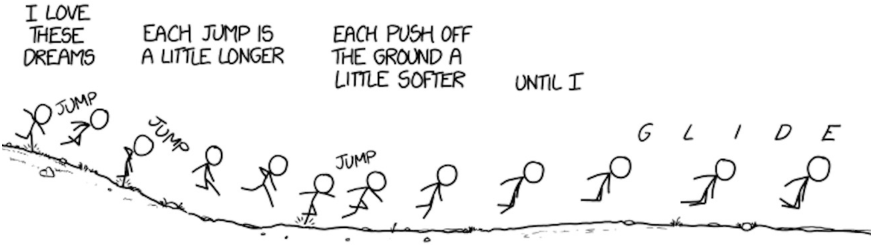
The doubly asymmetric simple exclusion process

Yuhan Jiang

Harvard



Based on arXiv: 2312.09427

Outline

- Background on the asymmetric simple exclusion process (ASEP) and multispecies ASEP
- The doubly ASEP (DASEP) and main results
- A combinatorial phenomenon called *homomesy*
- Lumping from the DASEP to *the colored Boolean process*
- Lumping from the colored Boolean process to *the restricted random growth model*
- A special case; combinatorics of (partial) matchings on graphs

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The asymmetric simple exclusion process (ASEP)

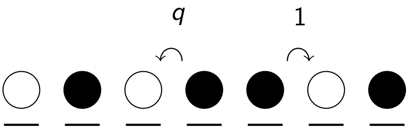

Figure: This state will be represented by the binary word 0101101

- A Markov chain for particles hopping on a 1d lattice
- Introduced independently in biology by (Macdonald-Gibbs-Pipkin 1968), and in mathematics by (Spitzer 1970)
- Many variations: multispecies, open boundary, half space, totally asymmetric, partially asymmetric ...
- Related to KPZ equation (Corwin-Shen-Tsai 2017), matrix ansatz (Derrida-Evans-Hakim-Pasquier 1993), multiline queues (Ferrari-Martin 2007), Macdonald polynomials (Corteel-Mandelshtam-Williams 2018) ...

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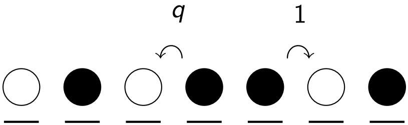

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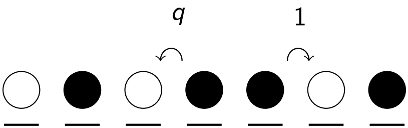

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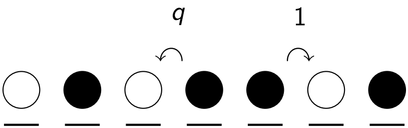

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The multispecies ASEP (mASEP)

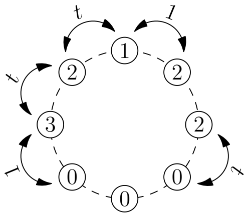

Figure: A state of ASEP$(3, 2, 2, 2, 1, 0, 0, 0)$. This state can be represented by the word 12200032.

- For constant $0 \leq t \leq 1$, a pair of neighboring particles i, j exchange with rate $\frac{t}{n}$ if $i > j$ and with rate $\frac{1}{n}$ if $i < j$
- Irreducible $\implies$ there exists a unique stationary distribution
- Order the labels into a partition λ. The set of states are the permutations of parts of λ, denoted $S_n(\lambda)$

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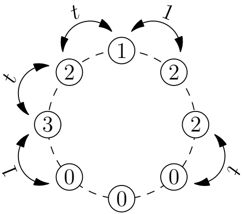

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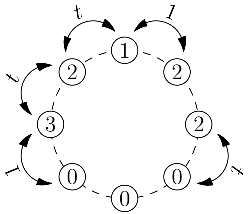

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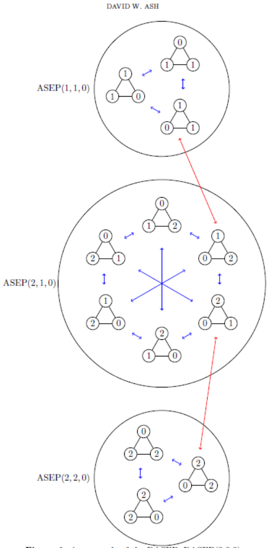

Figure 1: An example of the DASEP: DASEP(3,2,2)

Definition (Ash 2023)

- $n = \#$ sites, $p = \#$ allowed species, and $q = \#$ particles
- mASEP with exchange rates $\frac{t}{3n}$ or $\frac{1}{3n}$ and *particles can spontaneously change their species*
- A particle's species can increase with rate $\frac{u}{3n}$, and decrease with rate $\frac{1}{3n}$
- $\text{parts}^+(\lambda) = \#$ positive parts of λ. The DASEP(n, p, q) is a Markov chain on

$$\Gamma_n^{p,q} = \bigcup_{\substack{\lambda_1 \le p, \\ \text{parts}^+(\lambda) = q}} S_n(\lambda).$$

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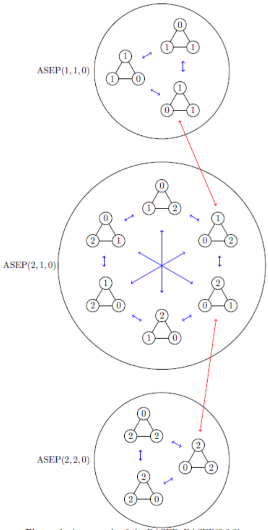
DAVID W. ASH



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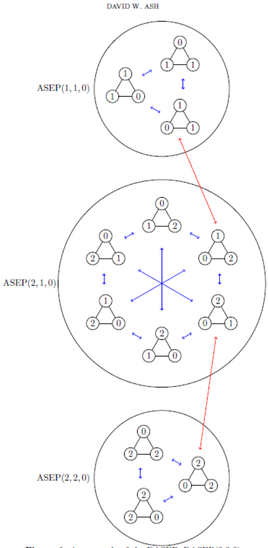

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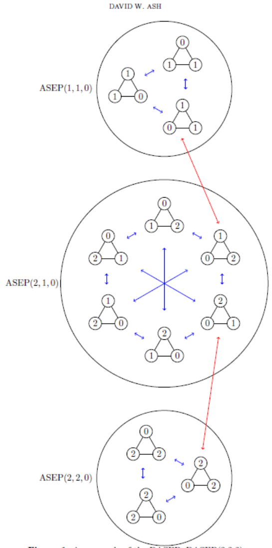

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One-line notation DASEP(3,2,2)

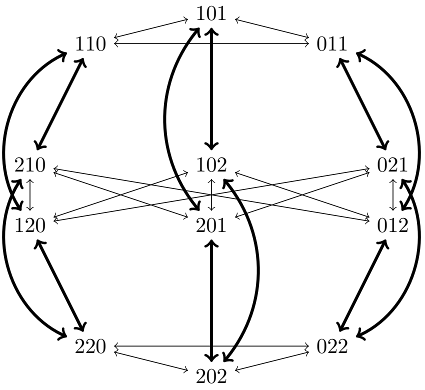

Figure: The state diagram of DASEP(3, 2, 2). Bold arrows represent the changes of species. There is an inherent cyclic symmetry such that the state 102, 021, 210 are essentially the same.

Overview of main results

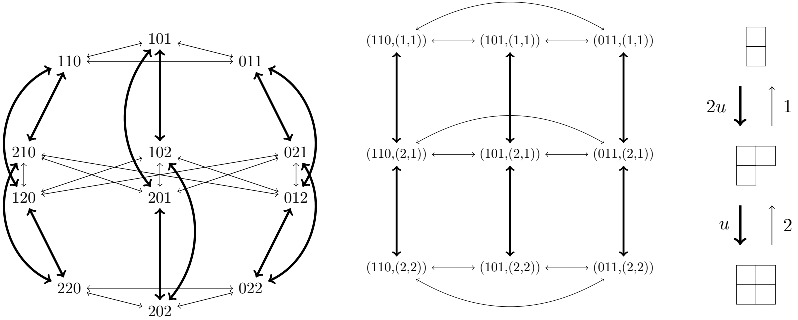

Figure: With the notion of *lumping*, a projection of Markov chains, we will turn the pictures on the left, to the middle, then to the right, which is a Markov chain on a set of Young diagrams. Our results generalize Theorem 5.2 in (Ash 2023).

Notation

- The stationary distribution consists of rational functions in t, u. Clear the denominators and denote the unnormalized steady state probability of μ by $\pi_{\mathrm{DASEP}}(\mu)$
- For each partition λ, let $m_i = m_i(\lambda)$ be the number of parts of λ that equal i. We can also write $\lambda = \langle 1^{m_1} 2^{m_2} \cdots \rangle$.
- For any binary word w, define $S_n^w(\lambda)$ as the set of all permutations of λ whose supports are aligned with w (positions of 0's are the same).

Example

- $S_3^{011}(2,1,0) = \{012, 021\}$
- $S_3^{011}(2,2,0) = \{022\}$

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The unnormalized steady state probabilities of DASEP(3, 2, 2)

μ	$\pi_{\text{DASEP}}(\mu)$
011	$u + 3t + 4$
012	$u(u + 4t + 3)$
021	$u(u + 2t + 5)$
022	$u^2(u + 3t + 4)$

Because of cyclic symmetry, we know the steady state probabilities of all states. Observation:

- $\pi_{\text{DASEP}}(012) + \pi_{\text{DASEP}}(021 = 2u\pi_{\text{DASEP}}(011)$
- $\pi_{\text{DASEP}}(022) = u^2\pi_{\text{DASEP}}(011)$

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Observation:

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Results

Theorem (J.)

Consider DASEP(n, p, q) *for any positive integers* n, p, q *with* $n > q$.

1. *For any two binary words* w, w' *with* q *ones and* $(n-q)$ *zeros, we have* $\pi_{\text{DASEP}}(w) = \pi_{\text{DASEP}}(w')$.
2. *For any binary word* w *and partition* $\lambda = \langle 1^{m_1} 2^{m_2} \cdots p^{m_p} \rangle$ *such that* $m_1 + \cdots + m_p = q$, *we have*

$$\sum_{\mu \in S_n^w(\lambda)} \pi_{\text{DASEP}}(\mu) = u^{|\lambda|-q} |S_n^w(\lambda)| \pi_{\text{DASEP}}(w)$$

$$= u^{|\lambda|-q} \binom{q}{m_1, m_2, \ldots, m_p} \pi_{\text{DASEP}}(w).$$

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For DASEP(n,p,q) *and two partitions* λ,μ *with* $\lambda_1 \le p, \mu_1 \le p, \text{parts}^+(\lambda) = \text{parts}^+(\mu) = q$, *we have*

$$\frac{\sum_{\nu \in S_n(\lambda)} \pi_{\text{DASEP}}(\nu)}{\sum_{\nu \in S_n(\mu)} \pi_{\text{DASEP}}(\nu)} = \frac{|S_n(\lambda)| u^{|\lambda|}}{|S_n(\mu)| u^{|\mu|}}.$$

Remark

If $\lambda = \langle 1^{m_1} 2^{m_2} \cdots p^{m_p} \rangle$, *then* $|S_n(\lambda)| = \binom{n}{n-q, m_1, \ldots, m_p}$.

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Homomesy

Definition (Propp-Roby 2015)

Given a set S, an invertible map $\tau : S \to S$ such that each τ-orbit is finite, and a function (or "statistic") $f : S \to K$ for some field K of characteristic zero, we say the triple (S, τ, K) exhibits *homomesy* if there exists a constant $c \in K$ such that for every τ-orbit $O \subset S$

$$\frac{1}{\#O} \sum_{x \in O} f(x) = c.$$

In this situation we say that f is *homomesic* under the action of τ on S, or more specifically c-mesic.

The average value of the statistic is the same across all orbits.

Example

10-words	inversions
0011	0
0110	2
1100	4
1001	2
average	2

10-words	inversions
0101	1
1010	3
average	2

Table: Cyclic shift of binary words

−+++	0
-++++-	0
+++−	1
++−+	0
+−++	0

-+-++	0
+-++-	0
-++-+	0
++-+-	1
+-+-+	0

Table: Cyclic shift of sequence of a (-1) and b (+1). The Boolean statistic output 1 if all initial segments add up to be positive. The average of this statistic is $\frac{b-a}{a+b}$.

Homomesy on polynomials with group action

We may define a more general form of homomesy where f takes values in a polynomial ring over a field of characteristic zero, and consider orbits of a group action instead of a single map τ.

Then, if we take S to be $S_n(\lambda)$ with S_q acting on the nonzero parts, and f to be the statistic π_{DASEP}, our theorem shows that $(S_n(\lambda), S_q, \pi_{\text{DASEP}})$ exhibits homomesy in this more general sense.

If we take S to be the state space $\Gamma_n^{p,q}$ with S_n acting on the sites, then $(\Gamma_n^{p,q}, S_n, \pi_{\text{DASEP}})$ also exhibits homomesy in this more general sense.

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If we take S to be the state space $\Gamma_n^{p,q}$ with S_n acting on the sites, then $(\Gamma_n^{p,q}, S_n, \pi_{\mathrm{DASEP}})$ also exhibits homomesy in this more general sense.

Homomesy on polynomials with group action

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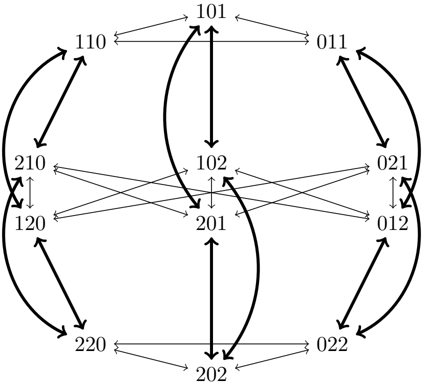
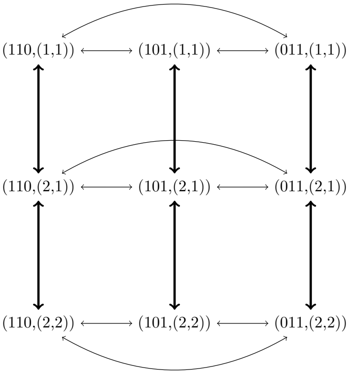
The colored Boolean process





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The *colored Boolean process* is a Markov chain on

$$\Omega_n^{p,q} = \{(w, \lambda) | w \in S_n(1^q 0^{n-q}), \lambda_1 \le p, \text{parts}^+(\lambda) = q\}$$

with transition probabilities:

$$(w, \dots \underbrace{i \cdots i}_{m_i} \dots) \xrightarrow{\frac{m_i}{3n}} (w, \dots \underbrace{i \cdots i}_{m_i - 1}, i-1 \dots) \qquad i > 1$$

$$(w, \dots \underbrace{i \cdots i}_{m_i} \dots) \xrightarrow{\frac{m_i u}{3n}} (w, \dots i+1, \underbrace{i \cdots i}_{m_i - 1} \dots) \qquad i < p$$

$$(\dots 01 \dots, \lambda) \xrightarrow{\frac{1}{3n}} (\dots 10 \dots, \lambda)$$

$$(\dots 10 \dots, \lambda) \xrightarrow{\frac{t}{3n}} (\dots 01 \dots, \lambda)$$

We denote the unnormalized steady state probabilities of $\Omega_n^{p,q}$ by π_{CBP}.

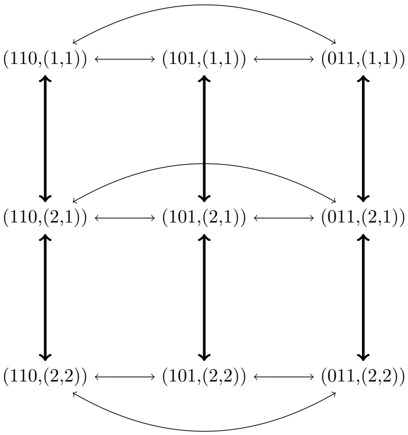

Figure: The state diagram of $\Omega_3^{2,2}$. For $n = 3, p = q = 2$, we have binary words $\{110, 101, 011\}$ and partitions $\{(1,1), (2,1), (2,2)\}$. The product of these two sets make $\Omega_3^{2,2}$.

Lumping

Definition (Kemeny, 1976)

Let $\{X_t\}$ be a Markov chain on state space Ω_X with transition matrix P, and let $f : \Omega_X \to \Omega_Y$ be a surjective map. Suppose there is an $|\Omega_Y| \times |\Omega_Y|$ matrix Q such that for all $y_0, y_1 \in \Omega_Y$, if $f(x_0) = y_0$, then

$$\sum_{x : f(x) = y_1} P(x_0, x) = Q(y_0, y_1).$$

Then $\{f(X_t)\}$ is a Markov chain on Ω_Y with transition matrix Q. We say that $\{f(X_t)\}$ is a *lumping* of $\{X_t\}$ and $\{X_t\}$ is a *lift* of $\{f(X_t)\}$.

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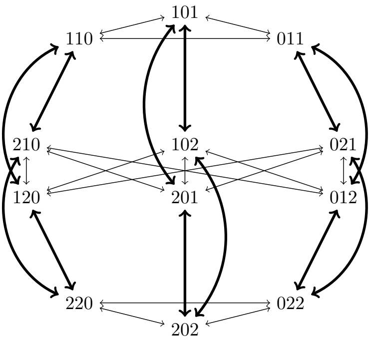
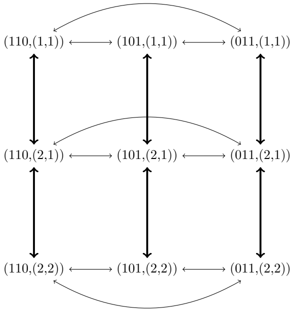
Lemma





Fix (w_0, λ_0) and (w_1, λ_1). For any $\mu_0 \in S_n^{w_0}(\lambda_0)$, the quantity

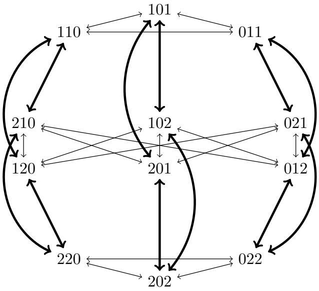
$$\sum_{\mu : f(\mu) \in S_n^{w_1}(\lambda_1)} P(\mu_0, \mu)$$

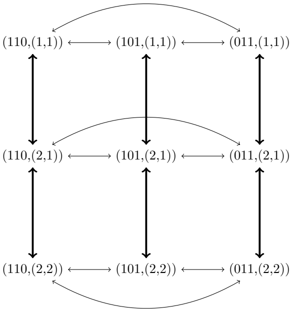
is independent of the choice of μ_0, and nonzero only in 4 cases.

DASEP lumps to the colored Boolean process

Theorem (J.)

The projection map $f : \Gamma_n^{p,q} \to \Omega_n^{p,q}$ *sending each* $\mu \in S_n^w(\lambda)$ *to* (w, λ) *is a lumping of* DASEP(n,p,q) *onto the colored Boolean process* $\Omega_n^{p,q}$.





The DASEP lumps to the colored Boolean process

Proposition (Kemeny, 1976)

Suppose p is a stationary distribution for $\{X_t\}$*, and let* π *be the measure on* Ω_Y *defined by* $\pi(y) = \sum_{x:f(x)=y} p(x)$*. Then* π *is a stationary distribution for* $\{f(X_t)\}$*.*

Corollary (J.)

The steady state probabilities of the colored Boolean process and the steady state probabilities of the DASEP are related as follows:

$$\pi_{\text{CBP}}(w, \lambda) \propto \sum_{\mu \in S_n^w(\lambda)} \pi_{\text{DASEP}}(\mu).$$

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The stationary distribution of the colored Boolean process

Theorem (J.)

Consider the colored Boolean process $\Omega_n^{p,q}$.

1. *The steady state probabilities of all binary words are equal, i.e., for any w, w' with q ones and $(n-q)$ zeros,*

$$\pi_{\mathrm{CBP}}(w, \langle 1^q 0^{n-q} \rangle) = \pi_{\mathrm{CBP}}(w', \langle 1^q 0^{n-q} \rangle).$$

2. *For an arbitrary state (w, λ), we have*

$$\pi_{\mathrm{CBP}}(w, \lambda) = u^{|\lambda|-q} \binom{q}{m_1, \ldots, m_p} \pi_{\mathrm{CBP}}(w, \langle 1^q 0^{n-q} \rangle).$$

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Example

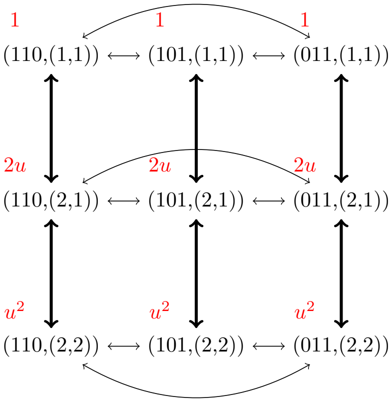
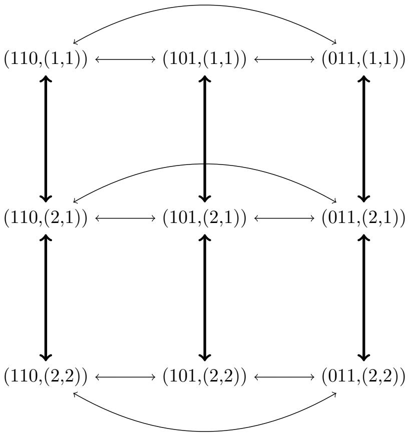
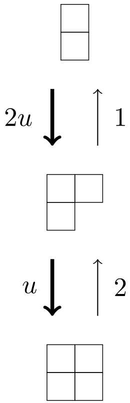


Figure: The unnormalized steady state probabilities of $\Omega_3^{2,2}$. The steady state probability of $(110,(2,1))$ is the same as that of $(101,(2,1))$ and $(011,(2,1))$, which is $u^{2+1-2}\binom{2}{1,1} = 2u$ times the steady state probability of $(110,(1,1))$.

A further lumping





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The *reverse lexicographic order* is a partial order on partitions such that $\nu \leq \lambda$ if either $\nu = \lambda$ or for some j,

$$\nu_1 = \lambda_1 \quad \dots \quad \nu_{j-1} = \lambda_{j-1} \qquad \nu_j < \lambda_j.$$

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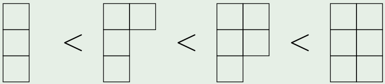

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- Let λ, ν be partitions, and let $m_i(\nu)$ denote the number of parts of ν equal to i.
- We say that λ *covers* ν *at* i if there exists a unique j such that $\lambda_j = \nu_j + 1 = i$ and for all $k \neq j$ we have $\lambda_k = \nu_k$, written $\lambda \gtrdot_i \nu$.
- We say that ν *is covered by* λ *at* i if there exists a unique j such that $\nu_j = \lambda_j - 1 = i$ and for all $k \neq j$ we have $\nu_k = \lambda_k$, written $\nu \lessdot_i \lambda$.
- In both cases, we have $m_i(\nu) = m_i(\lambda) + 1$

Example

$(1, 1, 1) \lessdot_1 (2, 1, 1)$, $(3, 2, 2) \gtrdot_2 (3, 2, 1)$, $(3, 2, 2) \gtrdot_3 (2, 2, 2)$
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The restricted random growth model

Definition (J.)

Define the *restricted random growth model* on

$$\chi^{p,q} = \{\lambda : \lambda_1 \leq p, \text{parts}^+(\lambda) = q\}$$

which includes all partitions that fit inside a $q \times p$ rectangle but do not fit inside a shorter rectangle, with transition probabilities $d^{(n)}_{\nu,\lambda}$:

- Let $m_i(\nu)$ denote the number of parts of ν equal to i. Transition probabilities $d^{(n)}_{\nu,\lambda}$ are:
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  - In all other cases where $\nu \neq \lambda$, $d^{(n)}_{\nu,\lambda} = 0$ and $d^{(n)}_{\lambda,\lambda} = 1 - \sum_{\nu:\nu\neq\lambda} d^{(n)}_{\nu,\lambda}$

We denote the unnormalized stationary distribution of $\chi^{p,q}$ by π_{RRG}.

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The doubly ASEP (DASEP)

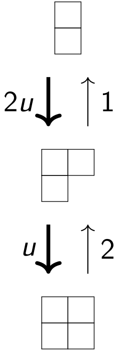

Figure: The state diagram of the *restricted random growth model* on $\chi^{2,2}$.

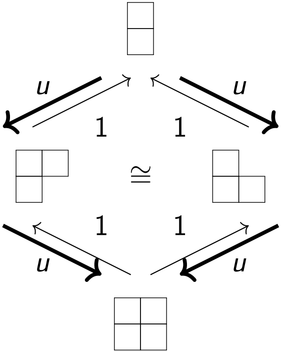

Figure: In this Markov chain, boxes are randomly added or removed from the right of any random row. It lumps to the restricted random growth model by rearranging parts in weakly decreasing order.

CBP lumps to the restricted random growth model

Theorem (J.)

The projection map on state spaces $g : \Omega_n^{p,q} \to \chi^{p,q}$ sending (w, λ) to λ (forgetting the positions of 0's) is a lumping of the colored Boolean process $\Omega_n^{p,q}$ to the restricted random growth model $\chi^{p,q}$.

Corollary (J.)

The unnormalized steady state probabilities of the restricted random growth model and the steady state probabilities of the colored Boolean process are related as follows:

$$\pi_{\mathrm{RRG}}(\lambda) \propto \sum_{w \in S_n(1^q 0^{n-q})} \pi_{\mathrm{CBP}}(w, \lambda).$$

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Stationarity of the restricted random growth model

Theorem (J.)

For DASEP(n, p, q) *and two partitions* λ, μ *with* $\lambda_1 \le p, \mu_1 \le p$ *and* $\text{parts}^+(\lambda) = \text{parts}^+(\mu) = q$, *we have*

$$\frac{\pi_{\text{RRG}}(\lambda)}{\pi_{\text{RRG}}(\mu)} = \frac{\sum_{w \in S_n(1^q 0^{n-q})} \pi_{\text{CBP}}(w, \lambda)}{\sum_{w \in S_n(1^q 0^{n-q})} \pi_{\text{CBP}}(w, \mu)}$$

$$= \frac{\sum_{\nu \in S_n(\lambda)} \pi_{\text{DASEP}}(\nu)}{\sum_{\nu \in S_n(\mu)} \pi_{\text{DASEP}}(\nu)} = \frac{|S_n(\lambda)| u^{|\lambda|}}{|S_n(\mu)| u^{|\mu|}}.$$

A close study of $p = q = 2$

- If there were only one species of particle, i.e. $p = 1$, the stationary distribution of DASEP$(n, 1, q)$ is uniform.
- If there were only one particle, i.e., $q = 1$, then the stationary distribution of DASEP$(n, p, 1)$ are given by powers of u, not involving t due to cyclic symmetry.
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The stationary distribution of DASEP(n,2,2)

The stationary distributions of DASEP$(n, 2, 2)$ are described by recurrence relations and differentiate the parity of n.

Let $(a_k)_{k\geq 0}$ and $(b_k)_{k\geq -1}$ be polynomial sequences in u, t satisfying the recurrence relation

$$a_k = (u + 2t + 3)a_{k-1} - (t + 1)^2 a_{k-2}$$
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with initial conditions $b_{-1} = 0, a_0 = b_0 = 1, a_1 = u + 3t + 4$.

Remark

If we set $u = t = 1$, the polynomial sequence a_k specializes to trinomial transform of Lucas number 8, 44, 232, 1216 ... and and b_k specializes to 6, 32, 168, 880 ... which is the binomial transform of the denominators of continued fraction convergents to $\sqrt{5}$.

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with initial conditions $b_{-1} = 0, a_0 = b_0 = 1, a_1 = u + 3t + 4$.

Remark

If we set $u = t = 1$, the polynomial sequence a_k specializes to trinomial transform of Lucas number 8, 44, 232, 1216 . . . and and b_k specializes to 6, 32, 168, 880 . . . which is the binomial transform of the denominators of continued fraction convergents to $\sqrt{5}$.

Fun facts about the two sequences

Consider matchings (subset of disjoint edges) of the cycle graph C_{2k+1} or line graph L_{2k+1} with odd number of vertices. Assign each matching M a weight of $(t+1)^{|M|}(u+1)^{k-|M|}$. Then a_k is the sum of weights over all matchings of a cycle, and b_k is that of the line. This can be seen via induction.

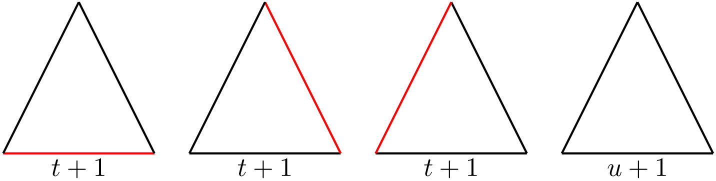


Figure: $a_1 = u + 3t + 4$

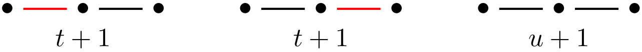


Figure: $b_1 = u + 2t + 3$

Fun facts about the two sequences

Consider matchings (subset of disjoint edges) of the cycle graph C_{2k+1} or line graph L_{2k+1} with odd number of vertices. Assign each matching M a weight of $(t+1)^{|M|}(u+1)^{k-|M|}$. Then a_k is the sum of weights over all matchings of a cycle, and b_k is that of the line. This can be seen via induction.

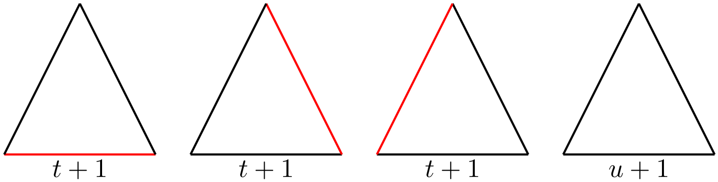


Figure: $a_1 = u + 3t + 4$

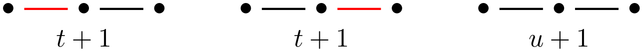


Figure: $b_1 = u + 2t + 3$

The stationary distribution of DASEP(n,2,2)

The unnormalized steady state probabilities of the infinite family DASEP$(n, 2, 2)$:

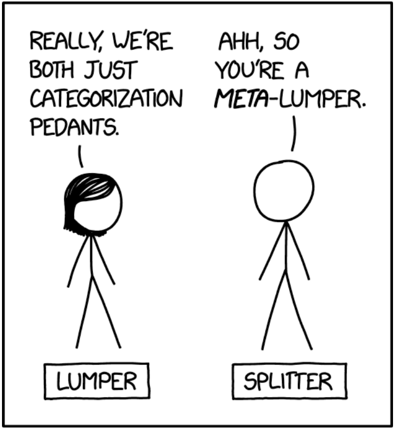
When $n = 2k + 1$ is odd, for $0 \leq m < k$,

μ	$\pi_{\text{DASEP}}(\mu)$
$S_n((1, 1, 0, \ldots, 0))$	a_k
$0 \ldots 010^m 20 \ldots 0$	$ua_k + u(t-1)(t+1)^m a_{k-m-1}$
$0 \ldots 020^m 10 \ldots 0$	$ua_k - u(t-1)(t+1)^m a_{k-m-1}$
$S_n((2, 2, 0, \ldots, 0))$	$u^2 a_k$

When $n = 2k + 2$ is even, for $0 \leq m \leq k$,

μ	$\pi_{\text{DASEP}}(\mu)$
$S_n((1, 1, 0, \ldots, 0))$	b_k
$0 \ldots 010^m 20 \ldots 0$	$ub_k + u(t-1)(t+1)^m b_{k-m-1}$
$0 \ldots 020^m 10 \ldots 0$	$ub_k - u(t-1)(t+1)^m b_{k-m-1}$
$S_n((2, 2, 0, \ldots, 0))$	$u^2 b_k$

Thank you!



References

- Jiang, The doubly asymmetric simple exclusion process, the colored Boolean process, and the restricted random growth model, arXiv:2312.09427 (2023).
- Ash, Introducing DASEP: the doubly asymmetric simple exclusion process, arXiv:2201.00040 (2023).