A basis of the alternating diagonal coinvariants

Yuhan Jiang

University of California, Berkeley

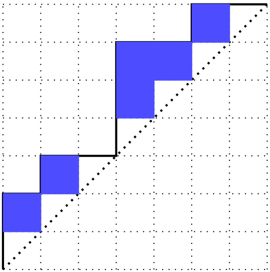
UC Davis Combinatorics Seminar

Overview

1. The ring of diagonal coinvariants
2. A basis for the alternating diagonal coinvariants
3. A conjectural basis for the alternating component of the generalized diagonal coinvariants
4. A conjectural basis of the space of generalized diagonal coinvariants

The q, t-Catalan number

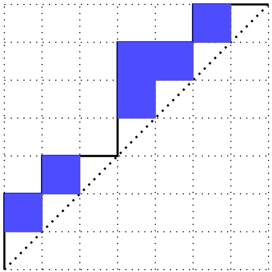
- A Dyck path is a lattice path from $(0, 0)$ to (n, n) which stays weakly above the diagonal $y = x$.
- The *area* of a Dyck path is the number of cells (complete squares) under the path and above the diagonal $y = x$.



area $= 6$

The q, t-Catalan number

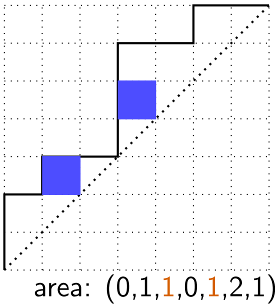
- The *area sequence* of a Dyck path π is $\textbf{area}(\pi) = (a_1, \ldots, a_n)$ where a_i is the number of cells between the path π and the diagonal on the i-th row.



area: (0,1,1,0,1,2,1)

The q, t-Catalan number

- Let $\textbf{area}(\pi) = (a_1, \ldots, a_n)$ be the area sequence of π.
- primary diagonal inversion (primary dinv): $i < j$ and $a_i = a_j$;
- secondary diagonal inversion: $i < j$ and $a_i = a_j + 1$.
- dinv $= \#$ primary dinv $+ \#$ secondary dinv.



(3,5) is a primary dinv pair

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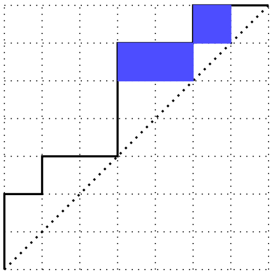

area: (0,1,1,0,1,2,1)

(6,7) is a secondary dinv pair

The q,t-Catalan number

- Let $\textbf{area}(\pi) = (a_1, \ldots, a_n)$ be the area sequence of π.
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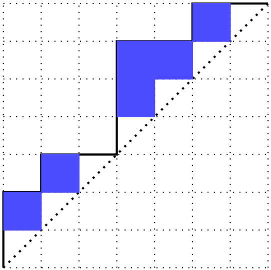


area: (0,1,1,0,1,2,1)

dinv $=$ 10

primary dinv pairs: (1,4), (2,3), (2,5), (2,7), (3,5), (3,7), (5,7).

secondary dinv pairs: (2,4), (3,4), (6,7).

The q, t-Catalan number

The q, t-Catalan number $C_n(q, t)$ was defined by Haiman as the *bigraded Hilbert series* of the ideal of alternating polynomials in the ring of *diagonal coinvariants*. Therefore, it is q, t-symmetric.

Haglund–Haiman–Loehr:

$$C_n(q, t) = \sum_{\pi \text{ Dyck path}} t^{\text{area}(\pi)} q^{\text{dinv}(\pi)}.$$

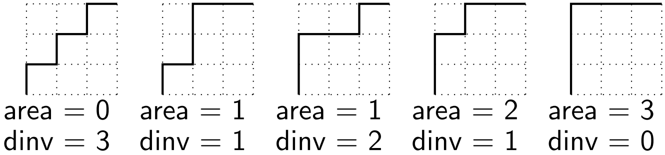


Figure: $C_3(q, t) = q^3 + q^2 t + qt^2 + t^3 + qt$.

- $C_n(q,t) = C_n(t,q)$, but there is no combinatorial proof.
- There is a bijection ζ, defined by Haglund and generalized by Thomas–Williams, on Dyck paths such that $\text{dinv}(\pi) = \text{area}(\zeta(\pi))$, so area and dinv are equidistributed. However, $\text{area}(\pi) = \text{bounce}(\zeta(\pi))$ so ζ does not provide a combinatorial proof of q,t-symmetry.
- However, as they were originally defined by Haiman as the *bigraded Hilbert series* in the context of diagonal coinvariants, q,t-symmetry is implied by algebra.

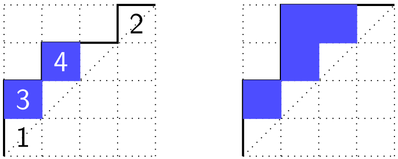


Figure: The area cells of $\zeta(\pi)$ are $(1,2),(2,3),(2,4),(3,4)$. They also label the dinv pairs in π.

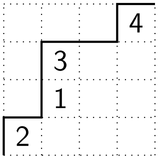
The ring of diagonal coinvariants

- The symmetric group S_n acts on $\mathbb{C}[\mathbf{x}, \mathbf{y}] := \mathbb{C}[x_1, y_1, \ldots, x_n, y_n]$ by permuting the indices on $\mathbf{x}$ and $\mathbf{y}$ simultaneously.
- Let
$$\mathfrak{m}_+^{S_n}(\mathbf{x}, \mathbf{y}) = \left\langle \sum_k x_k^i y_k^j : (i, j) \neq (0, 0) \right\rangle$$
be the ideal of invariant polynomials.
- The *diagonal coinvariant ring* is defined to be
$$DR_n := \mathbb{C}[\mathbf{x}, \mathbf{y}]/\mathfrak{m}_+^{S_n}(\mathbf{x}, \mathbf{y}).$$

For example, $DR_2 = \mathbb{C}[x_1, x_2, y_1, y_2]/\langle x_1 + x_2, y_1 + y_2, x_2^2, x_2 y_2, y_2^2 \rangle$.

The ring of diagonal coinvariants

Haiman showed that DR_n has dimension equal to $(n+1)^{n-1}$, the number of parking functions of length n.



A parking function is a Dyck path with a labelling of its vertical steps which is increasing along each wall. The name comes from cars with preference parking positions. Here the preference vector is $(2,1,2,4)$.

The Frobenius character of DR_n was studied as the *shuffle conjecture*, raised by Haglund–Haiman–Loehr–Remmel–Ulyanov, proven by Carlsson–Mellit.

Carlsson–Oblomkov gave a monomial basis of DR_n labeled by parking functions. This parking function, for example, gives the monomial $x_1 y_3$.

The ideal of alternating polynomials

- Let $\mathcal{A}$ be the ideal generated by alternating polynomials

$$\mathcal{A} = \langle f : (\sigma f) = \operatorname{sgn}(\sigma) f \rangle.$$

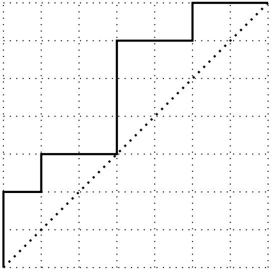
- E.g. when $n = 2$, we have $\mathcal{A} = \langle x_1 - x_2, y_1 - y_2 \rangle$.
- The dimension of $\mathcal{A}/\langle \mathbf{x}, \mathbf{y} \rangle \mathcal{A}$ is the Catalan number.
- Christian Stump asked if there should be a vector space basis labeled by Dyck paths.
- The q,t-Catalan numbers were first defined by Haiman as the bigraded Hilbert series of $\mathcal{A}/\langle \mathbf{x}, \mathbf{y} \rangle \mathcal{A}$, that is,

$$\sum \mathcal{A}_{i,j} q^i t^j = C_n(q,t) = \sum_{\pi \text{ Dyck path}} t^{\operatorname{area}(\pi)} q^{\operatorname{dinv}(\pi)}.$$

- We construct such a basis which manifest the q,t-Catalan combinatorics.

Dinv sequence

We define the *dinv sequence* of π to be $\mathbf{dinv}(\pi) = (d_1, \ldots, d_n)$ where d_i is the number of j for which $(i < j)$ is a dinv pair.



area: (0,1,1,0,1,2,1)

dinv: (1,4,3,0,1,1,0)

Figure: The primary dinv pairs are (1,4), (2,3), (2,5), (2,7), (3,5), (3,7), (5,7). The secondary dinv pairs are (2,4), (3,4), (6,7).

Main theorem

Theorem (J.)

For any Dyck path π *from* $(0,0)$ *to* (n,n)*, let* $\textbf{area}(\pi) = (a_1, \ldots, a_n)$ *and* $\textbf{dinv}(\pi) = (d_1, \ldots, d_n)$*. The set of bivariate Vandermonde determinants*

$$\Delta_\pi = \det(x_i^{d_j} y_i^{a_j})_{i,j=1}^n$$

form a vector space basis for the ideal $\mathcal{A}/\langle \mathbf{x}, \mathbf{y} \rangle \mathcal{A}$ *of alternating polynomials in* DR_n*.*

For example, for the Dyck path shown below, we have $\textbf{area}(\pi) = (0,1,0)$ and $\textbf{dinv}(\pi) = (1,1,0)$.

Δ 

$$= \begin{vmatrix} x_1 & x_1 y_1 & 1 \\ x_2 & x_2 y_2 & 1 \\ x_3 & x_3 y_3 & 1 \end{vmatrix}$$

Main theorem

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For any Dyck path π from $(0,0)$ to (n,n), let $\mathbf{area}(\pi) = (a_1, \ldots, a_n)$ *and* $\mathbf{dinv}(\pi) = (d_1, \ldots, d_n)$. *The set of bivariate Vandermonde determinants*

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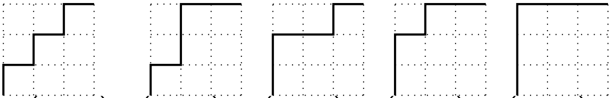
form a vector space basis for the ideal $\mathcal{A}/\langle \mathbf{x}, \mathbf{y} \rangle \mathcal{A}$ of alternating polynomials in DR_n.

We recover the formula for q, t-Catalan numbers as a corollary

$$C_n(q,t) = \sum_{\pi \text{ Dyck path}} t^{\mathrm{area}(\pi)} q^{\mathrm{dinv}(\pi)},$$

because $\deg_{\mathbf{x}}(\Delta_\pi) = \mathrm{dinv}(\pi), \deg_{\mathbf{y}}(\Delta_\pi) = \mathrm{area}(\pi)$.
Our basis is the antisymmetrization of the monomial basis of DR_n given by Carlsson and Oblomkov.

Example



area:	(0,0,0)	(0,0,1)	(0,1,0)	(0,1,1)	(0,1,2)
dinv:	(2,1,0)	(1,0,0)	(1,1,0)	(0,1,0)	(0,0,0)

$$(x_1 - x_2)(x_2 - x_3)(x_1 - x_3),$$

$$x_1y_2 - x_1y_3 - x_2y_1 + x_2y_3 + x_3y_1 - x_3y_2,$$

$$x_1x_2y_1 - x_1x_2y_2 - x_1x_3y_1 + x_1x_3y_3 + x_2x_3y_2 - x_2x_3y_3,$$

$$-x_1y_1y_2 + x_1y_1y_3 + x_2y_1y_2 - x_2y_2y_3 - x_3y_1y_3 + x_3y_2y_3,$$

$$(y_1 - y_2)(y_2 - y_3)(y_1 - y_3).$$

The space of generalized diagonal coinvariants

- Recall that DR_n is the ring of diagonal coinvariants and $\mathcal{A}$ is the ideal of alternating polynomials.
- Garsia and Haiman defined *the space of generalized diagonal coinvariants*
$$DR_n^{(m)} \cong (\mathcal{A}^{m-1}/\mathcal{A}^{m-1}\mathfrak{m}_+^{S_n}(\mathbf{x},\mathbf{y})).$$
- For example, $DR_2^{(2)}$ is generated by $x_1(x_1-x_2), x_1(y_1-y_2), y_1(y_1-y_2)$.

The space of generalized diagonal coinvariants

- Haiman showed that the dimension of $DR_n^{(m)}$ is $(mn+1)^{n-1}$, the number of parking functions on m-Dyck paths.
- The Frobenius character of $DR_n^{(m)}$ was studied as a special case of the *(extended) rational shuffle conjecture*, proven by Mellit.

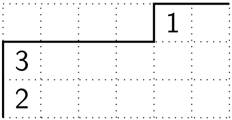


Figure: This is a parking function on a 2-Dyck path.

The Fuß-Catalan number

- The Fuß-Catalan number is defined as

$$\frac{1}{mn+1}\binom{(m+1)n}{n},$$

the number of lattice paths from $(0,0)$ to (mn, n) staying weakly above the diagonal $my = x$, or m-Dyck paths.

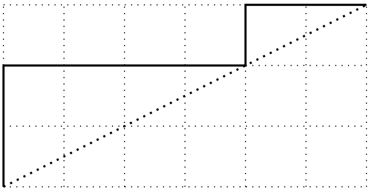


Figure: This is an m-Dyck path.

The alternating component of generalized diagonal coinvariants

- The alternating component of $DR_n^{(m)}$ is isomorphic to

$$(\mathcal{A}^m / \langle \mathbf{x}, \mathbf{y} \rangle \mathcal{A}^m) \, .$$

  whose bigraded Hilbert series was defined by Haiman as the q, t*-Fuß-Catalan number*, a refinement of the Fuß-Catalan number.
- For example, when $m = n = 2$, $\mathcal{A}^m / \langle \mathbf{x}, \mathbf{y} \rangle \mathcal{A}^m$ is generated by $x_1 - x_2, y_1 - y_2, (x_1 - x_2)^2, (x_1 - x_2)(y_1 - y_2), (y_1 - y_2)^2$.

The q, t-Fuß-Catalan number

- The extended rational shuffle conjecture, raised by Bergeron–Garsia–Leven–Xin, and proven by Mellit, implies that

$$C_n^{(m)}(q,t) = \sum_{\pi \text{ m-Dyck path}} t^{\text{area}(\pi)} q^{\text{dinv}(\pi)}.$$

- The area is defined just as before, but the dinv is more involved to define.
- For example,

$$C_3^{(2)} = q^6+q^5t+q^4t^2+q^3t^3+q^2t^4+qt^5+t^6+q^4t+q^3t^2+q^2t^3+qt^4+q^2t^2$$

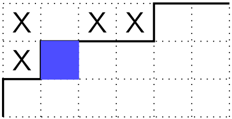


Figure: This is a 2-Dyck path with area 1 and dinv 4.

The q, t-Fuß-Catalan number

- Loehr defined the *bounce* of an m-Dyck path and proposed

$$C_n^{(m)}(q,t) = \sum_{\pi \text{ m-Dyck path}} q^{\mathrm{area}(\pi)} t^{\mathrm{bounce}(\pi)}.$$

- With the generalized ζ-map of Thomas–Williams, Loehr's formula of (bounce,area) is equivalent to the (area,dinv) formula.

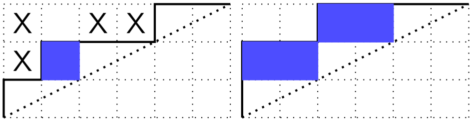


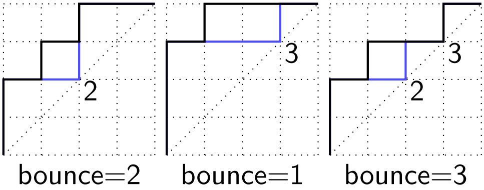
Figure: We have $\mathrm{dinv}(\pi) = \mathrm{area}(\zeta(\pi)) = 4$, but how to define bounce?

Bounce

Haglund defined the bounce path of a Dyck path:

- Start at $(0, 0)$ and travel north along π until you encounter the beginning of an east step. Then turn east and travel until you hit the diagonal. Then turn north again, etc. Continue this way until you arrive at (n, n).
- The bounce path will hit the diagonal at places $0, j_0, j_1, \ldots, j_{b-1}, n$.
- The bounce is defined as

$$\text{bounce}(\pi) := \sum (n - j_i).$$



Bounce

- The bounce path will hit the diagonal at places $0, j_0, j_1, \ldots, j_{b-1}, n$.
- The *bounce sequence* of π is the increasing n-tuple $\mathbf{bounce}(\pi)$ with j_0 many zeroes, $j_1 - j_0$ many ones, ..., and $(n - j_{b-1})$ many b's.

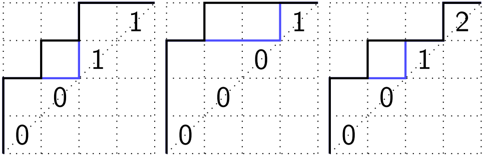


Figure: This labelling scheme, or the idea of bounce sequence, appeared in Carlsson–Mellit's proof the shuffle conjecture.

Bounce of m-Dyck paths

Loehr generalized the bounce statistic to m-Dyck paths. The bounce path is defined as follows:

- Start at $(0,0)$ and travel north along π until you encounter the beginning of an east step; the distance traveled is v_0. Then travel v_0 units east, so $h_0 = v_0$.

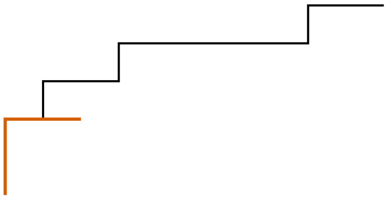


Figure: The vertical steps in the bounce path are 2, and horizontal steps 2,

Bounce of m-Dyck paths

- Next, travel v_1 units north until you encounter an east step, and travel $v_0 + v_1$ units east.

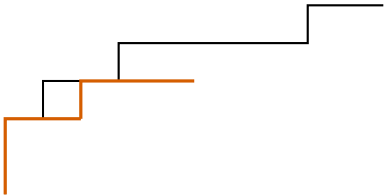
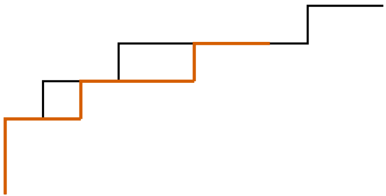


Figure: The vertical steps in the bounce path are 2,1, and horizontal steps 2,3,

Bounce of m-Dyck paths

- For vertical steps, we always travel north v_i units until we are blocked by the path.
- For horizontal steps, when $i < m$, we move $h_i = v_i + v_{i-1} + \cdots + v_0$ units east.
- When $i \geq m$, we move $h_i = v_i + v_{i-1} + \cdots + v_{i-(m-1)}$ units east.



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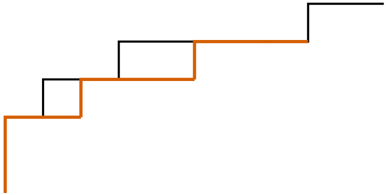

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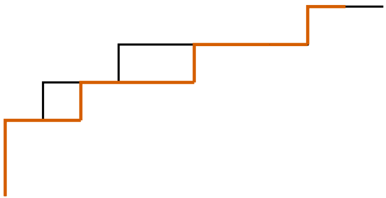

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Bounce of m-Dyck paths

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- For horizontal steps, when $i < m$, we move $h_i = v_i + v_{i-1} + \cdots + v_0$ units east.
- When $i \geq m$, we move $h_i = v_i + v_{i-1} + \cdots + v_{i-(m-1)}$ units east.
- Loehr defined bounce$(\pi) = \sum(n - v_0 - v_1 - \cdots - v_i)$.

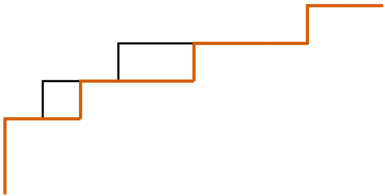


Figure: The vertical steps in the bounce path are 2,1,1,0,1,0 and horizontal steps 2,3,2,1,1,1.

Bounce of m-Dyck paths

- Let $v_0, v_1, v_2, \ldots$ be the vertical steps in the bounce path of π.
- We define the bounce sequence of π to be the increasing n-tuple $\mathbf{bounce}(\pi)$ with v_0 many zeroes, v_1 many ones, v_2 many twos, etc.

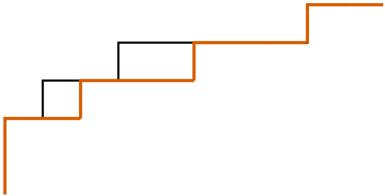
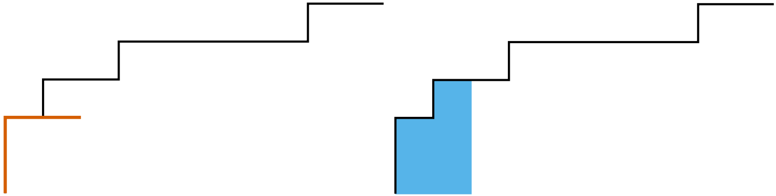


Figure: The vertical steps in the bounce path are 2,1,1,0,1,0 and horizontal steps 2,3,2,1,1,1. The bounce sequence is $(0, 0, 1, 2, 4)$.

Decomposition algorithm

- Let $i \in \{0, 1, \ldots, m-1\}$. Given an m-Dyck path π with bounce path given by horizontal steps $h_0, h_1, \ldots$.
- We define π^i to be the Dyck path obtained as the concatenation of the parts under π in the columns specified by the horizontal steps h_{km+i} for $k \geq 0$.



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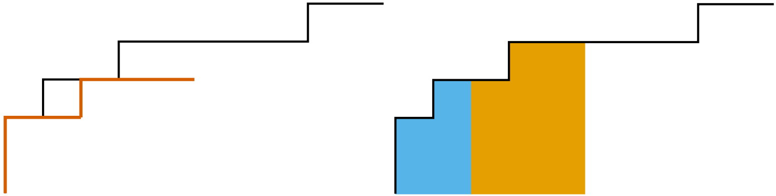

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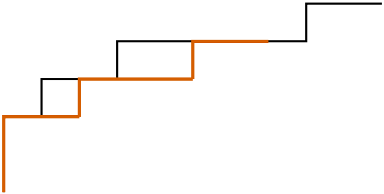
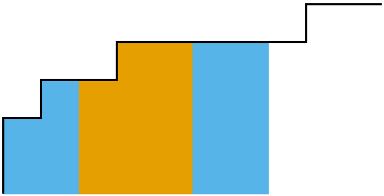
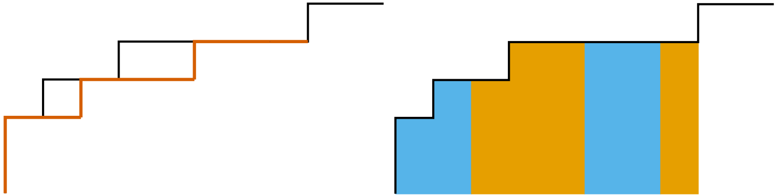




Decomposition algorithm

- Let $i \in \{0, 1, \ldots, m-1\}$. Given an m-Dyck path π with bounce path given by horizontal steps $h_0, h_1, \ldots$.
- We define π^i to be the Dyck path obtained as the concatenation of the parts under π in the columns specified by the horizontal steps h_{km+i} for $k \geq 0$.



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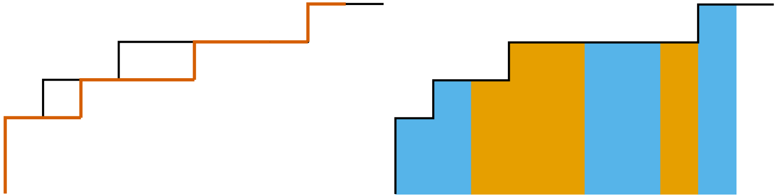

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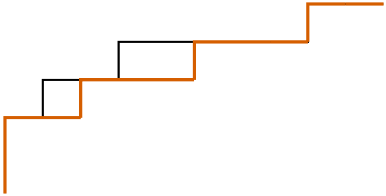
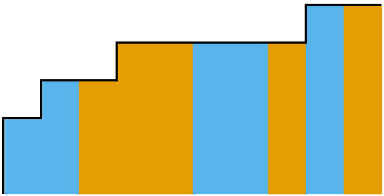
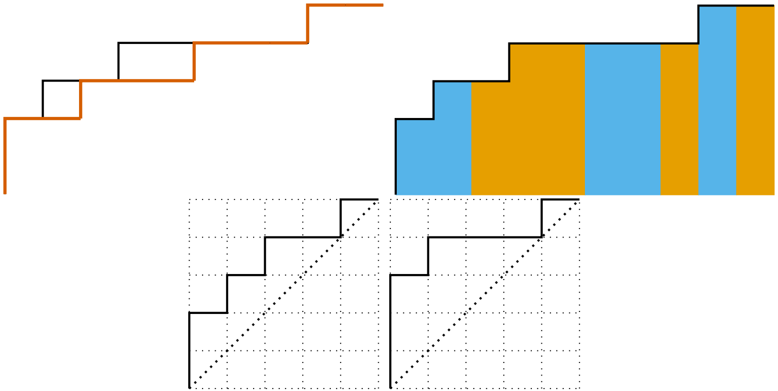




Decomposition algorithm

- π^i has bounce composition $(h_i, h_{m+i}, h_{2m+i}, \ldots)$.



Decomposition theorem

Theorem (J.)

Let π be an m-Dyck path whose bounce path has horizontal steps $h_0, h_1, \ldots$. For any $i \in \{0, 1, \ldots, m-1\}$, let π^i be the Dyck path obtained as the concatenation of the parts under π in the columns specified by the horizontal steps h_{km+i} for $k \geq 0$. We have

$$\mathbf{area}(\pi) = \sum_{i=0}^{m-1} \mathbf{area}(\pi^i),$$

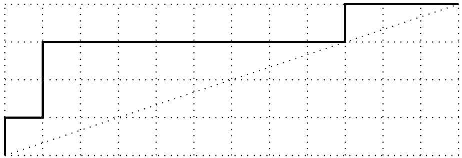
$$\mathbf{bounce}(\pi) = \sum_{i=0}^{m-1} \mathbf{bounce}(\pi^i).$$

In particular, the set $\{h_{km+i}\}_{k \geq 0}$ form the bounce composition of π^i.

Idea for the proof

For the bounce, Loehr characterized the vertical steps $\vec{v}$ by:

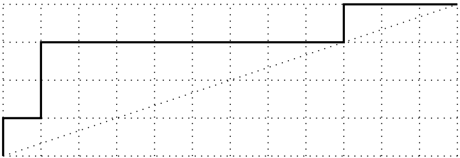
- $v_0 > 0$;
- $\sum v_i = n$;
- there is never a string of m or more consecutive zeroes in $\vec{v}$.



We have vertical steps (1,2,0,0,1).

Idea for the proof

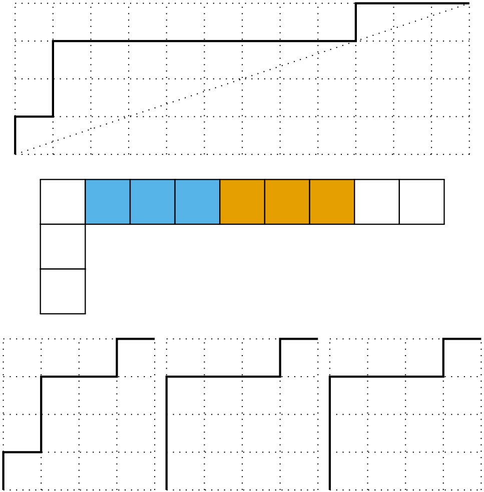
In particular, $h_i \neq 0$ for any i.
Also, $\sum_{k\geq 0} h_{km+i} = \sum v_i = n$.



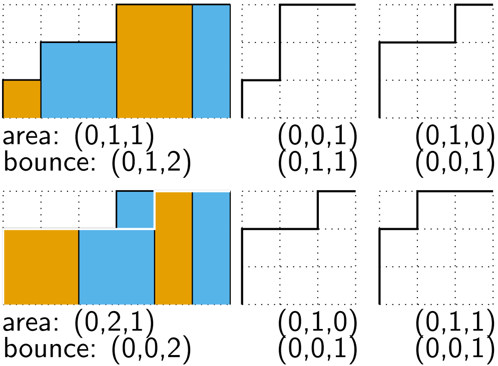
We have horizontal steps (1,3,3,2,1,1,1), and
$1+2+1=3+1=3+1=4$.

Idea for the proof

For the area, note that the squares above π in the $mn \times n$ rectangle is a concatenation of the squares above π^i in the $n \times n$ squares.



Examples


area: (0,1,1)
bounce: (0,1,2)
(0,0,1)
(0,1,1)
(0,1,0)
(0,0,1)
area: (0,2,1)
bounce: (0,0,2)
(0,1,0)
(0,0,1)
(0,1,1)
(0,0,1)

Main conjecture

Conjecture (J.)

For any m-Dyck path π, let $(\pi^0, \ldots, \pi^{m-1})$ be the decomposition of π defined before. Then $\{\prod_{i=0}^{m-1} \Delta_{\zeta^{-1}(\pi^i)}\}$ over all m-Dyck paths form a basis of the alternating component of the space of generalized diagonal coinvariants.

Our conjecture aligns with the q, t-Fuß-Catalan statistics of Loehr

$$C_n^{(m)}(q,t) = \sum_{\pi \text{ m-Dyck path}} q^{\text{area}(\pi)} t^{\text{bounce}(\pi)},$$

because

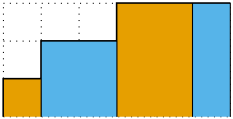
$$\deg_{\mathbf{x}}(\prod_{i=0}^{m-1} \Delta_{\zeta^{-1}(\pi^i)}\}) = \text{area}(\pi)$$

and

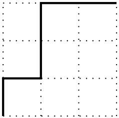
$$\deg_{\mathbf{y}}(\prod_{i=0}^{m-1} \Delta_{\zeta^{-1}(\pi^i)}\}) = \text{bounce}(\pi).$$

Example

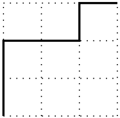
We show the decomposition of a 2-Dyck path.



area: (0,1,1)
bounce: (0,1,2)



area: (0,0,1)
bounce: (0,1,1)

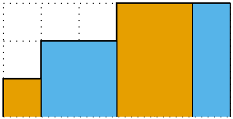


area: (0,1,0)
bounce: (0,0,1)

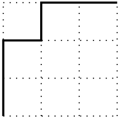
Example

The corresponding $\Delta_{\zeta^{-1}(\pi^0)}\Delta_{\zeta^{-1}(\pi^1)}$ is equal to

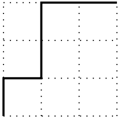
$$(-x_1y_1y_2 + x_1y_1y_3 + x_2y_1y_2 - x_2y_2y_3 - x_3y_1y_3 + x_3y_2y_3)$$
$$(x_1y_2 - x_1y_3 - x_2y_1 + x_2y_3 + x_3y_1 - x_3y_2).$$



area: (0,1,1)
bounce: (0,1,2)



area: (0,1,1)
dinv: (0,1,0)



area: (0,0,1)
dinv: (1,0,0)

Question

Can we extend our decomposition to give a vector space basis of $DR_n^{(m)}$?

Thank you!

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