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First think about the following nonsense proof:
Exercise. Prove that every integer is equal to 0.

Proof. Suppose $1 = 0$. Then $x = x \cdot 1 = x \cdot 0 = 0$. $\square$

The mistake of this proof is USING THE DESIRED RESULT AS OUR FIRST ASSUMPTION / USING THE WRONG ASSUMPTION. Although this exercise is hilarious, but this is the mistake that I made in our class. Let's look over the proof that I have done yesterday:
Exercise. Translate $\lim_{x \to 0} \frac{2x+1}{x+1} = 1$ and show that it is true.

Proof. <u>Translation</u>: For every $\epsilon > 0$, there exists $\delta > 0$ such that $|x - 0| < \delta$ implies $|\frac{2x+1}{x+1} - 1| < \epsilon$.
<u>Goal</u>: Find a δ such that $|x - 0| < \delta \Rightarrow |\frac{2x+1}{x+1} - 1| < \epsilon$.
First simplify $|\frac{2x+1}{x+1} - 1| < \epsilon \Leftrightarrow |\frac{2x+1}{x+1} - \frac{x+1}{x+1}| < \epsilon \Leftrightarrow |\frac{x}{x+1}| < \epsilon \Leftrightarrow |x| < |x+1|\epsilon$.
Guess $\delta = 1$. Then, $|x| < 1 \Leftrightarrow -1 < x < 1 \Leftrightarrow 0 < x + 1 < 2$.
So, $|x| < |x+1|\epsilon < 2\epsilon$.
δ should be chosen as $\min\{1, 2\epsilon\}$. $\square$

See the red line part. This is exactly how the mistake happens: you use the desired result $|x| < |x+1|\epsilon$ to deduce $|x| < 2\epsilon$! So, you should try to modify the proof as follows:

Proof. <u>Translation</u>: For every $\epsilon > 0$, there exists $\delta > 0$ such that $|x - 0| < \delta$ implies $|\frac{2x+1}{x+1} - 1| < \epsilon$.
<u>Goal</u>: Find a δ such that $|x - 0| < \delta \Rightarrow |\frac{2x+1}{x+1} - 1| < \epsilon$.
<u>Rewrite result</u>: $|\frac{2x+1}{x+1} - 1| < \epsilon \Leftrightarrow |\frac{2x+1}{x+1} - \frac{x+1}{x+1}| < \epsilon \Leftrightarrow |\frac{x}{x+1}| < \epsilon$.
<u>Assumption 1</u>: Assume $|x| < \frac{1}{2}$. Then, $|x| < \frac{1}{2} \Leftrightarrow -\frac{1}{2} < x < \frac{1}{2} \Leftrightarrow \frac{1}{2} < x + 1 < \frac{3}{2}$.
<u>Assumption 2</u>: Since we want $|\frac{x}{x+1}| < \epsilon$, under the above assumption we already know $|x+1| < \frac{3}{2}$, DO ANOTHER ASSUMPTION $|x| < \epsilon \cdot \frac{3}{2}$. Then, we can prove what we want:
<u>Argument</u>: If $|x| < \frac{1}{2}$ and $|x| < \epsilon \cdot \frac{3}{2}$, then we have $|\frac{x}{x+1}| < \epsilon \cdot \frac{3}{2} \cdot \frac{3}{2} = \epsilon$. Exactly what we want!
<u>Conclusion</u>: Hence, δ should be chosen as $\min\{\frac{1}{2}, \frac{3}{2}\epsilon\}$. $\square$

Let's do another exercise using the correct argument as shown above:
Exercise. Translate $\lim_{x \to a} \sqrt{x} = \sqrt{a}$ and show that it is true.

Proof. <u>Translation</u>: For every $\epsilon > 0$, there exists $\delta > 0$ such that $|x - a| < \delta$ implies $|\sqrt{x} - \sqrt{a}| < \epsilon$.
<u>Goal</u>: Find a δ such that $|x - a| < \delta \Rightarrow |\sqrt{x} - \sqrt{a}| < \epsilon$.
<u>Rewrite result</u>: $|\sqrt{x} - \sqrt{a}| < \epsilon \Leftrightarrow \frac{|x-a|}{\sqrt{x}+\sqrt{a}} < \epsilon$.
<u>Assumption 1</u>: Assume $|x - a| < \frac{a}{2}$. Then, $|x - a| < \frac{a}{2} \Leftrightarrow \frac{a}{2} < x < \frac{3a}{2} \Leftrightarrow \sqrt{\frac{a}{2}} < \sqrt{x} < \sqrt{\frac{3a}{2}} \Leftrightarrow \sqrt{\frac{a}{2}} + \sqrt{a} < \sqrt{x} + \sqrt{a} < \sqrt{\frac{3a}{2}} + \sqrt{a} \Leftrightarrow \frac{1}{\sqrt{\frac{a}{2}}+\sqrt{a}} > \frac{1}{\sqrt{x}+\sqrt{a}} < \frac{1}{\sqrt{\frac{3a}{2}}+\sqrt{a}}$.
<u>Assumption 2</u>: Since we want $\frac{|x-a|}{\sqrt{x}+\sqrt{a}} < \epsilon$, under the above assumption we already know $\frac{1}{\sqrt{x}+\sqrt{a}} < \frac{1}{\sqrt{\frac{a}{2}}+\sqrt{a}}$, DO ANOTHER ASSUMPTION $|x - a| < \epsilon \cdot (\sqrt{\frac{a}{2}} + \sqrt{a})$. Then, we can prove what we want:
<u>Argument</u>: If $|x - a| < \frac{a}{2}$ and $|x - a| < \epsilon \cdot (\sqrt{\frac{a}{2}} + \sqrt{a})$, then we have $\frac{|x-a|}{\sqrt{x}+\sqrt{a}} < \epsilon \cdot (\sqrt{\frac{a}{2}} + \sqrt{a}) \cdot \frac{1}{\sqrt{\frac{a}{2}}+\sqrt{a}} = \epsilon$. Exactly what we want!
<u>Conclusion</u>: Hence, δ should be chosen as $\min\{\frac{a}{2}, (\sqrt{\frac{a}{2}} + \sqrt{a})\epsilon\}$. $\square$

Remark. Actually the correct translation is: For every $\epsilon > 0$, there exists $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|\sqrt{x} - \sqrt{a}| < \epsilon$. Why doesn't this matter our proof?