d \to c} \frac{f(d) - f(c)}{d-c}$. Then $f'(b) \leq f'(c)$, i.e., $f'$ is (non-strictly) increasing.
84. Consider $f(x) = x^3$. Then it does not have local maximum and local minimum at 0. Because $f'''$ is continuous, as $f'''(c) > 0$, $f'''$ must be positive in a small neighborhood containing $c$. Then by 1/D test, we know $f''$ is increasing in this small neighborhood. With the fact that $f''(c) = 0$, $f'''(x)$ will be negative in the small neighborhood which is left to $c$ and be positive in the small neighborhood which is right to $c$. Then by concavity test, $f$ is concave upward on one side and concave downward on another side, i.e., $c$ is an inflection point.

87. Denote $f_k(x) = \begin{cases} x^4(k + \sin \frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$. Then $f(x) = f_0(x), g(x) = f_2(x), h(x) = f_{-2}(x)$.

(a) $f_k'(0) = \lim_{x \to 0} \frac{x^4(k + \sin \frac{1}{x}) - 0}{x - 0} = \lim_{x \to 0} x^3(k + \sin \frac{1}{x}) = 0$ by squeeze theorem. (∵ $x^3(k + \sin \frac{1}{x})$ is bounded by $x^3(k + 1)$ and $x^3(k - 1)$, you can use squeeze here) $f_k'(x) = 4x^3(k + \sin \frac{1}{x}) + x^4 \cos \frac{1}{x} \cdot -\frac{1}{x^2} = 4x^3(k + \sin \frac{1}{x}) - x^2 \cos \frac{1}{x}$ for $x \neq 0$. Now consider we have a sequence $x_n = \frac{1}{n\pi}$, where $n \in \mathbb{Z}$. Then,

$$f_k'(x_n) = 4x_n^3(k + \sin \frac{1}{x_n}) - x_n^2 \cos \frac{1}{x_n} = 4 \left( \frac{1}{n\pi} \right)^3 (k + \sin n\pi) - \left( \frac{1}{n\pi} \right)^2 \cos n\pi$$

$$= -\frac{1}{n^2\pi^2} (-1)^n = (-1)^{n+1} \frac{1}{n^2\pi^2}$$

which is positive when $n$ is odd and negative when $n$ is even. Hence the derivatives indeed change sign infinitely often on both sides of 0.

(b) $f_k(x)$ is bounded by $(k + 1)x^4$ and $(k - 1)x^4$ and keeps oscillating between them. So for $k = 0$, $f(x)$ have nor a local maximum nor local minimum at 0. But for $k = 2/-2$, $f_k(x)$ is positive/negative when $x$ is small enough, therefore $f_k(x)$ is smaller/bigger than $f_k(0)$ when $x$ is small enough, i.e., $f_k(x)$ has a local max/min.