A BASIS OF THE ALTERNATING DIAGONAL COINVARIANTS

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ABSTRACT. We construct an explicit vector space basis in terms of bivariate Vandermonde determinants for the alternating component of the diagonal coinvariant ring DR_n, answering a question of Stump. As a Corollary, we recover the combinatorial formula of the q, t-Catalan numbers. Moreover, we construct a decomposition of an m-Dyck path into an m-tuple of Dyck paths such that the area sequence and bounce sequence of the m-Dyck path is entrywise the sum of the area sequences and bounce sequences of the Dyck paths in the tuple.

1. INTRODUCTION

The symmetric group S_n acts diagonally on $\mathbb{C}[\mathbf{x}, \mathbf{y}] := \mathbb{C}[x_1, y_1, \ldots, x_n, y_n]$. The *diagonal coinvariant ring* is defined to be

$$DR_n = \mathbb{C}[\mathbf{x}, \mathbf{y}]/\mathfrak{m}_+^{S_n}(\mathbf{x}, \mathbf{y}), \quad \mathfrak{m}_+^{S_n}(\mathbf{x}, \mathbf{y}) = \left\langle \sum_k x_k^i y_k^j : (i, j) \neq (0, 0) \right\rangle$$

the quotient space by the ideal generated by nonconstant diagonally symmetric polynomials. This space was studied by Haiman [Hai94] for $n!$-conjecture [Hai02], and its Frobenius character was studied as the shuffle conjecture [Mel21, CM18].

Let

$$\mathcal{A} = \langle f : (\sigma f) = \operatorname{sgn}(\sigma) f, \ \sigma \in S_n \rangle$$

be the ideal generated by all *alternating polynomials*. The q, t-Catalan numbers were first defined by Haiman as the *bigraded Hilbert series* of $\mathcal{A}$, that is, $\sum_{i,j \geq 0} \dim(\mathcal{A}_{i,j}) q^i t^j$ where $\mathcal{A}_{i,j}$ is the bihomogeneous component of $\mathcal{A}$ in bidegree (i, j), for which Haglund [Hag03] later gave combinatorial formulas in terms of Dyck path statistics.

In [Stu10], Stump asked for an explicit vector space basis of $\mathcal{A}$ given by a maximal linearly independent subset of the *bivariate Vandermonde determinants*

$$\Delta_X(\mathbf{x}, \mathbf{y}) := \det(x_i^{\alpha_j} y_i^{\beta_j})_{i,j=1}^n, \quad , X = ((\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)) \subseteq \mathbb{N}^2,$$

with coefficients in $\mathbb{C}[x, y]_+^{S_n}$. We address Stump's problem by constructing such a basis of $\mathcal{A}$ coming from antisymmetrizing the monomial basis of DR_n given by Carlsson and Oblomkov [CO25] in Section 2.

Our first main theorem construct a bais for the alternating component of the diagonal coinvariants in terms of bivariate Vandermonde determinants.

Theorem 1. *For any Dyck path π from $(0, 0)$ to (n, n), let $(a_1(\pi), \ldots, a_n(\pi))$ be the area sequence of π and let $(d_1(\pi), \ldots, d_n(\pi))$ be the dinv sequence (see Definition 1) of π. Let*

$$\Delta_\pi = \det(x_i^{d_j(\pi)} y_i^{a_j(\pi)}).$$

The set of bivariate Vandermonde determinants $\{\Delta_\pi\}$ over all Dyck paths π of semilength n form a vector space basis for $\mathcal{A}/\langle \mathbf{x}, \mathbf{y} \rangle \mathcal{A}$ in DR_n.

As $\deg_{\mathbf{x}}(\Delta_\pi) = \operatorname{dinv}(\pi)$ and $\deg_{\mathbf{y}}(\Delta_\pi) = \operatorname{area}(\pi)$, we recover the combinatorial formula for q,t-Catalan numbers given by [HL05].

Corollary 1. *The q,t-Catalan number is given by*

$$C_n(q,t) = \sum_{\pi \in L^+_{n,n}} t^{\operatorname{area}(\pi)} q^{\operatorname{dinv}(\pi)}.$$

Furthermore, Garsia and Haiman [GH96] defined the *space of generalized diagonal coinvariants* as

$$DR_n^{(m)} := (\mathcal{A}^{m-1}/\mathcal{A}^{m-1}\mathfrak{m}_+^{S_n}(\mathbf{x},\mathbf{y})) \otimes \epsilon^{\otimes(m-1)},$$

where ϵ is the 1-dimensional sign representation. The alternating component of $DR_n^{(m)}$ is

$$M^{(m)} \cong (\mathcal{A}^m/\langle \mathbf{x},\mathbf{y}\rangle \mathcal{A}^m) \otimes \epsilon^{\otimes(m-1)}.$$

The Frobenius character of this ring was studied as the rational shuffle conjecture. [GM13, Hik14, GNt15, BGSLX16, Mel21]. The *q,t-Fuß-Catalan numbers* were defined by Haiman [Hai94] as the bigraded Hilbert series of the alternating component of the space of generalized diagonal coinvariants.

In [Stu10], Stump further asked if one could obtain a minimal generating set of $\mathcal{A}^m$ as a $\mathbb{C}[\mathbf{x},\mathbf{y}]$-module, as their image in the quotient form a vector space basis for $M^{(m)}$ by the graded Nakayama's lemma. We construct a decomposition of rational Dyck paths and conjecture that such a basis is given by products of bivariate Vandermonde determinants coming from our decomposition in Conjecture 1, Section 3.

Acknowledgements

The author would like to thank Christian Stump for suggesting the problem. We thank Christian Stump, Alexei Oblomkov, Mike Zabrocki, François Bergeron, Sylvie Corteel, Lauren K. Williams, Nupur Jain, and Mitsuki Hanada for helpful discussions. We thank Alexei Oblomkov for his code. We thank Sylvie Corteel and Mitsuki Hanada for their comments on the writing.

2. q,t-Catalan combinatorics

We give a basis in terms of bivariate Vandermonde determinants for $\mathcal{A}/\langle \mathbf{x},\mathbf{y}\rangle\mathcal{A}$ labeled by Dyck paths.

2.1. Main theorem.

A Dyck path of semilength n is a lattice path from $(0,0)$ to (n,n) that stays above the diagonal $y = x$. We denote the set of Dyck paths of semilength n by $L^+_{n,n}$.

Let $\pi \in L^+_{n,n}$ be a Dyck path. Let (i,j) be a cell in the $n \times n$-square. We say that (i,j) is an area cell of π if $j > i$ and (i,j) lies below π. The *area sequence* of π is the tuple

$$\mathbf{area}(\pi) = (a_1(\pi), \ldots, a_n(\pi))$$

such that $a_i(\pi)$ is the number of area cells of π on the i-th row [Hag08]. We say that (i,j) is a *primary diagonal inversion* (or primary dinv pair) of π if $i < j$ and $a_i = a_j$; we call (i,j) a *secondary dinv pair* if $i < j$ and $a_i = a_j + 1$.

Definition 1. For a Dyck path $\pi \in L^+_{n,n}$, the *dinv sequence* of π is the tuple

$$\mathbf{dinv}(\pi) = (d_1(\pi), \dots, d_n(\pi))$$

such that $d_i(\pi) = |\{1 \leq i < j \leq n \mid a_i = a_j \text{ or } a_i = a_j + 1\}\}|$. In other words, $d_i(\pi)$ is the number of $j > i$ such that (i, j) form a diagonal inversion in π.

We recover the dinv statistic of [Hag08] by adding up the dinv sequence.

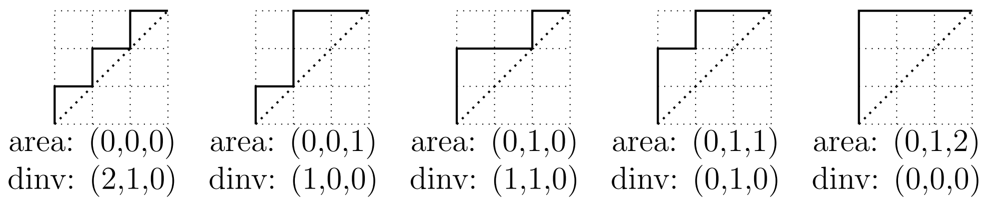

FIGURE 1. The area sequence and dinv sequence of all Dyck paths in $L^+_{3,3}$.

Example 1. When $n = 3$, the basis elements are

$$(x_1 - x_2)(x_2 - x_3)(x_1 - x_3), \tag{1}$$
$$x_1y_2 - x_1y_3 - x_2y_1 + x_2y_3 + x_3y_1 - x_3y_2, \tag{2}$$
$$x_1x_2y_1 - x_1x_2y_2 - x_1x_3y_1 + x_1x_3y_3 + x_2x_3y_2 - x_2x_3y_3, \tag{3}$$
$$-x_1y_1y_2 + x_1y_1y_3 + x_2y_1y_2 - x_2y_2y_3 - x_3y_1y_3 + x_3y_2y_3, \tag{4}$$
$$(y_1 - y_2)(y_2 - y_3)(y_1 - y_3). \tag{5}$$
$$\tag{6}$$

They correspond to Dyck paths in Figure 1 in order. The third q, t-Catalan number is therefore equal to

$$C_3(q, t) = q^3 + q^2t + qt^2 + t^3 + qt.$$

2.2. **Carlsson–Oblomkov basis.** To prove our theorem, we need to introduce the monomial basis of DR_n given by Carlsson and Oblomkov in [CO25].

A parking function $P = (\pi, \sigma)$ of length n consists of a Dyck path $\pi \in L^+_{n,n}$ together with the labelling of the rows by a permutation $\sigma \in S_n$, that are decreasing along each vertical wall, often called the *labelling permutation*. Denote the set of parking functions of length n by $\mathrm{PF}(n)$. We follow the conventions in [CO25].

Definition 2. Let $P = (\pi, \sigma) \in PF(n)$. The *area sequence* of P is defined to be the area sequence of π. We say that (σ_i, σ_j) is a *primary dinv pair* of P if $i < j, a_i = a_j$ and $\sigma_i < \sigma_j$; we call (σ_i, σ_j) a *secondary dinv pair* if $i > j, a_j = a_i + 1$ and $\sigma_i < \sigma_j$. The *dinv sequence* of P is $\mathbf{dinv}(P) = (d_1(P), \dots, d_n(P))$ such that

$$d_i = |\{1 \leq i < j \leq n \mid a_i = a_j \text{ and } \sigma_i < \sigma_j, \text{ or } a_i = a_j + 1 \text{ and } \sigma_i > \sigma_j\}|. \tag{7}$$

The integer vector $\mathbf{area}(P)_{\sigma^{-1}}$ is called the *major sequence* of P, denoted by $\mathbf{maj}(P)$.

The following definition appeared implicitly in Section 2.4 of [CM18].

Definition 3 (lex-labelling)**.** We define a map $\phi : L^+_{n,n} \to \mathrm{PF}(n)$ as follows. Given a Dyck path π, we order its rows in increasing order of the area; for rows with the same area, we order them in increasing order among themselves. Then, we label the vertical steps of ϕ from 1 through n with respect to this ordering.

σ	123	132	213	231	312	321
major index	000	101	010	011	001	012
schedule	321	111	121	211	221	111

TABLE 1. The majors and schedules of permutations $\sigma \in S_3$.

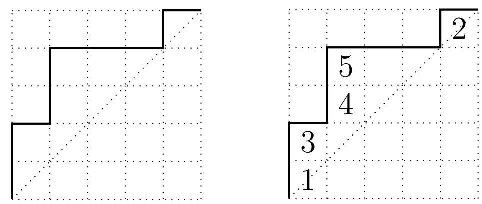

FIGURE 2. We draw a Dyck path and its image under ϕ.

Proposition 1. *For any Dyck path $\pi \in L^+_{n,n}$, we have $\mathbf{dinv}(\pi) = \mathbf{dinv}(\phi(\pi))$.*

Proof. Let $(a_1, \ldots, a_n)$ be the area sequence of π, and let σ be the labelling permutation of $\phi(\pi)$. If $a_i = a_j$, then by the definition of ϕ, we have $\sigma_i < \sigma_j$ if and only if $i < j$. If $a_i > a_j$, then by the definition of ϕ, we have $\sigma_i > \sigma_j$ always. These coincide Equation (7) with Definition 1. □

Definition 4 ([Hag08])**.** For a permutation $\sigma \in S_n$ with descents at places $i_1 < i_2 < \cdots < i_k$, we define the *runs*, denoted $\mathbf{r}(\sigma) = (r_1(\sigma), \ldots, r_k(\sigma))$ as the maximal consecutive increasing subsequences of σ. By convention, let $r_{k+1}(\sigma) = 0$. We define the *schedule* of σ to be the sequence $\mathrm{sch}(\sigma) = (\mathrm{sch}_1(\sigma), \ldots, \mathrm{sch}_n(\sigma))$ where

$$\text{if } \sigma_i \in r_j(\sigma), \text{ then } \mathrm{sch}_i(\sigma) = |\{k \in r_j(\sigma) \mid k > \sigma_i\}| + |\{k \in r_{j+1}(\sigma) \mid k < \tau_i\}|.$$

Given a permutation $\sigma \in S_n$, we define the *major index table* is given by $\mathbf{maj}(\sigma) = (a_1, \ldots, a_n)$ where

$$a_{\sigma_i} = |\{i \leq j \leq n-1 \mid \sigma_j > \sigma_{j+1}\}|.$$

Let $\mathrm{cars}(\sigma)$ denote the set of parking functions whose cars in rows of area i consists of elements of the $(k+1-i)$-th run of σ.

Theorem 2 ([Hag08, CO25])**.** *For any $\sigma \in S_n$, and for any $P \in \mathrm{cars}(\sigma)$, we have $\mathbf{maj}(P) = \mathbf{maj}(\sigma)$. Moreover, if $\mathbf{dinv}(P) = (k_1, \ldots, k_n)$, then we have $0 \leq k_i \leq \mathrm{sch}_i(\sigma) - 1$ for all $i \in [n]$.*

Carlsson and Oblomkov gives a monomial basis for DR_n with these statistics:

Theorem 3 (Theorem B,[CO25])**.** *We have a vector space basis of DR_n given by*

$$\{\mathbf{y}^{\mathbf{maj}(\sigma)} x_{\sigma_1}^{k_1} \cdots x_{\sigma_n}^{k_n}\}$$

ranging over $\sigma \in S_n$ and $0 \leq k_i \leq \mathrm{sch}_i(\sigma) - 1$.

Example 2. When $n = 3$, we tabulate permutations and their majors and schedules in Table 1. The Carlsson–Oblomkov basis is (when $n = 3$):

$$x_1^2x_2, x_1^2, x_1x_2, x_1, x_2, 1, \tag{8}$$
$$y_1y_3, \tag{9}$$
$$x_1y_2, y_2, \tag{10}$$
$$x_2y_2y_3, y_2y_3, \tag{11}$$
$$x_1x_3y_3, x_1y_3, x_3y_3, y_3, \tag{12}$$
$$y_2y_3^2. \tag{13}$$

As a corollary, we derive an alternative Carlsson–Oblomkov basis more explicitly expressed in parking functions.

Corollary 2. *Ranging over all parking functions $P = (\pi, \sigma) \in \mathrm{PF}_n$, we have another vector space basis of DR_n given by $\{\sigma(\mathbf{y}^{\mathbf{area}(P)}\mathbf{x}^{\mathbf{dinv}(P)})\}$, where σ acts on the indices of $\mathbf{y}$ and $\mathbf{x}$ simultaneously.*

Proof. By Theorem 2, we have $\mathbf{maj}(\sigma) = \mathbf{maj}(P) = \mathbf{area}(P)_{\sigma^{-1}}$. By Theorem 3, we conclude the proof by permuting the indices of $\mathbf{x}$ and $\mathbf{y}$ simultaneously with σ. □

2.3. **Proof of the main theorem.** Now we proceed to prove Theorem 1.

Lemma 1. *Sort the y-exponent vector in increasing lex-order and then the x-exponent vector in decreasing lex-order. For a Dyck path π with area sequence $a(\pi) = (a_1, \ldots, a_n)$ and dinv sequence $d(\pi) = (d_1, \ldots, d_n)$, let σ be the labelling permutation of the parking function $\phi(\pi)$ from Definition 3 that orders rows by nondecreasing area (ties broken as there). Then $\sigma(\mathbf{y}^{\mathbf{area}(\phi(\pi))}\mathbf{x}^{\mathbf{dinv}(\phi(\pi))})$ is the unique leading term in the expansion of Δ_π.*

Proof. Sorting by $a(\pi)$ maximizes the y-exponent vector in the chosen order; among equal a's, the tie-break in Definition 3 makes the x-vector lexicographically maximal. Any $w \neq \sigma(\pi)$ either permutes a pair with a out of order or ties broken the other way, lowering the (x, y) exponent vector. Hence $\sigma(\mathbf{y}^{\mathbf{area}(\phi(\pi))}\mathbf{x}^{\mathbf{dinv}(\phi(\pi))})$ is the unique initial term with coefficient ± 1. □

Proof of Theorem 1. Write Δ_π in the alternative Carlsson–Oblomkov basis

$$\{\sigma(\mathbf{y}^{\mathbf{area}(P)}\mathbf{x}^{\mathbf{dinv}(P)})\} \tag{14}$$

of DR_n (Corollary 2). Then the coefficient of $\sigma(\mathbf{y}^{\mathbf{area}(\phi(\pi))}\mathbf{x}^{\mathbf{dinv}(\phi(\pi))})$ is ± 1, and all the other terms have strictly larger exponent vector in our graded lex order by Lemma 1. Thus the change-of-basis matrix from $\{\Delta_\pi\}$ to $\{\sigma(\mathbf{y}^{\mathbf{area}(P)}\mathbf{x}^{\mathbf{dinv}(P)})\}$ is unitriangular with ± 1's on the diagonal. □

3. q, t-Fuss-Catalan combinatorics

We conjecture a basis for $\mathcal{A}^m / \langle \mathbf{x}, \mathbf{y} \rangle \mathcal{A}^m$ labeled by *m-Dyck paths.* An m-Dyck path is a lattice path from $(0, 0)$ to (mn, n) that stays weakly above the diagonal $my = x$. In particular, we introduce the *bounce sequence* and define a decomposition of m-Dyck paths into an m-tuple of Dyck paths which is additive on both the area sequence and the bounce sequence.

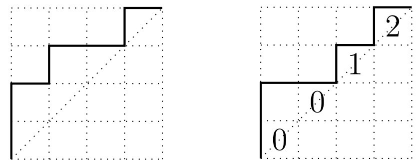

FIGURE 3. A Dyck path on the left with bounce path on the right, with bounce sequence $(0, 0, 1, 2)$.

3.1. **Bounce.** To state our conjectural basis, we switch from (area,dinv) to (bounce,area).

We first introduce the *bounce* defined by Hagund on Dyck paths. To define the bounce statistic, we need to define the *bounce path.*

Definition 5 ([Hag08])**.** Given a Dyck path $\pi \in L^+_{n,n}$, define the *bounce path* of π to be the path described by the following algorithm.

Start at $(0, 0)$ and travel north along π until you encounter the beginning of an east step. Then turn east and travel until you hit the diagonal. Then turn north again, etc. Continue this way until you arrive at (n, n). The bounce path will hit the diagonal at places $(0, 0), (j_0, j_0), (j_1, j_1), \dots, (j_{b-1}, j_{b-1}), (n, n)$. The j_i's are called *touch points.* We define the *bounce statistic* to be the sum

$$\text{bounce}(\pi) = \sum_{i=0}^{b-1} n - j_i$$

The idea of *bounce sequence* is implicit in [CM18].

Definition 6. Given a Dyck path $\pi \in L^+_{n,n}$ whose bounce path has touch points $j_0, \dots, j_b$, the *bounce sequence* of π is an increasing n-tuple $\mathbf{bounce}(\pi)$ with j_0 many zeroes, $j_1 - j_0$ many ones, ..., and $(n - j_{b-1})$ many b's. The sequence $(j_0, j_1 - j_0, \dots, n - j_{b-1})$ of vertical steps in the bounce path are also called the *bounce composition* of π.

Example 3. In Figure 3, we draw a Dyck path and its bounce path, with bounce sequence labelling the diagonal blocks. This labelling scheme appeared in [CM18].

Loehr extended the bounce statistic to m-Dyck paths.

Definition 7 ([Loe05])**.** Given an m-Dyck path $\pi \in L^+_{mn,n}$, define the *bounce path* of π to be an alternating sequence of vertical moves of lengths $v_0, v_1, \dots, v_s$ and horizontal moves of lengths $h_0, h_1, \dots$. These lengths are calculated as follows.

Start at $(0, 0)$ and travel north along π until you encounter the beginning of an east step; the distance traveled is v_0. Then travel v_0 units east, so $h_0 = v_0$. Next, travel v_1 units north until you encounter an east step, and travel $v_0 + v_1$ units east. In general, we always travel north v_i units north until we are blocked by the path. Afterwards, for $i < m$, we travel $h_i = v_0 + v_1 + \cdots + v_i$ units east; for $i \geq m$, we travel $h_i = v_i + v_{i-1} + \cdots + v_{i-(m-1)}$ units east. Finally, we define the *bounce statistic* to be

$$\text{bounce}(\pi) = \sum_{k \geq 0} k v_k = \sum_{k=0}^{s} (n - v_0 - v_1 - \cdots - v_k), \tag{15}$$

a weighted sum of the lengths of the vertical segments in the bounce path. The vertical steps $(v_0, \dots, v_s)$ are also called the *bounce composition* of π.

Conversely, a vector $(v_0, \dots, v_s)$ is a bounce composition for an m-Dyck path only if

(1) $v_0 > 0$, $v_s > 0$;
(2) $\sum v_i = n$;
(3) there is never a string of m or more consecutive zeroes.

We generalize our definition of *bounce sequence* to m-Dyck paths.

Definition 8. Given an m-Dyck path $\pi \in L^+_{mn,n}$, with bounce composition $(v_0, \dots, v_s)$, we define its bounce sequence to be the increasing n-tuple $\mathbf{bounce}(\pi)$ with v_0 many zeroes, v_1 many ones, ..., and v_s many s's. The bounce sequence sums up to the bounce statistic.

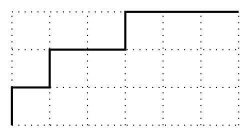

FIGURE 4. A rational Dyck path in $L^+_{mn,m}$ for $m = 2, n = 3$, with bounce sequence (0,1,2).

Example 4. The Dyck path in Figure 4 is equal to its own bounce path, with $v_0 = v_1 = v_2 = 1$ and bounce sequence $(0, 1, 2)$. We have $h_0 = v_0 = 1, h_1 = v_0 + v_1 = 2, h_2 = v_1 + v_2 = 2, h_3 = v_2 = 1$.

3.2. **The decomposition algorithm.**

Lemma 2. *For any rational Dyck path $\pi \in L^+_{mn,n}$ for which the bounce path has vertical steps $v_0, v_1, \dots, v_{s-1}$ and horizontal steps $h_0, h_1, \dots$ (the tail terms are conventionally set to 0), for any $i \in \{0, 1, \dots, m-1\}$, and for any $r \geq 1$, we have that*

$$\sum_{k=0}^{r} h_{km+i} = \sum_{j=0}^{rm+i} v_j. \tag{16}$$

In particular, $\sum_{k \geq 0} h_{km+i} = n$.

Proof. By definition, for $j < m$, we have

$$h_j = v_j + \cdots + v_1 + v_0.$$

For $j \geq m$, we have

$$h_j = v_j + v_{j-1} + \cdots + v_{j-m+1}.$$

Therefore,

$$\sum_{k=0}^{r} h_{km+i} = (v_i + \cdots + v_0) + \sum_{k=0}^{r-1} (v_{km+m+i} + \cdots + v_{km+i+1}) = v_0 + \cdots + v_{rm+i}. \tag{17}$$

Setting r to be sufficiently large, we have that $\sum_{k \geq 0} h_{km+i} = \sum v_i = n$. ☐

Definition 9. Let $\pi \in L^+_{mn,n}$ be an m-Dyck path of which the bounce path has horizontal steps $h_0, h_1, \dots$. For any $i \in \{0, 1, \dots, m-1\}$, we define $\pi^i \in L^+_{n,n}$ to be the concatenation of the parts under π in the columns specified by h_{km+i} for $k \geq 0$. In particular, the bounce composition of π^i is given by $(h_i, h_{m+i}, \dots)$.

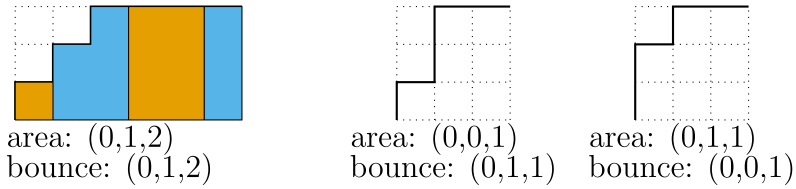

FIGURE 5. A rational Dyck path (left) with bounce path shown in Figure 4, bounce sequence $(0, 1, 2)$, and area sequence $(0, 1, 2)$, and its decomposition into two Dyck paths (right).

Example 5. The 2-Dyck path in Figure 4 is the bounce path of the 2-Dyck path on the left in Figure 5. In particular, we have the same horizontal steps $(1, 2, 2, 1)$ as in Example 4. We decompose the 2-Dyck path with respect to the horizontal steps and color them alternatingly. Then we collect regions of the same color and concatenate them into two Dyck paths on the right.

Theorem 4. *For any m-Dyck path $\pi \in L^+_{mn,n}$ and for any $i \in \{0, 1, \ldots, m-1\}$, let π^i be defined as in Definition 9. Then, the bounce (resp. area) sequence of π is equal to the the entrywise sum of the bounce (resp. area) sequences of π_i, i.e.,*

$$\mathbf{area}(\pi) = \sum_{i=0}^{m-1} \mathbf{area}(\pi^i) \tag{18}$$

$$\mathbf{bounce}(\pi) = \sum_{i=0}^{m-1} \mathbf{bounce}(\pi^i). \tag{19}$$

To prove this theorem, we need the following definitions

Definition 10. A *partition* $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0)$ is a weakly decreasing sequence of nonnegative integers. The *conjugate partition* of λ is denoted as λ' where $\lambda'_j = |\{i : \lambda_i \geq j\}|$.

Definition 11. Let $\pi \in L^+_{n,n}$ be a Dyck path with area sequence $\mathbf{area}(\pi) = (a_1, \ldots, a_n)$. The *diagram* of π is the partition λ such that $\lambda_i = n - i - a_{n-i+1}$.

Let $\pi \in L^+_{mn,n}$ be an m-Dyck path with area sequence $\mathbf{area}(\pi) = (a_1, \ldots, a_n)$. The *diagram* of π is the partition λ such that $\lambda_i = m(n-i) - a_{n-i+1}$.

Proof. By Definition 9, we see that for any $k \in [mn]$, there exists $i \in \{0, \ldots, m-1\}$ and $j \in [n]$ such that $\mathrm{dg}(\pi)'_k = \mathrm{dg}(\pi^i)'_j$. Conversely, for any $i \in [0, m-1]$ and any $j \in [n]$, there exists $k \in [mn]$ such that $\mathrm{dg}(\pi^i)'_j = \mathrm{dg}(\pi)'_k$. In other words, $\mathrm{dg}(\pi)$ is a concatenation of the parts of π^is. Then, $\mathrm{dg}(\pi)$ as the conjugate of $\mathrm{dg}(\pi)'$, satisfies $\mathrm{dg}(\pi)_\ell = |\{k : \mathrm{dg}(\pi)'_k \geq \ell\}| = \sum |\{k : \mathrm{dg}(\pi^i)'_j \geq \ell\}| = \sum_{i=0}^{m-1} \mathrm{dg}(\pi^i)_\ell$. As the area sequence is determined by the diagram, we have the first equality.

For the second equality, for any $i \in [0, m-1]$, as (h_{km+i}) form the bounce composition of π^i, we have that $\mathrm{bounce}(\pi^i)$ is a weakly increasing sequence with h_{km+i} many k's. Now we count the number of zeroes on the RHS. We have that $\mathrm{bounce}(\pi^i)$ starts with h_i number of zeroes, and $h_0 = v_0 = \min\{h_0, \ldots, h_{m-1}\}$, so there are v_0 many zeroes on the RHS. Then we count the number of ones on the RHS. We know that $\mathrm{bounce}(\pi^0)$ will have h_m consecutive ones after v_0 many zeroes, and we get ones at position k as far as $\mathrm{bounce}(\pi^1)_k, \ldots, \mathrm{bounce}(\pi^{m-1})_k$ stay zero. We have $h_1 = v_0 + v_1 = \min\{h_m, h_1, \ldots, h_{m-1}\}$, so $\mathrm{bounce}(\pi^1)_k = 0$ but $\mathrm{bounce}(\pi^1)_{k+1} = 1$, and $\mathrm{bounce}(\pi^1)_k = \cdots = \mathrm{bounce}(\pi^{m-1})_k = 0$. Therefore, there are v_1 many ones on the RHS. The same argument generalizes to prove the second equality. □

Remark 1. Athanasiadis [Ath05] showed a way to decompose an m-Dyck path into an m-tuple of Dyck paths for which the area sequence is additive. Our decomposition is essentially different.

3.3. **The sweep map.** Haglund introduced the ζ-map for Dyck paths[HL05], and then Thomas and Williams generalized it to rational Dyck paths and even broader context [Hag08]. We follow the conventions in [CM18]. It is implied by classical results that the ζ-map, combined with the Definition 3 lex-labelling of Dyck paths, is a bijection between the bounce sequences, area sequences, and dinv sequences.

An (m, n)-Dyck path is a lattice path from $(0, 0)$ to (m, n) staying weakly above the diagonal $y = \frac{n}{m}x$. Thomas–Williams definition applies to (m, n)-Dyck paths for arbitrary positive integers m, n.

Definition 12. Given an (m, n)-Dyck path π, let $m' = m/\gcd(m, n)$ and $n' = n/\gcd(m, n)$. We define the *rank* of a lattice point (x, y) in the $m \times n$-rectangle as $\mathrm{rank}(x, y) = m'y - n'x$. Let the rank of a vertical step in π be the rank of its south end, and let the rank of a horizontal step be the rank of its west end. Then $\zeta(\pi)$ is obtained as we reorder the steps in π by increasing rank, and order the steps of the same rank from northeast to southwest.

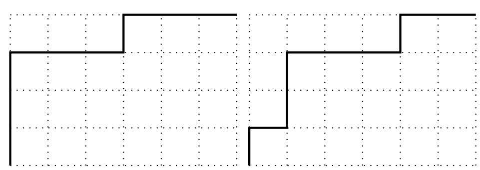

FIGURE 6. A (6,4)-Dyck path on the left and its ζ-image on the right.

Example 6. A Dyck word of a Dyck path $\pi \in L^+_{m,n}$ is a binary word in $\{0, 1\}^{m+n}$ where 1 represents a north step and 0 represents an east step. Consider the (6,4)-Dyck path with Dyck word $w = (1, 1, 1, 0, 0, 0, 1, 0, 0, 0)$ and rank displayed in the second row of the array below. We obtain $w' = (1, 0, 1, 1, 0, 0, 0, 1, 0, 0)$ as we order the elements based on their ranks.

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 6 & 9 & 7 & 5 & 3 & 6 & 4 & 2 \end{pmatrix}$$

Theorem 5 ([Hag03, HL05, TW18])**.** *For any Dyck path* π*, we have* $\mathrm{dinv}(\pi) = \mathrm{area}(\zeta(\pi))$ *and* $\mathrm{area}(\pi) = \mathrm{bounce}(\zeta(\pi))$.

Remark 2. The reversal of a Dyck path π is obtained by flipping it across the diagonal, denoted $\mathrm{rev}(\pi)$. The convention we follow here results in the reversal of the ζ-map defined in [Hag08] and in `SageMath`.

Carlsson and Mellit gave an alternative description for the ζ-map of Dyck paths. Let $\mathrm{Dinv}(\pi)$ be the set of dinv pairs of π and let $\mathrm{Area}(\pi)$ be the set of area cells of π. Let σ be the labelling permutation of the parking function $\phi(\pi)$. Then $\zeta(\pi)$ is defined by

$$\mathrm{Area}(\zeta(\pi)) = \{(\sigma_j, \sigma_{j'} : (j, j') \in \mathrm{Dinv}(\pi)\}. \tag{20}$$

Their results further implied the following lemma.

Lemma 3. *Let $\pi \in L^+_{n,n}$ be a Dyck path such that $\phi(\pi) = (\pi, \sigma)$, that is, σ is the labelling permutation of the parking function obtained from π through its lex-labelling, see Definition 3. Then* $\mathbf{bounce}(\zeta^{-1}(\pi)) = \mathbf{maj}(\pi)$, *and* $\mathbf{area}(\mathrm{rev}(\zeta^{-1}(\pi))) = \mathrm{rev}(\mathbf{dinv}(\pi))$.

Moreover, primary dinv pairs in π map to area cells below the bounce path in $\zeta(\pi)$, and secondary dinv pairs in π map to area cells above the bounce path.

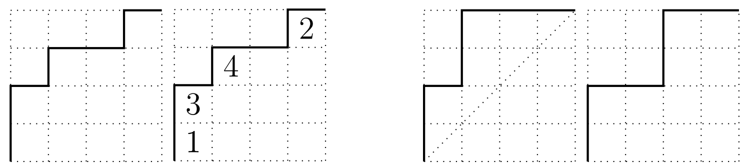

FIGURE 7. A Dyck path π (left), its lex-labelling $\phi(\pi)$, and its image under the ζ-map (right) and the bounce path of $\zeta(\pi)$.

Example 7. Consider the Dyck path π on the left in Figure 7 with area sequence $\mathbf{area}(\pi) = (0, 1, 1, 0)$. The dinv pairs for $\phi(\pi)$ are $(1, 2), (2, 3), (2, 4), (3, 4)$, which are exactly the area cells for the Dyck path on the right $\zeta(\pi)$ in Figure 7. The bounce sequence of $\zeta(\pi)$ is $\mathbf{bounce}(\zeta(\pi)) = (0, 0, 1, 1)$. The lex-labelling of π gives the permutation $\sigma = (1, 3, 4, 2)$, and $\sigma(\mathbf{bounce}(\zeta(\pi))) = (0, 1, 1, 0) = \mathbf{area}(\pi)$. The area sequence of the reversal of $\zeta(\pi)$ is $\mathbf{area}(\mathrm{rev}(\zeta(\pi))) = (0, 1, 2, 1)$, which is also the reversal of $\mathbf{dinv}(\pi) = (1, 2, 1, 0)$.

3.4. **A conjectural basis of alternating generalized diagonal coinvariants.** We conjecture that our decomposition provides a basis for the alternating component of the generalized diagonal coinvariants.

Conjecture 1. *For any rational Dyck path $\pi \in L^+_{mn,n}$, for any $i \in \{0, 1, \ldots, m-1\}$, let $\pi^i \in L^+_{n,n}$ be the Dyck path defined in Definition 9. Then $\{\prod_{i=0}^{m-1} \Delta_{X(\zeta^{-1}(\pi^i))}\}$ over all m-Dyck paths form a basis of the alternating component of the space of generalized diagonal coinvariants.*

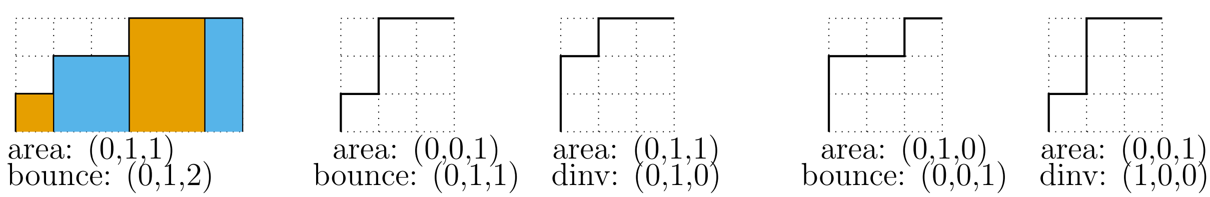

FIGURE 8. The decomposition of a 2-Dyck path and the ζ^{-1}-image of this decomposition.

Example 8. In Figure 8, we show the decomposition of a 2-Dyck path. The corresponding $\Delta_{\zeta^{-1}(\pi^1)}\Delta_{\zeta^{-1}(\pi^2)}$ is equal to

$$(-x_1y_1y_2+x_1y_1y_3+x_2y_1y_2-x_2y_2y_3-x_3y_1y_3+x_3y_2y_3)(x_1y_2-x_1y_3-x_2y_1+x_2y_3+x_3y_1-x_3y_2).$$

An m-parking function consists of an m-Dyck path π and a permutation $\sigma \in S_n$ which is increasing along each vertical wall.

We analyze the synergy between our decomposition algorithm Definition 9 and the ζ-map on parking functions.

Proposition 2. *Let (π, σ) be an m-parking function, and let $(r_1, \dots, r_n)$ be the permuted rank vector where r_i is the rank of the vertical step in π with label i (see Definition 12). Let $(v_0, v_1, \dots)$ be the bounce composition of $\zeta(\pi)$. Then the labels of $\zeta(\pi)$ to the left of the v_j vertical steps in the bounce path of $\zeta(\pi)$ are exactly those labels i with $r_i = j$.*

Proof. The most northeast step of rank j is always horizontal, because the path eventually returns to rank 0. Therefore, we want to show that the number of horizontal steps of rank j is equal to h_{j-1}. We prove by induction. The base case asserts that the number of horizontal steps of rank 1 is equal to $h_0 = v_0$. A horizontal step has rank 1 if and only if its west end has rank 1 and its east end has rank 0. Either its east end is (mn, n), or it is followed by some vertical step of rank 0. There is always a vertical step of rank 0 at $(0, 0)$. Therefore, we have proven the base case.

For the induction step, a horizontal step has rank j if and only if its west end has rank 1 and its east end has rank $j-1$, which can be followed by either vertical steps or horizontal steps of rank $j-1$. However, for every vertical step of rank $j-m-1$, the north end of such a vertical step has rank $j-1$, and must be followed by some step of rank $j-1$. Therefore, the number of horizontal steps of rank j is equal to, by induction hypothesis, $v_{j-1} + h_{j-2} - v_{j-m-1} = h_{j-1}$. □

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