Midterm Examination

MATH 110 (Summer 2016)

July 14, 2016

8:00 - 9:00AM

Name: ____________________

SID: ____________________

Please put away all books, calculators, electronic games, cell phones, pagers, .mp3 players, and other electronic devices. You may refer to a single 1-sided cheat sheet. Explain your answers in full English sentence as is customary and appropriate. Whenever you try to use some theorem, it would be better if you write down the statement of the theorem you are going to use. Otherwise, if the theorem is applied in some extremely implicit way, I would treat as no explanation. Your paper is your ambassador when it is graded.

Remember it is not necessary to do the extra problem. Although the total score is the sum of the points you get from each problem, you will get the full score for this midterm if you get full points for all the first four problems. Even if your total score is higher than 100, you still only get 100.

Problem	Your Score	Total Points
1		30 points
2		30 points
3		20 points
4		20 points
Extra		15 points
Total		100 points

Good luck!

Problem 1 (30 points) Consider $L \in \mathcal{L}(\mathcal{P}_2(\mathbb{C}), \mathbb{C}^2)$ defined by $p \mapsto (p(1), p(0))$.

(1) (10 points) Find a basis B of $\mathcal{P}_2(\mathbb{C})$ such that

$$\mathcal{M}(L, B, \mathrm{std}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

where std is the standard basis of $\mathbb{C}^2$.

(2) (10 points) Describe all the linear functionals in the dual basis of B.

(3) (10 points) Find a basis C of $\mathcal{P}_2(\mathbb{C})/\operatorname{Null} L$ such that

$$\mathcal{M}(L/\operatorname{Null} L, C, \mathrm{std}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

where std is the standard basis of $\mathbb{C}^2$. (Recall that if a subspace U is invariant under T, then T/U sends $v + U$ to $Tv + U$.)

Problem 2 (30 points) Let's define $\mathcal{P}_{1,1}(\mathbb{R})$ be the set of polynomials in two variables x, y with coefficients in $\mathbb{R}$ such that the degree of each monomial is at most 1. That is, $\mathcal{P}_{1,1}(\mathbb{R}) = \{a + bx + cy | a, b, c \in \mathbb{R}\}$. It can be equipped with addition and scalar multiplication in the usual way. ($f(x, y), g(x, y) \in \mathcal{P}_{1,1}(\mathbb{R})$, $\alpha \in \mathbb{R}$, then $(f + g)(x, y) := f(x, y) + g(x, y)$ and $(\alpha f)(x, y) := \alpha f(x, y)$.)

(1) (6 points) Show that $\mathcal{P}_{1,1}(\mathbb{R})$ is a vector space over $\mathbb{R}$.

(2) (6 points) Find out $\dim \mathcal{P}_{1,1}(\mathbb{R})$.

(3) (6 points) Consider $T \in \mathcal{L}(\mathcal{P}_1(\mathbb{R}) \times \mathbb{R}, \mathcal{P}_{1,1}(\mathbb{R}))$ defined by $(a + bx, c) \mapsto 2a + (a + b)x + 3cy$ and $S \in \mathcal{L}(\mathcal{P}_{1,1}(\mathbb{R}), \mathcal{P}_1(\mathbb{R}) \times \mathbb{R})$ defined by $a + bx + cy \mapsto (a + bx, c)$. Find out $\det(TS)$ and $\operatorname{Tr}(TS)$.

(4) (6 points) Describe $(\operatorname{Null}(TS))^0$, $(\operatorname{Range}(TS))^0$ and find out $\operatorname{Rank}(TS)$.

(5) (6 points) Find out all the eigenvalues and the corresponding eigenvectors of TS.

Problem 3 (20 points) Prove or find a counterexample for each following statement:

(1) (10 points) Suppose $V = U + W$ and B_U, B_W are bases of U, W, respectively. Then $B_U \cup B_W$ is a basis of V.

(2) (10 points) Suppose U and W are subspaces of V with $U \subset W$. (V can be infinite-dimensional.) Then $W^0 \subset U^0$.

Problem 4 (20 points) Suppose $p \in \mathcal{P}(\mathbb{R})$. Prove that there exists a polynomial $q \in \mathcal{P}(\mathbb{R})$ such that $5q'' + 3q' = p$, where q' is the derivative of q and q'' is the second derivative of q.