

THE UNIVERSAL MONODROMIC ARKHIPOV–BEZRUKAVNIKOV EQUIVALENCE

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ABSTRACT. We identify equivariant quasicoherent sheaves on the Grothendieck alteration of a reductive group G with universal monodromic Iwahori–Whittaker sheaves on the enhanced affine flag variety of the Langlands dual group G . This extends a similar result for equivariant quasicoherent sheaves on the Springer resolution due to Arkhipov–Bezrukavnikov. We further give a monoidal identification between adjoint equivariant coherent sheaves on the group G itself and bi-Iwahori–Whittaker sheaves on the loop group of G . These results are used in the sequel to this paper to prove the tame local Betti geometric Langlands conjecture of Ben-Zvi–Nadler.

Our proof of fully faithfulness provides an alternative to the argument of Arkhipov–Bezrukavnikov. Namely, while they localize in unipotent directions, we localize in semi-simple directions, thereby reducing fully faithfulness to an order of vanishing calculation in semi-simple rank one.

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1. INTRODUCTION

1.1. Overview.

In the paper [BZN18], Ben-Zvi–Nadler introduced the Betti geometric Langlands program, a version of geometric Langlands in which the spectral side concerns the usual character varieties of topological surfaces, i.e., the representation varieties of their fundamental groups.¹

Among their many interesting conjectures, the basic local assertion is an identification of two universal versions of the affine Hecke category; this statement in fact had appeared in their earlier work [BZN07]. Here, in terms of character varieties, local means near a given puncture in the surface, and universal means that one does not fix the monodromy of local systems around the puncture, but rather allows them to vary through all conjugacy classes in the group.

As they emphasized in their initial paper, when one restricts from all conjugacy classes to unipotent ones, their local conjecture was already a celebrated theorem of Bezrukavnikov [B16]. This result of Bezrukavnikov, in turn, built on an earlier work joint with Arkhipov [AB09].

The goal of this paper and its sequel is to supply a proof of Ben-Zvi–Nadler’s local conjecture, i.e., tame local Betti geometric Langlands. Accordingly, in this first paper, we provide the deformation of the results of [AB09] over all conjugacy classes, and in its sequel, we provide the deformation of [B16].

A direct deformation of the arguments of Arkhipov and Bezrukavnikov seems to present nontrivial challenges at some key steps, cf. Remark 1.6.1 below. Partly for this reason, we have chosen to provide some alternative arguments at various points; we hope these provide a useful supplement to their perspective.

In the remainder of this introduction, we review for context the statement of tame local Betti geometric Langlands in Section 1.2, provide a precise statement of our results in Section 1.3, and discuss some aspects of the argument in Section 1.4.

1.2. Statement of tame local Betti geometric Langlands.

It is both helpful as context and requires less technical input to formulate the deformation of Bezrukavnikov’s equivalence, rather than that of Arkhipov–Bezrukavnikov, so this is where we begin.

In what follows, let G be a pinned complex reductive group. Fix an auxiliary coefficient field k of characteristic zero, and write \mathbf{G} for Langlands dual group defined over k .

On the automorphic side, we denote by Fl the enhanced affine flag variety of G , viewed as an ind-analytic variety over the complex numbers. As such, we may consider the derived category of weakly Iwahori–constructible sheaves of k -vector spaces on Fl in the analytic topology. We denote this category, which is naturally monoidal under convolution, by $\mathrm{Shv}_{(T)}(\mathrm{Fl})$.

Remark 1.2.1. One can get a good feel for this category, in particular its fine dependence on the chosen sheaf theory, simply by considering the monoidal unit. This object, for the usual reasons, is supported on the trivial Iwahori orbit. The full subcategory of $\mathrm{Shv}_{(T)}(\mathrm{Fl})$ supported on this orbit is canonically equivalent to the category of all local systems of k -vector spaces on the abstract Cartan T . Under the tautological identification of the group algebra $R := k[\pi_1(T)]$ with the ring of functions on the dual torus \mathbf{T} , we may equivalently think of this category as all quasicohherent sheaves on \mathbf{T} . Under this correspondence, the monoidal unit corresponds to the structure sheaf of \mathbf{T} , and individual character sheaves on T correspond to the skyscraper sheaves at closed points of \mathbf{T} . In this sense, the monoidal unit is the direct integral of all the character sheaves on T , and is a prototypical object with universal monodromy.

On the dual side, we write \mathbf{B} for the Borel subgroup of \mathbf{G} , and consider the tautological fiber product of adjoint quotient stacks $\mathbf{B}/\mathbf{B} \times_{\mathbf{G}/\mathbf{G}} \mathbf{B}/\mathbf{B}$. Its category of ind-coherent sheaves, which we denote by

$$\mathrm{IndCoh}(\mathbf{B}/\mathbf{B} \times_{\mathbf{G}/\mathbf{G}} \mathbf{B}/\mathbf{B})$$

¹In the case of surfaces without punctures, i.e., the global unramified case, their conjecture is now a theorem [ABC+24].

is naturally monoidal under convolution.

Having introduced notation, let us state Ben-Zvi–Nadler’s local conjecture, and explain its relation to Bezrukavnikov’s equivalence. The conjecture reads as follows.

Conjecture 1.2.2 ([BZN07, BZN18]). *There is a monoidal equivalence*

$$\mathrm{Shv}_{(I)}(\mathrm{Fl}) \simeq \mathrm{IndCoh}(\mathbf{B}/\mathbf{B} \times_{\mathbf{G}/\mathbf{G}} \mathbf{B}/\mathbf{B}).$$

To explain the connection with Bezrukavnikov’s equivalence, we need to first recall that both categories are naturally linear over $R \otimes R$. On the automorphic side, the linearity over $R \otimes R$ is afforded by the monodromy of sheaves along left and right Iwahori cosets. On the spectral side, the linearity over $R \otimes R$ comes from the two projections to \mathbf{B}/\mathbf{B} , postcomposed with the tautological map to the invariant theory quotient $\mathbf{B}/\mathbf{B} \rightarrow \mathbf{B}/\!\!/ \mathbf{B} \simeq \mathbf{T}$. We recall this again has an interpretation in terms of monodromy. Namely, the Steinberg stack is the moduli space of \mathbf{G} local systems on a cylinder with \mathbf{B} reductions around the two punctures. Under this identification, the maps to \mathbf{T} simply record the semi-simplified monodromy around each of the two punctures.

Let us specialize to the identity in $\mathbf{T} \times \mathbf{T}$, i.e., unipotent monodromy. With this, the automorphic side becomes Iwahori equivariant sheaves on the usual affine flag variety, and the spectral side becomes ind-coherent sheaves with nilpotent singular support on the (derived) stack $\mathbf{U}/\mathbf{B} \times_{\mathbf{G}/\mathbf{G}} \mathbf{U}/\mathbf{B}$. The matching of these two specializations as monoidal categories is precisely the result of Bezrukavnikov [B16].

In [DT24] we prove Conjecture 1.2.2 using the results of the present paper, which we now turn to.

1.3. Statement of results.

Our two main results are Theorems 1.3.1 and 1.3.3 below. Before giving their formal statements, we would like to first recall some relevant context.

At a high level, the argument of Bezrukavnikov establishing the main result of [B16] parallels an earlier argument of Kazhdan–Lusztig at the function theoretic level, identifying two Langlands dual realizations of the affine Hecke algebra [KL87].

We recall that the basic pattern of both arguments is visible in the following toy model. Consider a map of finite sets $X \rightarrow Y$. If we pass to the algebras of k -valued functions, pullback defines a map of algebras $\mathrm{Fun}(Y) \rightarrow \mathrm{Fun}(X)$, and one has

$$\mathrm{End}_{\mathrm{Fun}(Y)}(\mathrm{Fun}(X)) \simeq \mathrm{Fun}(X \times_Y X).$$

That is, the convolution algebra $\mathrm{Fun}(X \times_Y X)$ emerges from the more basic action of $\mathrm{Fun}(Y)$ on $\mathrm{Fun}(X)$. We emphasize that similar mechanisms, e.g. the action of a group on its flag variety, produce the appearances of Hecke algebras and categories throughout representation theory. In any case, given an algebra A , the datum of a map of algebras $\phi : A \rightarrow \mathrm{Fun}(X \times_Y X)$ is the same as an action of A on $\mathrm{Fun}(X)$ commuting with the action of $\mathrm{Fun}(Y)$, and for ϕ to be an isomorphism is the assertion that the action is faithful, with image the full centralizer of the action of $\mathrm{Fun}(Y)$. That is, in general, to prove an isomorphism of algebras $A_1 \simeq A_2$, it is of course enough to realize them as the same set of operators acting on the same vector space. However, when one of the algebras is presented as functions on a fiber product $\mathrm{Fun}(X \times_Y X)$, it admits a preferred choice of representation, namely $\mathrm{Fun}(X)$, and one additionally has an explicit description of the centralizer of the action, namely the image of $\mathrm{Fun}(Y)$ in $\mathrm{Fun}(X \times X)$.

When we replace $X \rightarrow Y$ as above with $\mathbf{U}/\mathbf{B} \rightarrow \mathbf{G}/\mathbf{G}$, and functions with Grothendieck groups of coherent sheaves, Kazhdan–Lusztig essentially found that $K_0(\mathbf{U}/\mathbf{B})$ is naturally identified with the underlying vector space of the antispherical representation of the affine Hecke algebra, i.e., Iwahori–Whittaker functions on affine flags. Moreover, under this identification, $K_0(\mathbf{G}/\mathbf{G})$ acted as the center of the affine Hecke algebra,

which then yielded the identification of the K -theory of the Steinberg with the affine Hecke algebra itself.² However, the double centralizer property did not hold literally in their setting, which complicates the argument. As we will see below, these complications disappear at the sheaf-theoretic level.

When we pass from Grothendieck groups of coherent sheaves to the categories of coherent sheaves themselves, it is again true that given a map $X \rightarrow Y$ between reasonable stacks, one has

$$\mathrm{End}_{\mathrm{QCoh}(Y)}(\mathrm{QCoh}(X)) \simeq \mathrm{QCoh}(X \times_Y X),$$

as shown by Ben-Zvi–Francis–Nadler [BZFN10]. In particular, given a monoidal category \mathcal{A} , the datum of a monoidal functor $\mathcal{A} \rightarrow \mathrm{QCoh}(X \times_Y X)$ is the same as an action of \mathcal{A} on $\mathrm{QCoh}(X)$ commuting with the action of $\mathrm{QCoh}(Y)$. With this in mind, to identify coherent sheaves on Steinberg with the affine Hecke category, one should in particular show that the natural categorifications of the antispherical representation on the automorphic and spectral sides, namely Iwahori–Whittaker sheaves on affine flags and $\mathrm{QCoh}(U/B)$, respectively, are still canonically equivalent. I.e., one again wants to identify both monoidal categories as the ‘same’ endofunctors of the ‘same’ category. The assertion that the two natural modules are still the ‘same’ is the main result of [AB09].

In the universal case, the preceding discussion suggests we should replace U/B with B/B , and match it with a suitably defined category of universal Iwahori–Whittaker sheaves on Fl . The latter is slightly delicate to set up, but we do so and denote it by $\mathrm{Shv}_{(I,\chi)}(\mathrm{Fl})$. The first task of this paper is then to prove the following.

Theorem 1.3.1. *There is a canonical equivalence of categories*

$$\mathrm{Shv}_{(I,\chi)}(\mathrm{Fl}) \simeq \mathrm{QCoh}(B/B).$$

To motivate our second main result, note that to obtain from Theorem 1.3.1 a monoidal functor

$$\mathrm{Shv}_{(I)}(\mathrm{Fl}) \rightarrow \mathrm{QCoh}(B/B \times_{G/G} B/B),$$

it remains to argue why the action of $\mathrm{Shv}_{(I)}(\mathrm{Fl})$ on $\mathrm{QCoh}(B/B)$ commutes with the action of $\mathrm{QCoh}(G/G)$.

Remark 1.3.2. We note this direct naive approach to producing the functor between the two sides³ of Conjecture 1.2.2, which is the one used in [DT24], mildly differs from the strategy of [B16]. But, we should emphasize, it differs even less mildly from unpublished approaches of Gaitsgory–Lurie, Ben-Zvi–Nadler, and Arinkin–Bezrukavnikov to derived geometric Satake; the first of these approaches was recently realized by Campbell–Raskin [CR23].

To produce a datum of commutativity between the actions of $\mathrm{Shv}_{(I)}(\mathrm{Fl})$ and $\mathrm{QCoh}(G/G)$, we should look for an automorphic realization of the action of $\mathrm{QCoh}(G/G)$ on $\mathrm{Shv}_{(I,\chi)}(\mathrm{Fl})$. In fact, we show it is simply the convolution action of bi-Iwahori–Whittaker sheaves on the loop group of G , which we denote by ${}_{\chi}\mathbf{H}_{\chi}$. We recall this is the most obvious candidate, i.e., is a monoidal category acting on $\mathrm{Shv}_{(I,\chi)}(\mathrm{Fl})$ by convolution on the ‘other side’ from the affine Hecke category. Moreover, this matching is actually forced by more general expectations in local geometric Langlands.

At all events, our second basic result in this paper is the following.

Theorem 1.3.3. *There is an equivalence of monoidal categories*

$${}_{\chi}\mathbf{H}_{\chi} \simeq \mathrm{QCoh}(G/G),$$

compatible with their actions on both sides of Theorem 1.3.1.

²In truth, we recall one needs to consider a further \mathbf{G}_m -grading, to account for the parameter q in the affine Hecke algebra.

³Strictly speaking, of course we are producing the spectral side up to renormalization issues, i.e., obtaining the further projection from IndCoh to QCoh .

In the case of unipotent monodromy, a version of Theorem 1.3.3 was first observed by Bezrukavnikov [B09]. Slightly more carefully, his work was not explicitly concerned with the monoidal structure, but rather matched the perverse t-structure on the automorphic side with the perverse coherent t-structure on the spectral side, and developed implications for two-sided cells in the affine Hecke category. An explicitly monoidal equivalence appeared later in the work of Chen and the first named author [CD23], building on unpublished ideas of Arinkin–Bezrukavnikov.

Given the two results of this paper, in [DT24] we essentially obtain Conjecture 1.2.2 by a double centralizer argument.

1.4. Structure of the argument.

We now discuss the proofs of Theorems 1.3.1 and 1.3.3, with a view towards to what we leave undisturbed from [AB09], and we have opted to change.

The argument of [AB09] roughly consists of two steps. In the first step, they construct a functor from the spectral side to the automorphic side, and in the second step they prove that it is an equivalence. In short, there are very good reasons not to disturb the first step, so we do not. However the second step can be modified, and we choose to give a different argument.

We now recall, in slightly more detail, the basic strategy of [AB09], stated *mutatis mutandis* in the universal setting. On the spectral side, the composition

$$\mathrm{QCoh}(\mathbf{B}/\mathbf{B}) \xrightarrow{\Delta_*} \mathrm{IndCoh}(\mathbf{B}/\mathbf{B} \times_{\mathbf{G}/\mathbf{G}} \mathbf{B}/\mathbf{B}) \xrightarrow{\pi_*} \mathrm{QCoh}(\mathbf{B}/\mathbf{B})$$

is tautologically equivalent to the identity, where Δ and π respectively denote the diagonal and second projection. Note that Δ_* is monoidal, with respect to the usual tensor product on the source and convolution on the target, and similarly π_* can be interpreted as convolution with the structure sheaf of \mathbf{B}/\mathbf{B} . To orient the reader, we emphasize that the basic pattern here follows its analogue in the toy model of $\mathrm{Fun}(X \times_Y X)$.

From the identification of the Grothendieck groups with the antispherical representation, we expect the structure sheaf of \mathbf{B}/\mathbf{B} should be exchanged with the ‘simplest’ object on the automorphic side, namely the Iwahori–Whittaker sheaf with minimal support. On the automorphic side, convolution with this object amounts to performing Whittaker averaging $\mathrm{Shv}_{(I)}(\mathrm{Fl}) \rightarrow \mathrm{Shv}_{(I,\chi)}(\mathrm{Fl})$. Therefore assuming Conjecture 1.2.2, we can rewrite this as

$$\mathrm{QCoh}(\mathbf{B}/\mathbf{B}) \xrightarrow{\Delta_*} \mathrm{IndCoh}(\mathbf{B}/\mathbf{B} \times_{\mathbf{G}/\mathbf{G}} \mathbf{B}/\mathbf{B}) \simeq \mathrm{Shv}_{(I)}(\mathrm{Fl}) \rightarrow \mathrm{Shv}_{(I,\chi)}(\mathrm{Fl}).$$

The first step of [AB09] is the direct construction of the monoidal functor $\mathrm{QCoh}(\mathbf{B}/\mathbf{B}) \rightarrow \mathrm{Shv}_{(I)}(\mathrm{Fl})$, and the second step is verifying that the composite $\mathrm{QCoh}(\mathbf{B}/\mathbf{B}) \rightarrow \mathrm{Shv}_{(I,\chi)}(\mathrm{Fl})$ is indeed an equivalence.

Although we closely follow the arguments of Arkhipov–Bezrukavnikov in Part I, our paper is logically self-contained when G is classical, in the sense that we do not assume the results of [AB09]. In particular, we bypass their use of the regular centralizer, cf. the discussion in Section 1.6 below. However for G exceptional, we do use a result of [BFO09], depending on Arkhipov–Bezrukavnikov’s regular centralizer arguments, to prove that Whittaker averaged central sheaves are tilting.

1.5. Structure of the argument: part I.

Arkhipov–Bezrukavnikov’s construction. Let us recall the idea of part I of the argument. Ignoring at first the monodromy of the local systems, one wants to construct a monoidal functor $\mathrm{Rep}(\mathbf{B}) \rightarrow \mathrm{Shv}_{(I)}(\mathrm{Fl})$. This is difficult essentially due to the complexity of $\mathrm{Rep}(\mathbf{B})$. The nearby categories $\mathrm{Rep}(\mathbf{G})$ and $\mathrm{Rep}(\mathbf{T})$ are somewhat simpler, and one may use them via the following maneuver, sometimes called *Drinfeld–Plücker formalism*. Namely, one considers the usual parabolic induction diagram $\mathbf{T} \leftarrow \mathbf{B} \rightarrow \mathbf{G}$, and notes that in terms of this

$$\mathrm{pt}/\mathbf{B} \simeq \mathbf{G} \backslash (\mathbf{G}/\mathbf{U}) / \mathbf{T},$$

compatibly with maps to pt/\mathbf{G} and pt/\mathbf{T} . Let us pretend that \mathbf{G}/\mathbf{U} is affine, so that $\mathrm{Rep}(\mathbf{B})$ is approximately the same as $\mathrm{Rep}(\mathbf{B})' := \mathrm{Fun}(\mathbf{G}/\mathbf{U})\text{-mod}(\mathrm{Rep}(\mathbf{G} \times \mathbf{T}))$. Given the simple structure of $\mathrm{Fun}(\mathbf{G}/\mathbf{U})$ as an algebra,

as dictated by the Peter–Weyl theorem, a monoidal functor out of $\text{Rep}(\mathbf{B})'$ is the same data as a monoidal functor from $\text{Rep}(\mathbf{G} \times \mathbf{T})$, along with appropriate highest weight arrows between the objects of $\text{Rep}(\mathbf{G})$ and $\text{Rep}(\mathbf{T})$ indexed by the same dominant weights.

In the setting of [AB09], one already has Gaiitgory’s functor $\text{Rep}(\mathbf{G}) \rightarrow \text{Shv}_{(I)}(\text{Fl})$, constructed via geometric Satake and the factorization action of the affine Grassmannian on affine flags, i.e., via nearby cycles. One similarly already has a functor $\text{Rep}(\mathbf{T}) \rightarrow \text{Shv}_{(I)}(\text{Fl})$, essentially obtained from the torus case by parabolic induction; the essential image are the Wakimoto sheaves. As [AB09] observe, one has natural highest weight arrows of the required form, essentially for support reasons, and this gives the desired functor from $\text{Rep}(\mathbf{B})$; the monodromy of nearby cycles then provides the desired lift to $\text{QCoh}(\mathbf{B}/\mathbf{B})$.

Remark 1.5.1. Some of the fundamental ideas in the above argument fit into subsequent language as follows: for an algebraic group \mathbf{H} , an E_2 -action of the symmetric monoidal category $\text{Rep}(\mathbf{H})$ is the same data as a usual, i.e., E_1 -, action of its integral over the circle, namely $\text{QCoh}(\mathbf{H}/\mathbf{H})$.

More truthfully, this argument is complicated by the non-affineness of \mathbf{G}/\mathbf{U} , and circumventing this is one of the technical feats of [AB09]. Their basic idea is as follows. While \mathbf{G}/\mathbf{U} is not affine, it is not so far from it. Namely, the map to the affinization $\mathbf{G}/\mathbf{U} \rightarrow (\mathbf{G}/\mathbf{U})^{\text{aff}}$ is an open embedding, and the argument so far has actually produced a functor F out of

$$\text{Rep}(\mathbf{B})' \simeq \text{QCoh}(\mathbf{G} \setminus (\mathbf{G}/\mathbf{U})^{\text{aff}} / \mathbf{T}).$$

To argue F really does factor through restriction to $\text{QCoh}(\mathbf{G} \setminus (\mathbf{G}/\mathbf{U})/\mathbf{T})$, one must further check that F kills all objects supported on the boundary of $(\mathbf{G}/\mathbf{U})^{\text{aff}}$, i.e., those objects whose restrictions to the open orbit vanish. By equivariance, this is the same as asking that the fiber at the identity coset in $(\mathbf{G}/\mathbf{U})^{\text{aff}}$ should vanish. The latter fiber retains an action of \mathbf{T} , and the resulting functor $\text{Rep}(\mathbf{B}) \rightarrow \text{Rep}(\mathbf{T})$ is given by semi-simplification, i.e., passing to the associated graded representation. For this reason, [AB09] show that highest weight arrows out of central sheaves are the last step in Wakimoto filtrations. They then inspect the functor of taking the associated graded and by a Tannakian argument identify it with the above fiber functor, which completes the first step of the argument.

Applications. In applications, the form of Arkhipov–Bezrukavnikov’s approach to part I of the argument is crucial. This is notably the case in the work of Frenkel–Gaiitgory on localization theory for affine Lie algebras at critical level, essentially due to the similar use of factorization, i.e. vertex operator algebraic, methods in that theory [FG09]. In extensions of their work to treat general highest weight modules, which will be performed elsewhere, one needs again the compatibility with factorization; see e.g. the interesting work of Færgeman on the motivicity of rigid local systems for recent uses of this expected extension [F24].

Relatedly, one expects similar nearby cycles constructions for affine Hecke categories of Bernstein blocks of arbitrary depth, and for local global compatibilities, as in the work of Genestier–Lafforgue in arithmetic [GL18], it will likely be important that spectral descriptions of the blocks are compatible with the action of $\text{QCoh}(\mathbf{G}/\mathbf{G})$.

Universal monodromic deformation. The practical consequence for us of the preceding discussion is that the first step of [AB09], i.e. the construction of the functor, should not be modified. Accordingly, in Part I of this paper we perform the necessary checks that this indeed works, i.e. deforms over all conjugacy classes in the desired way.

Here, almost all the arguments and constructions of [AB09], as well as those of Gaiitgory [G01], essentially deform straightforwardly. We now briefly describe the two closely related exceptions. The first is the verification of the Plücker relations, performed in Section 7. The problem here is that writing down a canonical highest weight arrow in the universal setting is slightly more delicate, because loop rotation no longer fixes the standard base points where one rigidifies the sheaves. The second exception is checking that the associated graded of the Wakimoto filtrations on central sheaves takes the expected form, performed in Section 8. The problem here is that, after running the Tannakian formalism, it remains to further identify

a certain G -bundle over T/T . In Arkhipov–Bezrukavnikov’s unipotent monodromic setting, the analogous G -bundle is over pt/T , so when the coefficient field is algebraically closed the problem is vacuous.

With that said, the reader already familiar with [AB09], or the wonderful book of Achar–Riche [AR24], may wish to simply take the first part of our paper on faith, and skip directly to the second part, which we now describe.

1.6. Structure of the argument: part II.

With the monoidal functor $\mathrm{QCoh}(\mathbf{B}/\mathbf{B}) \rightarrow \mathrm{Shv}_{(I)}(\mathbf{Fl})$ in hand, it remains to do the following two things.

- (i) First, one needs to construct an appropriate Iwahori–Whittaker category $\mathrm{Shv}_{(I,\chi)}(\mathbf{Fl})$, equipped with an action of $\mathrm{Shv}_{(I)}(\mathbf{Fl})$ and an averaging functor $\mathrm{Shv}_{(I)}(\mathbf{Fl}) \rightarrow \mathrm{Shv}_{(I,\chi)}(\mathbf{Fl})$. I.e., we need the analogue of the antispherical representation along with its canonical generator.
- (ii) Second, one needs to show the composite map $\mathrm{QCoh}(\mathbf{B}/\mathbf{B}) \rightarrow \mathrm{Shv}_{(I,\chi)}(\mathbf{Fl})$ is an equivalence.

Let us describe our approach to both points.

Whittaker sheaves. In our estimation, step (i) is deceptively subtle, and may well be why the equivalences proven in this paper and its sequel were not obtained earlier. The problem is as follows. At a fixed semi-simplified monodromy, one has many standard recipes for defining Whittaker categories. Namely, for ℓ -adic sheaves, one may work with an affine flag variety over a field of positive characteristic, and use the Artin–Schreier sheaf. For D -modules, one may use the exponential D -module. For Betti sheaves at a fixed monodromy, Gaitsgory introduced a version of the Kirillov model, familiar from the representation theory of $GL_2(\mathbb{F}_q)$, which has the desired behavior [G18].

When we work with Betti sheaves with varying monodromy, none of the above approaches work: ℓ -adic, de Rham, and Betti families of local systems give genuinely different moduli spaces, which rules out the first two approaches. For the remaining Kirillov model, one needs to impose a certain \mathbf{C}^\times -equivariance in its construction, which is incompatible with universal monodromy. Constructing the desired universal Whittaker model is therefore one of the basic problems one must confront here.

Fortunately, the solution was essentially supplied in [LNY24, IY23, T23], where the authors defined the Whittaker functional on universal monodromic Hecke categories in terms of vanishing cycles, and proved that it is monoidal; such a microlocal approach to Whittaker functionals appears also in [N06, NT24, FR22]. We use monoidality of the Whittaker functional to construct the Whittaker module for the universal finite Hecke category. The desired category $\mathrm{Shv}_{(I,\chi)}(\mathbf{Fl})$ is then obtained by induction, exactly as one defines the antispherical representation of the affine Hecke algebra.

Fully faithfulness. Let us now turn to step (ii), i.e., arguing the obtained functor is an equivalence. Here, it is helpful to recall two extremely influential ideas of Soergel, first deployed together in the Category \mathcal{O} [S90]. The first idea, known as Soergel’s Struktursatz, is that to calculate homomorphisms between tilting objects or their variants, the answers are unchanged after passing to the quotient by all objects whose singular support does not touch the regular nilpotent orbit. The second idea, known as Soergel’s deformation philosophy, is essentially that the answers will also be flat in the highest weight, so that one may perform calculations after deforming to nearby simpler categories.

In some sense, [AB09] apply Soergel’s first idea to prove step (ii), whereas we apply Soergel’s second idea. That is, they localize in the nilpotent directions, whereas we localize in the semi-simple ones. Let us say this all more precisely, beginning with the approach of [AB09].

Localization in the nilpotent directions. Momentarily, suppose one already knows an equivalence between $\mathrm{IndCoh}_{\mathrm{nilp}}(\mathbf{U}/\mathbf{B} \times_{\mathbf{G}/\mathbf{G}} \mathbf{U}/\mathbf{B})$ and the automorphic affine Hecke category. Within \mathbf{U}/\mathbf{B} , one has the unique open orbit of regular unipotent elements $\mathbf{U}^{\mathrm{reg}}/\mathbf{B} \simeq \mathrm{pt}/\mathbf{Z}$, where \mathbf{Z} is the centralizer of any particular regular unipotent element e . It follows that one has a monoidal localization

$$\mathrm{IndCoh}_{\mathrm{nilp}}(\mathbf{U}/\mathbf{B} \times_{\mathbf{G}/\mathbf{G}} \mathbf{U}/\mathbf{B}) \rightarrow \mathrm{IndCoh}_{\mathrm{nilp}}(\mathrm{pt}/\mathbf{Z} \times_{\mathbf{G}/\mathbf{G}} \mathrm{pt}/\mathbf{Z}).$$

The latter open substack has much simpler geometry - as the centralizer of e in G is again given by Z , the latter fiber product is essentially pt/Z , up to a derived nil-thickening arising from the self-intersection.⁴

To reduce calculations to inspecting this regular locus, and in particular the fully faithful embedding from $\text{Rep}(Z)$ to the Iwahori–Whittaker category, one would like to explicitly identify this simple quotient in automorphic terms. This was done, among other things, by Bezrukavnikov in the work [B04], motivated by work of Lusztig and Vogan, particularly the work of Lusztig on two-sided cells in the affine Hecke category, where it corresponds to the smallest two sided cell [Lus89, V91]. We recall that these considerations also have a direct analogue in arithmetic, namely how the center of the affine Hecke algebra acts on subquotients of unramified principal series, with the principal orbit corresponding to the character of the Steinberg representation.

This was the localization in nilpotent directions used by [AB09]. Namely, they prove fully faithfulness via further restriction to this microlocalization, which they identify with $\text{Rep}(Z)$ using a Tannakian argument.

Remark 1.6.1. Extending the strategy of [AB09] to the universal monodromic case, i.e., directly proving an equivalence between $\mathbf{B}^{\text{reg}}/B$ with a suitable microlocalization of $\text{Shv}_{(I)}(\mathbb{F}1)$, does not appear to be straightforward. The obstruction is explained by Bezrukavnikov–Riche in the discussion following Theorem 1.3 of [BR22]. Namely the general result, Proposition 1 of [B04], that was used in [AB09] to construct the fiber functor out of the microlocalized category, does not appear to work in families. We expect, however, that it may be possible to factor the fiber functor through microlocalization using an affine nondegenerate Whittaker model as in [DLYZ].

In contrast to the approach of [AB09], the approach we will describe below does not seem to have a direct analogue at the function theoretic level, as characters of an Iwahori form a discrete group, i.e., no longer have moduli.

Localization in the semi-simple directions. In the present work, we will perform calculations following Soergel’s second idea, i.e., the deformation philosophy. Namely, we first check that various spaces of homomorphisms on both sides of the equivalence are vector bundles over \mathbb{T} , i.e., do not jump as the semi-simplified monodromy varies. As such, to verify fully faithfulness, by Hartogs’ lemma it is enough to work on an open subset of \mathbb{T} whose complement is codimension two. This allows us to throw away the intersections of root hyperplanes in \mathbb{T} , after which, working locally around each wall separately, we essentially reduce to the case of G being semi-simple rank one.

In the rank one case, we then give a direct analysis of the relevant spaces of homomorphisms. The main content is a certain order of vanishing calculation, involving the associated graded of the Wakimoto filtration on central sheaves and its spectral analogue. On the spectral side, we perform the order of vanishing calculation in Section 11.6 by further restricting to the regular locus \mathbf{B}^{reg} . On the automorphic side, we perform the order of vanishing calculation in Section 14 using the weight filtration.

These calculations seem to be new, are particular to the affine case, and in our view are interesting in their own right. With that said, we also recall that a similar idea, namely controlling homomorphisms between tilting objects in monodromic Hecke categories via the associated graded, appears in earlier work of Bezrukavnikov–Riche [BR21].

The bi-Whittaker equivalence. Finally, let us briefly comment on the argument for Theorem 1.3.3, i.e., the equivalence

$${}_x\mathbf{H}_\chi \simeq \text{QCoh}(G/G).$$

We construct a monoidal functor from $\text{QCoh}(G/G)$ to ${}_x\mathbf{H}_\chi$ as follows. Gaitsgory’s central sheaves already provide a monoidal functor from $\text{QCoh}(G/G)$ to the center of $\text{Shv}_{(I)}(\mathbb{F}1)$, and in particular to the equivariant

⁴In fact, the derived thickening comes exactly from the semi-simple deformation directions we are concerned with.

endomorphisms of any $\mathrm{Shv}_{(I)}(\mathrm{Fl})$ -module. Specializing to the Whittaker module $\mathrm{Shv}_{(I,\chi)}(\mathrm{Fl})$, we obtain the desired monoidal functor

$$\mathrm{QCoh}(\mathbf{G}/\mathbf{G}) \rightarrow \mathrm{End}_{\mathrm{Shv}_{(I)}(\mathrm{Fl})}(\mathrm{Shv}_{(I,\chi)}(\mathrm{Fl})) \simeq {}_{\chi}\mathrm{H}_{\chi}.$$

That this is an equivalence of categories then follows from Theorem 1.3.1 essentially by faithfully flat descent along $\tilde{\mathbf{G}}^{\mathrm{aff}} \rightarrow \mathbf{G}$, where $\tilde{\mathbf{G}}^{\mathrm{aff}}$ denotes the affinization of $\tilde{\mathbf{G}}$.

Remark 1.6.2. In the course of the latter analysis, we use a description of the affinization $\tilde{\mathbf{G}}^{\mathrm{aff}}$ which we were unable to locate in the literature. Namely, if \mathbf{G} has simply connected derived subgroup, then the natural map $\tilde{\mathbf{G}}^{\mathrm{aff}} \rightarrow \mathbf{G} \times_{\mathbf{G}/\mathbf{G}} \mathbf{T}$ is an isomorphism. In general, this is not true, and instead one has the following. If we choose a finite isogeny $1 \rightarrow \mathbf{Z} \rightarrow \mathbf{G}^{\mathrm{sc}} \rightarrow \mathbf{G} \rightarrow 1$, wherein \mathbf{G}^{sc} has simply connected derived subgroup, then

$$\tilde{\mathbf{G}}^{\mathrm{aff}} \simeq \mathbf{G} \times_{(\mathbf{G}^{\mathrm{sc}}//\mathbf{G}^{\mathrm{sc}})/\mathbf{Z}} \mathbf{T}.$$

1.7. Organization of the paper.

Let us briefly indicate the contents of the subsequent sections. As discussed above, we have divided the paper into two parts, corresponding to the first and second steps of the argument.

- In Section 2, we collect some mostly standard notation.

Contents of Part I.

- In Section 3, we provide an overview of Part I of the paper.
- In Section 4 we gather some basic results about the automorphic category.
- In Section 5, we discuss the universal monodromic Wakimoto sheaves.
- In Section 6, we discuss the universal monodromic version of Gaitsgory’s construction of central sheaves via nearby cycles, and establish that central sheaves have Wakimoto filtrations.
- In Section 7, we produce the highest weight arrows between the central sheaves and Wakimoto sheaves.
- In Section 8, we give a Tannakian description of the associated graded of the Wakimoto filtration on central sheaves.
- Finally, in Section 9, we assemble the previous ingredients to construct a monoidal functor from $\mathrm{QCoh}(\mathbf{B}/\mathbf{B})$ to $\mathrm{Shv}_{(I)}(\mathrm{Fl})$, thereby completing Part I.

Contents of Part II.

- In Section 10, we provide an overview of Part II of the paper.
- In Section 11, we study the global sections of certain vector bundles on $\tilde{\mathbf{G}}$, and perform the order of vanishing calculation on the spectral side.
- In Section 12, we define the category of universal monodromic Whittaker sheaves in the finite and affine cases.
- In Section 13, we show that the Whittaker averages of central sheaves are tilting.
- In Section 14, we perform the order of vanishing calculation on the automorphic side.
- In Section 15, study the Wakimoto filtrations on central sheaves after localizing away from all but one wall.
- In Section 16, we prove our first main Theorem 1.3.1, the Whittaker equivalence.
- In Section 17 we prove our second main Theorem 1.3.3, the bi-Whittaker equivalence, thereby completing Part II.

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2. NOTATION

2.1. Reductive groups. Fix a coefficient field k of characteristic 0.

Let G be a pinned complex reductive group. Let $U \subset B$ be the unipotent radical of the Borel. Write Λ for the coweight lattice of the maximal torus $T \subset G$. Let Λ^+ be the dominant coweights, and Λ^{++} be the strictly dominant coweights. Let W^f denote the finite Weyl group.

Let \mathbf{G} be the Langlands dual group defined over k . Let V_λ be the irreducible \mathbf{G} -module of highest weight $\lambda \in \Lambda^+$. Let $\mathbf{U} \subset \mathbf{B}$ the unipotent radical of the *negative* Borel.⁵ Let $\tilde{\mathbf{G}} := \mathbf{G} \times^{\mathbf{B}} \mathbf{B}$ be the universal Grothendieck–Springer variety. Let $(\mathbf{G}/\mathbf{U})^{\text{aff}} := \text{Spec}(\mathcal{O}(\mathbf{G}/\mathbf{U}))$ be the affine closure of base affine space. Write $R := k[\Lambda]$ for the ring of functions on the maximal torus $\mathbf{T} \subset \mathbf{G}$.

Let $Z \subset G$ be the center of the derived subgroup of G ; note that Z is a finite central subgroup of G . Let $\mathbf{Z} := \text{Hom}(Z, \mathbf{C}^\times)$ be its Cartier dual. Then $G^{\text{ad}} := G/Z$ is the product of a torus and an adjoint reductive group. Let \mathbf{G}^{sc} be the Langlands dual group to G^{ad} . Then \mathbf{G}^{sc} is the product of a torus and a simply connected reductive group, $\mathbf{Z} \subset \mathbf{G}^{\text{sc}}$ is a finite central subgroup, and $\mathbf{G} = \mathbf{G}^{\text{sc}}/\mathbf{Z}$.

If the derived subgroup of \mathbf{G} is not simply connected, then the naïve characteristic polynomial map should be replaced by

$$\mathbf{G} = \mathbf{G}^{\text{sc}}/\mathbf{Z} \rightarrow \mathbf{C} := (\mathbf{G}^{\text{sc}}//\mathbf{G}^{\text{sc}})/\mathbf{Z}.$$

2.2. Loop groups. Let \mathring{I} be the pro-unipotent radical of the Iwahori subgroup $I \subset G[[t]]$. The enhanced affine flag variety $\text{Fl} := G((t))/\mathring{I}$ is a G/U -torsor over the affine Grassmannian $\text{Gr} := G((t))/G[[t]]$.⁶

Let $W := W^f \rtimes \Lambda$ be the extended affine Weyl group. Let $\ell(w)$ denote the length of an element $w \in W$. Let \mathbf{R} be the set of affine coroots and \mathbf{R}^+ be the set of positive affine coroots.

2.3. Constructible sheaves. Suppose H is a complex algebraic group and $Y = \text{colim } Y_i$ an ind-complex analytic H -space. Our coefficient field k is assumed to be characteristic 0.

Let $\text{Shv}_{(H)}(Y) = \lim^! \text{Shv}_{(H)}(Y_i)$ be the limit under $!$ -pullback of the weakly H -constructible unbounded derived categories of sheaves of k -vector spaces on Y_i . Weakly H -constructible means locally constant along the H -orbit. We work in the analytic topology and impose no finiteness conditions on stalks.

Let $\text{Perv}_{(H)}(Y)$ be the abelian subcategory of perverse sheaves. We allow perverse sheaves to have infinite dimensional stalks, but require that they are compact in $\text{Shv}_{(H)}(Y)$.

All derived categories are interpreted in the DG sense.

2.4. Coherent sheaves. Suppose \mathbf{H} is an algebraic group acting on a stack \mathbf{Y} , both defined over k . We use the following notation for some associated categories of sheaves.

- Let $\text{Rep}(\mathbf{H})$ be the abelian category of finite dimensional \mathbf{H} -modules.
- Let $\text{Free}_{\mathbf{H}}(\mathbf{Y})$ be the additive (underived) category of \mathbf{H} -equivariant free coherent sheaves.

⁵We choose the negative Borel so that the line bundle $\mathbf{G} \times_{\mathbf{B}} k_\lambda \simeq \mathcal{O}(\lambda)$ is ample when $\lambda \in \Lambda^{++}$.

⁶Beware that the enhanced affine flag variety is usually denoted with a tilde in the literature.

- Let $\text{Perf}_{\mathbb{H}}(\mathcal{Y})$ be the derived category of perfect complexes, the bounded homotopy category of $\text{Free}_{\mathbb{H}}(\mathcal{Y})$.
- Let $\text{QCoh}_{\mathbb{H}}(\mathcal{Y})$ be its ind-completion, the unbounded derived category of quasi-coherent sheaves.
- Let $\text{Coh}_{\mathbb{H}}(\mathcal{Y})$ be the full subcategory in $\text{QCoh}_{\mathbb{H}}(\mathcal{Y})$ of bounded coherent complexes.
- Let $\text{IndCoh}_{\mathbb{H}}(\mathcal{Y})$ be the ind-completion of $\text{Coh}_{\mathbb{H}}(\mathcal{Y})$.

Part I. Construction of the functor

3. OUTLINE OF PART I

The goal of this first part of the paper is to construct, in the present universal monodromic setting, the Arkhipov–Bezrukavnikov functor, a certain monoidal functor

$$F : \text{QCoh}_{\mathbb{G}}(\tilde{\mathbb{G}}) \rightarrow \text{Shv}_{(I)}(\mathbb{F}\ell).$$

The result itself appears in Section 9, let us now indicate the steps used in the construction.

After collecting some preliminaries in Sections 4 and 5, the starting point is a universal monodromic version of Gaitsgory’s central functor, sending representations of \mathbb{G} to Wakimoto filtered perverse sheaves on the enhanced affine flag variety

$$Z : \text{Rep}(\mathbb{G}) \rightarrow \text{Perv}_{(I)}^{\text{waki}}(\mathbb{F}\ell).$$

This is constructed in Section 6 using nearby cycles from the affine Grassmannian to the affine flag variety, exactly as in the original work [G01]. Our proof that central sheaves admit Wakimoto filtrations is similar to the proof of Proposition 5 of [AB09], with the mild caveat that we need the extra notion of universal perversity.

In Section 7 we provide the extra data needed to extend the central functor to $\text{Rep}(\mathbb{B})$, that is, the highest weight arrows from central to Wakimoto sheaves. Checking the relevant Plücker relations between these highest weight arrows is more delicate in the universal monodromic setting, because a certain trivialization of the relevant stratum in Gaitsgory’s family does not extend globally.

Using monodromy of nearby cycles and highest weight arrows, we upgrade the central functor to a monoidal functor

$$F : \text{Free}_{\mathbb{G} \times \mathbb{T}}(\mathbb{G} \times (\mathbb{G}/\mathbb{U})^{\text{aff}} \times \mathbb{T}) \rightarrow \text{Perv}_{(I)}^{\text{waki}}(\mathbb{F}\ell)$$

such that

- (\star) after taking associated graded for the Wakimoto filtration, $\text{gr } F$ is isomorphic to restriction along $\mathbb{T} \rightarrow \mathbb{G} \times (\mathbb{G}/\mathbb{U})^{\text{aff}} \times \mathbb{T}$ sending $t \mapsto (t, 1, t)$.

The proof of (\star) requires additional arguments not necessary in the setting of [AB09], and we defer a discussion of this to the final paragraph of this section.

Assuming (\star), to get a functor out of $\text{QCoh}_{\mathbb{G}}(\tilde{\mathbb{G}})$ there are two remaining steps, which are performed in Section 9.

- (i) First we factor F through a closed subscheme $\tilde{\mathbb{G}}^{\text{aff}} \subset \mathbb{G} \times (\mathbb{G}/\mathbb{U})^{\text{aff}} \times \mathbb{T}$ cut out by a certain compatibility (33) between the monodromy and highest weight arrows.
- (ii) Second we factor F through an open subscheme $\tilde{\mathbb{G}}^{\text{un}} \subset \tilde{\mathbb{G}}^{\text{aff}}$, a \mathbb{T} -torsor over $\tilde{\mathbb{G}}$. Here we use (\star) to show that, after passing to homotopy categories, F kills complexes supported on the closed boundary.

Here, the arguments for (i) and (ii) are largely similar to ones that appeared in [AB09], [B16], and [AR24].

Finally, let us comment on the assertion (\star), whose proof appears in Sections 8 and 9. The proof crucially uses that $\text{gr } Z : \text{Rep}(\mathbb{G}) \rightarrow \text{Free}_{\mathbb{T}}(\mathbb{T})$ is *monoidally* isomorphic to restriction to the maximal torus.⁷ Following

⁷It is possible to construct such an isomorphism of plain functors using Braden’s theorem, but it does not appear straightforward to prove directly that this isomorphism intertwines the monoidal structures.

Arkhipov–Bezrukavnikov, $\text{gr } Z$ is a tensor functor and therefore identifies under Tannakian formalism with pullback along some map $\mathbb{T}/\mathbb{T} \rightarrow \text{pt}/G$. If k is not algebraically closed, it is not immediate how to identify this G -torsor on \mathbb{T}/\mathbb{T} , unlike in the setting of [AB09]. In Section 8.2, we use the Plücker relations relations to construct a \mathbb{B} -reduction of the G -torsor. We then observe there is a unique isomorphism class of \mathbb{B} -torsor on \mathbb{T}/\mathbb{T} whose associated graded \mathbb{T} -torsor is isomorphic to the pullback of the universal \mathbb{T} -torsor from pt/\mathbb{T} , thereby allowing us to identify the G -torsor.

4. UNIVERSAL MONODROMIC SHEAVES

Here we collect some basic results about universal monodromic sheaves on the enhanced affine flag variety. We recall the usual convolution formulas, and introduce the notion of universal perversity which combines perversity with flatness.

4.1. Standard and costandard extensions. If $w \in W := W^f \rtimes \Lambda$, let $j_w : \text{Fl}_w := I\dot{w} \hookrightarrow \text{Fl}$ be the Iwahori orbit through a lift $\dot{w} \in G((t))$. Let R_{Fl_w} be the perverse universal local system on $\text{Fl}_w \simeq T \times \mathbf{C}^{\ell(w)}$, i.e., the regular representation of the fundamental group. Define the universal standard and costandard sheaves by

$$\Delta_w := j_{w!} R_{\text{Fl}_w} \quad \text{and} \quad \nabla_w := j_{w*} R_{\text{Fl}_w} \in \text{Perv}_{(I)}(\text{Fl}).$$

Without having chosen a base point, these sheaves are defined uniquely, but only up to non-canonical isomorphism. After choosing a lift $\dot{w} \in G((t))$, they become unique up to canonical isomorphism, and we denote these canonical lifts by $R_{\text{Fl}_{\dot{w}}}$, $\Delta_{\dot{w}}$, $\nabla_{\dot{w}}$.

We prove in Lemma 4.2.4 below that these universal standard and costandard sheaves are indeed perverse. By adjunction

$$(1) \quad \text{RHom}(\Delta_{\dot{w}}, \nabla_{\dot{v}}) \simeq R \quad \text{and} \quad \text{RHom}(\Delta_w, \nabla_v) \simeq 0 \quad \text{for} \quad w \neq v.$$

Let s be a simple reflection and α the corresponding simple coroot. As explained in [T23], there is a short exact sequence of perverse sheaves

$$(2) \quad 0 \rightarrow \Delta_{\dot{s}} \rightarrow \nabla_{\dot{s}} \rightarrow \nabla_1 / (e^\alpha - 1) \rightarrow 0.$$

If $A, B \in \text{Shv}_{(I)}(\text{Fl})$ define the convolution

$$A * B := m_!(A \tilde{\boxtimes} B)[\dim T] \simeq m_*(A \tilde{\boxtimes} B) \quad \text{where} \quad m : \text{Fl} \tilde{\times} \text{Fl} := G((t)) \times^I \text{Fl} \rightarrow \text{Fl}.$$

Proposition 4.1.1. *Let $v, w \in W$.*

- (a) *There are canonical isomorphisms $\Delta_{\dot{v}} * \nabla_{\dot{v}^{-1}} \simeq \Delta_1$.*
- (b) *If $\ell(vw) = \ell(v) + \ell(w)$, there are canonical isomorphisms $\Delta_{\dot{v}} * \Delta_{\dot{w}} \simeq \Delta_{\dot{v}\dot{w}}$ and $\nabla_{\dot{v}} * \nabla_{\dot{w}} \simeq \nabla_{\dot{v}\dot{w}}$.*

Proof. Part (a) is similar to [T23]. Part (b) follows by commutativity of

$$\begin{array}{ccc} \text{Fl}_v \tilde{\times} \text{Fl}_w & \longrightarrow & \text{Fl}_{vw} & & (\dot{v}, \dot{w}) & \longmapsto & \dot{v}\dot{w} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ T \times T & \longrightarrow & T & & (1, 1) & \longmapsto & 1. \end{array}$$

The downward maps were induced by the choices of lifts $\dot{v}, \dot{w} \in G((t))$. □

Remark 4.1.2. Using the Whittaker module, to be defined in Section 12, it is possible to canonically define those standard and costandard sheaves that are indexed by the finite Weyl group [LY20]. Namely if $v \in W^f$, then Δ_v and ∇_v can be canonically defined by the condition that there are canonical isomorphisms ${}^x \Delta_1 * \Delta_v \simeq {}^x \Delta_1 * \nabla_v \simeq {}^x \Delta_1$ in $\text{Shv}_{(B, \chi)}(G/U)$. Thereby, for $v, w \in W^f$, the isomorphisms in Proposition 4.1.1 become canonical independently of the chosen lifts.

4.2. Universal perversity. First we define some (not stable) subcategories of $\mathrm{Shv}_{(I)}(\mathrm{Fl})$.

- Let $\langle \Delta[\geq 0] \rangle$ be generated under extensions by $\Delta_v[i]$ for $v \in W$ and $i \geq 0$.
- Let $\langle \nabla[\geq 0] \rangle$ be generated under extensions by $\nabla_v[i]$ for $v \in W$ and $i \geq 0$.
- Let $\langle \Delta[\leq 0] \rangle$ be generated under extensions by $\Delta_v[i]$ for $v \in W$ and $i \leq 0$.
- Let $\langle \nabla[\leq 0] \rangle$ be generated under extensions by $\nabla_v[i]$ for $v \in W$ and $i \leq 0$.

Lemma 4.2.1. *If $w \in W$ then*

- (a) *convolution by ∇_w preserves $\langle \Delta[\geq 0] \rangle$ and $\langle \nabla[\geq 0] \rangle$,*
- (b) *convolution by Δ_w preserves $\langle \nabla[\leq 0] \rangle$ and $\langle \Delta[\leq 0] \rangle$.*

Proof. Let $s, v \in W$ such that s is a simple reflection.

- If $vs < v$ then $\Delta_v * \nabla_s \simeq \Delta_{vs}$.
- If $v < vs$ then by (2) there is a triangle $\Delta_{vs} \rightarrow \Delta_v * \nabla_s \rightarrow \Delta_v / (e^\alpha - 1)$.

In both cases $\Delta_v * \nabla_s \in \langle \Delta[\geq 0] \rangle$. Therefore induction on $\ell(w)$ shows that $\langle \Delta[\geq 0] \rangle$ is preserved by all $- * \nabla_w$. The other claims are similar. \square

Definition 4.2.2. We call $A \in \mathrm{Shv}_{(I)}(\mathrm{Fl})$ universally perverse if it satisfies the following equivalent conditions.

Proposition 4.2.3. *For $A \in \mathrm{Shv}_{(I)}(\mathrm{Fl})$ the following are equivalent.*

- (a) *$M \otimes_R A \in \mathrm{Perv}_{(I)}(\mathrm{Fl})$ is perverse, for any finitely generated R -module M .*
- (b) *$A \in \langle \Delta[\geq 0] \rangle \cap \langle \nabla[\leq 0] \rangle$.*

Proof. Suppose that A satisfies (a). Since A is perverse, it belongs to $\langle \Delta[\geq 0] \rangle$. Moreover $j_w^! A$ has Tor-amplitude $\geq -\ell(w)$ as a complex of R -modules. Therefore $j_w^! A$ can be represented by a finite complex of finitely generated flat R -modules in degrees $\geq -\ell(w)$ by [Stacks, 0651]. The Laurent polynomial Quillen–Suslin theorem [S78] says that every finitely generated flat R -module is free.⁸ Therefore $A \in \langle \nabla[\leq 0] \rangle$ by the Cousin filtration. \square

Lemma 4.2.4. *For all $v, w \in W$, the sheaf $\Delta_w * \nabla_v$ is universally perverse.*

Proof. Lemma 4.2.1 implies that $\Delta_w * \nabla_v$ is contained in both $\langle \Delta[\geq 0] \rangle * \nabla_v \subset \langle \Delta[\geq 0] \rangle$ and $\Delta_w * \langle \nabla[\leq 0] \rangle \subset \langle \nabla[\leq 0] \rangle$. \square

Definition 4.2.5. We call $A \in \mathrm{Shv}_{(I)}(\mathrm{Fl})$ universally tilting if it satisfies the following equivalent conditions.

Proposition 4.2.6. *For $A \in \mathrm{Shv}_{(I)}(\mathrm{Fl})$ the following are equivalent.*

- (a) *A admits universal standard and costandard filtrations.*
- (b) *$A \in \langle \nabla[\geq 0] \rangle \cap \langle \Delta[\leq 0] \rangle$.*

Proof. Suppose that A satisfies (b). Lemma 4.2.1 implies that $\langle \Delta[\leq 0] \rangle \subset \langle \nabla[\leq 0] \rangle$. Therefore $A \in \langle \nabla[\geq 0] \rangle \cap \langle \nabla[\leq 0] \rangle$, so $j_w^! A$ is a finitely generated flat R -module in degree $-\ell(w)$. Therefore A admits a universal costandard filtration by [S78]. Similarly A admits a universal standard filtration. \square

⁸The Laurent polynomial Quillen–Suslin theorem is not really essential for this paper, but it mildly simplifies the notation by allowing us to avoid tensoring with finitely generated flat R -modules.

Proposition 4.2.7. *The product of universally tilting sheaves is universally tilting.*

Proof. Let $A, B \in \text{Shv}_{(T)}(\text{Fl})$ be universally tilting sheaves. Then $A * B$ admits a filtration with graded pieces $\Delta_w * B$. Lemma 4.2.1 implies $\Delta_w * B \in \Delta_w * \langle \Delta[\leq 0] \rangle \subset \langle \Delta[\leq 0] \rangle$, therefore $A * B \in \langle \Delta[\leq 0] \rangle$. A similar argument shows that also $A * B \in \langle \nabla[\geq 0] \rangle$. Therefore $A * B$ is universally tilting. \square

4.3. Loop rotation. Let \mathbf{C}^\times act on Fl by loop rotation $z \cdot t^\lambda := \lambda(z)t^\lambda$. Every sheaf in $\text{Shv}_{(T)}(\text{Fl})$ is locally constant along the loop rotation orbits. Monodromy makes $\text{Shv}_{(T)}(\text{Fl})$ an $(R \otimes R)[m^{\pm 1}]$ -linear category, where m acts by monodromy around the loop rotation orbits. The following is similar to Lemmas 6.6 and 6.7 of [BR22].

Lemma 4.3.1. *Let $v = xe^\lambda$ and $w = ye^\mu$ in W , where $x, y \in W^f$ and $\lambda, \mu \in \Lambda$.*

(a) *The monodromy actions on Δ_v and ∇_v factor through*

$$(R \otimes R)[m^{\pm 1}]/(x(r) \otimes 1 - 1 \otimes r, 1 \otimes e^\lambda - m).$$

(b) *If $v \neq w$ then,*

$$\text{Hom}(\Delta_v, \Delta_w) = \text{Hom}(\Delta_v, \nabla_w) = \text{Hom}(\nabla_v, \Delta_w) = \text{Hom}(\nabla_v, \nabla_w) = 0.$$

Proof. For part (a), the left and right T -actions on $\mathring{I} \setminus \text{Fl}_{xe^\lambda}$ differ by x . Moreover loop rotation acts on $\mathring{I} \setminus \text{Fl}_{xe^\lambda}$ by right multiplication via $\lambda : \mathbf{C}^\times \rightarrow T$.

For part (b), let $\alpha \in \text{Hom}(\Delta_v, \Delta_w)$. Observe that for $r \neq 0 \in R$, the monodromy action of $1 \otimes r$ on the image $\text{im } \alpha$ is injective. Indeed $\text{im } \alpha$ is a perverse subsheaf of Δ_w , and $1 \otimes r$ acts injectively on Δ_w by Proposition 4.2.4.

Since loop rotation monodromy acts by $1 \otimes e^\lambda$ on Δ_v and by $1 \otimes e^\mu$ on Δ_w , it follows that $1 \otimes (e^\lambda - e^\mu)$ acts by zero on $\text{im } \alpha$. The above observation therefore implies $\lambda = \mu$. If $s \in R$ then $1 \otimes (x(s) - y(s))$ acts by zero on $\text{im } \alpha$. The above observation implies $x(s) - y(s) = 0$ for all $s \in R$, hence $x = y$. \square

5. WAKIMOTO SHEAVES

Here we define universal monodromic Wakimoto sheaves W_λ on the enhanced affine flag variety. They correspond to vector bundles on the diagonal of the Steinberg stack, that are pulled back from characters of T . On the lattice subgroup of the affine Weyl group, the length function is only additive on the dominant cone. For this reason, the definition of W_λ involves expressing λ as a sum of dominant and anti-dominant weights.

5.1. Canonical lifts of the lattice elements. The moduli of B -local systems on the cylinder is only identified with B/B after choosing a coordinate. Accordingly, to canonically define Wakimoto sheaves we needed to choose a coordinate on the disc.

The choice of coordinate yields canonical lifts $\dot{\lambda} := t^\lambda \in G((t))$ of the lattice elements of the affine Weyl group. These lifts are multiplicative, in the sense that $t^\lambda t^\mu = t^{\lambda+\mu}$. For $\lambda, \mu \in \Lambda^+$ dominant, Proposition 4.1.1 gives a canonical isomorphisms

$$(3) \quad \Delta_{\dot{\lambda}} * \Delta_{\dot{\mu}} \simeq \Delta_{\dot{\lambda}+\dot{\mu}} \quad \text{and} \quad \nabla_{\dot{\lambda}} * \nabla_{\dot{\mu}} \simeq \nabla_{\dot{\lambda}+\dot{\mu}},$$

satisfying the associativity identity.

5.2. Definition of Wakimoto sheaves. The above equation 3 gives a canonical monoidal functor

$$W : \text{Rep}(T) \rightarrow \text{Shv}_{(T)}(\text{Fl}), \quad k_\lambda \mapsto W_\lambda.$$

Uniquely defined by $W_\lambda \simeq \nabla_{\dot{\lambda}}$ for $\lambda \in \Lambda^+$. Writing $\lambda = \mu - \eta$ for $\mu, \eta \in \Lambda^+$, there is a canonical isomorphism $W_\lambda \simeq \nabla_{\dot{\mu}} * \Delta_{\dot{\eta}}$. Lemma 4.2.4 shows that Wakimoto sheaves are universally perverse. The following is similar to Lemma 4.3.1.

Proposition 5.2.1. *Wakimoto sheaves satisfy the following properties.*

(a) *The monodromy action on W_λ factors through*

$$(R \otimes R)[m^{\pm 1}]/(r \otimes 1 - 1 \otimes r, 1 \otimes e^\lambda - m).$$

(b) *If $\mu \neq \lambda$ then $\mathrm{Hom}(W_\lambda, W_\mu) = 0$.*

(c) *If $\mu \not\leq \lambda$ then $\mathrm{RHom}(W_\lambda, W_\mu) = 0$.*

5.3. Wakimoto filtered perverse sheaves. Let $\mathrm{Perv}_{(I)}^{\mathrm{waki}}(\mathrm{Fl})$ be the full additive subcategory of perverse sheaves that admit a filtration with graded pieces W_λ . Such a filtration is necessarily unique. Note that $\mathrm{Perv}_{(I)}^{\mathrm{waki}}(\mathrm{Fl})$ is closed under convolution, unlike the larger category $\mathrm{Perv}_{(I)}(\mathrm{Fl})$. The following is similar to Corollary 6.3 of [BR21].

Proposition 5.3.1. *Associated graded gives a faithful monoidal functor*

$$\mathrm{gr} : \mathrm{Perv}_{(I)}^{\mathrm{waki}}(\mathrm{Fl}) \rightarrow \mathrm{Free}_{\mathbb{T}}(\mathbb{T}).$$

Proof. Proposition 5.2.1 says $\mathrm{Hom}(W_\lambda, W_\mu) = 0$ for $\mu \not\leq \lambda$. Therefore any map of Wakimoto filtered perverse sheaves preserves the Wakimoto filtration, hence gr is functorial. Moreover Proposition 5.2.1 says $\mathrm{Hom}(W_\lambda, W_\mu) = 0$ for $\mu < \lambda$, hence gr is faithful. Moreover gr is monoidal by Lemma 16 of [AB09]. \square

6. CENTRAL SHEAVES

Gaitsgory constructed a deformation from the affine Grassmannian to the affine flag variety, and Gaitsgory’s functor is defined by nearby cycles in this family. The resulting central sheaves correspond to vector bundles on the diagonal of the Steinberg stack, that are pulled back from representations of \mathbf{G} . Closely following [G01], we verify that universal monodromic central sheaves are convolution exact and Gaitsgory’s functor is central. Moreover we check that loop rotation monodromy induces a tensor automorphism, providing the extra data needed to lift Gaitsgory’s functor to \mathbf{G}/\mathbf{G} . Finally we prove that central sheaves admit universal Wakimoto filtrations following [AB09].

6.1. Gaitsgory’s central functor. Gaitsgory’s central functor is defined by nearby cycles from the affine Grassmannian to the affine flag variety.

Let $\mathrm{Fl}_{\mathbf{C}}$ be the moduli of: a G -bundle E on \mathbf{C} , a U -reduction of $E|_0$, a point $z \in \mathbf{C}$, and a trivialization of $E|_{\mathbf{C}-z}$. The choice of coordinate gives a trivialization $\mathrm{Fl}_{\mathbf{C}^\times} \simeq \mathrm{Gr} \times G/U \times \mathbf{C}^\times$ away from $z = 0$. Gaitsgory’s functor is nearby cycles in this family

$$Z : \mathrm{Rep}(\mathbf{G}) \rightarrow \mathrm{Shv}(\mathrm{Fl}), \quad V_\lambda \mapsto Z_\lambda := \psi(\mathrm{IC}_\lambda \boxtimes R \boxtimes k_{\mathbf{C}^\times}).$$

Here $R \in \mathrm{Shv}_{(B)}(G/U)$ is the perverse universal local system supported on B/U . We identified $\mathrm{Rep}(\mathbf{G}) \simeq \mathrm{Perv}_{G[\mathfrak{t}]}(\mathrm{Gr})$ exchanging $V_\lambda \leftrightarrow \mathrm{IC}_\lambda$ by geometric Satake [MV07]. Let $\mu_A : Z(A) \rightarrow Z(A)$ denote monodromy of nearby cycles.

6.2. Variations on Gaitsgory’s family. In the proofs that follow, we will need several variations on Gaitsgory’s family [G01], that also appear in Section 3.5 of [B16]. We will provide definitions when needed, but for now the following table summarizes their generic and special fibers.

Moduli space	Generic fiber	Special fiber
$\mathrm{Fl}_{\mathbf{C}}$	$\mathrm{Gr} \times G/U$	Fl
$\mathrm{Fl}'_{\mathbf{C}}$	$\mathrm{Gr} \times \mathrm{Fl}$	Fl
$(\mathrm{Fl} \widetilde{\times} \mathrm{Fl})_{\mathbf{C}}$	$\mathrm{Gr} \widetilde{\times} \mathrm{Gr} \times G/U \times G/U$	$\mathrm{Fl} \widetilde{\times} \mathrm{Fl}$
$\mathrm{Fl}_{\mathbf{C}} \widetilde{\times} \mathrm{Fl}$	$\mathrm{Gr} \times \mathrm{Fl} \times G/U$	$\mathrm{Fl} \widetilde{\times} \mathrm{Fl}$

6.3. Properties of the central functor.

Proposition 6.3.1. *Gaitsgory's central functor satisfies the following properties.*

- (a) *Central sheaves are weakly I -constructible.*
- (b) *Z admits a monoidal structure.*
- (c) *Monodromy of nearby cycles is a tensor automorphism of Z .*
- (d) *Loop rotation monodromy equals nearby cycles monodromy.*
- (e) *Central sheaves are universally perverse.*

Proof. Part (a) is similar to Section 2.3 of [G01]. Let $I_{\mathbf{C}} := G[t] \times_G B$, where $G[t] \rightarrow G$ is evaluation at 0. Gaitsgory's functor takes values in $\mathrm{Shv}_{(I)}(\mathrm{Fl})$ because

- (i) $\mathrm{IC}_{\lambda} \boxtimes R \boxtimes k_{\mathbf{C}^{\times}}$ is weakly constructible for $I_{\mathbf{C}} \curvearrowright \mathrm{Fl}_{\mathbf{C}^{\times}}$ by changing the trivialization of $E|_{\mathbf{C}-z}$,
- (ii) and Z_{λ} is supported on a subvariety of Fl on which I acts through a finite dimensional quotient, that is surjected onto by $I_{\mathbf{C}}$.

The proof of monoidality will use a family that deforms convolution for affine flags to convolution for the affine Grassmannian. Let $(\mathrm{Fl} \widetilde{\times} \mathrm{Fl})_{\mathbf{C}}$ be the moduli of: G -bundles E, E' on \mathbf{C} , U reductions of $E|_0$ and $E'|_0$, a point $z \in \mathbf{C}$, a trivialization of $E'|_{\mathbf{C}-z}$, and an isomorphism $E|_{\mathbf{C}-z} \simeq E'|_{\mathbf{C}-z}$. The convolution map $m : (\mathrm{Fl} \widetilde{\times} \mathrm{Fl})_{\mathbf{C}} \rightarrow \mathrm{Fl}_{\mathbf{C}}$ forgets E' .

Parts (b) and (c) are similar to Theorems 1(c) and 2 of [G01]. Namely Lemma 6.3.2 gives an isomorphism

$$Z(A) * Z(B) \simeq m_* \psi(A \widetilde{\boxtimes} B \boxtimes R \boxtimes R \boxtimes k_{\mathbf{C}^{\times}}) \simeq \psi m_*(A \widetilde{\boxtimes} B \boxtimes R \boxtimes R \boxtimes k_{\mathbf{C}^{\times}}) \simeq Z(A * B),$$

such that monodromy acts by $\mu_{A*B} = \mu_A * \mu_B$.

We used that nearby cycles commutes with pushforward along the convolution map, compatibly with monodromy. As m is not proper, this needs the following justification. We use the polar decomposition of T as the product of a compact real torus and a real vector space. Thus m factors through projection to the base of a real vector bundle followed by a proper map. Nearby cycles commutes with pushforward along projection to the base of the vector bundle, because our sheaves are constant along the fibers. Moreover nearby cycles always commutes with proper pushforward. In future we will use this argument in place of properness without further comment.

Associativity can be proved using a triple convolution version of Gaitsgory's family. Since nearby cycles is exact for the perverse t-structure,⁹ we need not check higher coherences.

Part (d) is similar to Theorem 7.8(6) of [BR22]. As in Lemma 4.3 of [BR22], inverse loop rotation acts $\mathbf{C}^{\times} \curvearrowright \mathrm{Fl}_{\mathbf{C}}$ such that

- (i) $\mathrm{IC}_{\lambda} \boxtimes R \boxtimes k_{\mathbf{C}^{\times}}$ is \mathbf{C}^{\times} -equivariant,
- (ii) $\mathrm{Fl}_{\mathbf{C}} \rightarrow \mathbf{C}$ is \mathbf{C}^{\times} -equivariant for the scaling action on \mathbf{C} ,
- (iii) and \mathbf{C}^{\times} acts on the special fiber $\mathrm{Fl}_0 = \mathrm{Fl}$ by inverse loop rotation.

Therefore claim 2 of [AB09] implies $\mu = m$.

Now we prove part (e). Since Z is monoidal it preserves dualizability, hence also compactness by Lemma 12.3.1. We already observed that nearby cycles is exact for the perverse t-structure. Therefore $Z_{\lambda} \otimes_R M \simeq \psi(\mathrm{IC}_{\lambda} \boxtimes M \boxtimes k_{\mathbf{C}^{\times}})$ is perverse, for any finitely generated R -module M . \square

We used the following argument from Section 5.2.3 of [G01].

⁹See Corollary 10.3.13 of [KS] for a proof in the weakly constructible analytic setting.

Lemma 6.3.2. *Nearby cycles in the family $(\mathrm{Fl} \widetilde{\times} \mathrm{Fl})_{\mathbf{C}}$ from $\mathrm{Gr} \widetilde{\times} \mathrm{Gr} \times G/U \times G/U \times \mathbf{C}^\times$ to $\mathrm{Fl} \widetilde{\times} \mathrm{Fl}$ sends*

$$(4) \quad \psi(A \widetilde{\boxtimes} B \boxtimes R \boxtimes R \boxtimes k_{\mathbf{C}^\times}) \simeq Z(A) \widetilde{\boxtimes} Z(B),$$

such that monodromy acts by $\mu_A \widetilde{\boxtimes} \mu_B$.

Proof. Projection onto the first factor $(\mathrm{Fl} \widetilde{\times} \mathrm{Fl})_{\mathbf{C}} \rightarrow \mathrm{Fl}_{\mathbf{C}}$ is given by forgetting E . We will pull back along a map $\mathcal{U} \rightarrow \mathrm{Fl}_{\mathbf{C}}$ such that

- (i) \mathcal{U} maps étale surjectively onto the closure in $\mathrm{Fl}_{\mathbf{C}}$ of the support of $A \boxtimes R \boxtimes k_{\mathbf{C}^\times}$ on $\mathrm{Fl}_{\mathbf{C}^\times}$,
- (ii) and the universal G -bundle E' on $\mathcal{U} \times \mathbf{C}$ is trivializable.

Such a map exists by Footnote 5 of [G01].

Let β, β' be two such trivializations of the universal G -bundle on $\mathcal{U} \times \mathbf{C}$, compatible with the U -reduction along $\mathcal{U} \times 0$. They induce two different splittings

$$(5) \quad b, b' : \mathcal{U} \times_{\mathrm{Fl}_{\mathbf{C}}} (\mathrm{Fl} \widetilde{\times} \mathrm{Fl})_{\mathbf{C}} \simeq \mathcal{U} \times_{\mathbf{C}} \mathrm{Fl}_{\mathbf{C}},$$

because, having trivialized E' , the data of $E|_{\mathbf{C}-z} \simeq E'|_{\mathbf{C}-z}$ is equivalent to a trivialization of $E|_{\mathbf{C}-z}$. The two trivializations differ by a map $\mathcal{U} \rightarrow \mathring{I}_{\mathbf{C}} := G[t] \times_G U$, and the corresponding splittings (5) differ by the action $\mathring{I}_{\mathbf{C}} \curvearrowright \mathrm{Fl}_{\mathbf{C}}$ on the second factor.

Let \mathcal{U}_0 be the special fiber. Restricting β to the formal disc around 0 gives a factorization

$$\begin{array}{ccc} \mathcal{U}_0 \times_{\mathrm{Fl}} (\mathrm{Fl} \widetilde{\times} \mathrm{Fl}) & \longrightarrow & \mathrm{Fl} \widetilde{\times} \mathrm{Fl} \\ b \downarrow & & \uparrow \\ \mathcal{U}_0 \times \mathrm{Fl} & \longrightarrow & G((t)) \times \mathrm{Fl} \end{array}$$

and hence an isomorphism

$$(Z(A) \widetilde{\boxtimes} Z(B))|_{\mathcal{U}_0 \times_{\mathrm{Fl}} (\mathrm{Fl} \widetilde{\times} \mathrm{Fl})} \simeq b^*(Z(A)|_{\mathcal{U}_0} \boxtimes Z(B)).$$

The other trivialization β' induces another such isomorphism, and the composition

$$b'^*(Z(A)|_{\mathcal{U}_0} \boxtimes Z(B)) \simeq (Z(A) \widetilde{\boxtimes} Z(B))|_{\mathcal{U}_0 \times_{\mathrm{Fl}} (\mathrm{Fl} \widetilde{\times} \mathrm{Fl})} \simeq b^*(Z(A)|_{\mathcal{U}_0} \boxtimes Z(B))$$

is induced by the $\mathring{I}_{\mathbf{C}}$ -equivariant structure on $Z(B)$.

Similarly, on the generic fibers, restricting β to the formal disc around z gives an isomorphism

$$(A \widetilde{\boxtimes} B \boxtimes R \boxtimes R \boxtimes k_{\mathbf{C}^\times})|_{\mathcal{U} \times_{\mathrm{Fl}_{\mathbf{C}^\times}} (\mathrm{Fl} \widetilde{\times} \mathrm{Fl})_{\mathbf{C}^\times}} \simeq b^*(A \boxtimes R \boxtimes k_{\mathbf{C}^\times})|_{\mathcal{U}} \boxtimes_{\mathbf{C}^\times} (B \boxtimes R \boxtimes k_{\mathbf{C}^\times}),$$

related to the isomorphism induced by β' by the $\mathring{I}_{\mathbf{C}}$ -equivariant structure on $(B \boxtimes R \boxtimes k_{\mathbf{C}^\times})$.

Nearby cycles on $\mathcal{U} \times_{\mathbf{C}} \mathrm{Fl}_{\mathbf{C}}$ sends

$$\psi((A \boxtimes R \boxtimes k_{\mathbf{C}^\times}) \boxtimes_{\mathbf{C}^\times} (B \boxtimes R \boxtimes k_{\mathbf{C}^\times})) \simeq Z(A)|_{\mathcal{U}_0} \boxtimes Z(B),$$

such that monodromy acts by $\mu_A|_{\mathcal{U}_0} \boxtimes \mu_B$.

Therefore nearby cycles on $\mathcal{U} \times_{\mathrm{Fl}_{\mathbf{C}}} (\mathrm{Fl} \widetilde{\times} \mathrm{Fl})_{\mathbf{C}}$ sends

$$(6) \quad \psi(A \widetilde{\boxtimes} B \boxtimes R \boxtimes R \boxtimes k_{\mathbf{C}^\times})|_{\mathcal{U}_0 \times_{\mathrm{Fl}} (\mathrm{Fl} \widetilde{\times} \mathrm{Fl})} \simeq (Z(A) \widetilde{\boxtimes} Z(B))|_{\mathcal{U}_0 \times_{\mathrm{Fl}} (\mathrm{Fl} \widetilde{\times} \mathrm{Fl})},$$

such that monodromy acts by $(\mu_A \widetilde{\boxtimes} \mu_B)|_{\mathcal{U} \times_{\mathrm{Fl}} (\mathrm{Fl} \widetilde{\times} \mathrm{Fl})}$.

Moreover (6) is independent of the choice of β , by the definition of the \mathring{I} -equivariant structure on $Z(B)$. Therefore it descends to the desired isomorphism (4). \square

6.4. Centrality. Now we prove that central sheaves are convolution exact, and construct the centrality isomorphisms.

Proposition 6.4.1. *Let $A \in \text{Perv}_{G[t]}(\text{Gr})$ and $B \in \text{Perv}_{(I)}(\text{Fl})$. Then $Z(A)$ is convolution exact, and there is an isomorphism of perverse sheaves $Z(A) * B \simeq B * Z(A)$, such that the following commutes*

$$(7) \quad \begin{array}{ccc} Z(A) * B & \longrightarrow & B * Z(A) \\ \mu_A * \text{id}_B \downarrow & & \downarrow \text{id}_B * \mu_A \\ Z(A) * B & \longrightarrow & B * Z(A). \end{array}$$

Proof. The proof follows Proposition 6 of [G01], although Gaitsgory does not explicitly check commutativity of (7).

The strategy is to identify both $Z(A) * B$ and $B * Z(A)$ with nearby cycles from $\text{Gr} \times \text{Fl} \times \mathbf{C}^\times$ to Fl in the following family. Let $\text{Fl}'_{\mathbf{C}}$ be the moduli of: a G -bundle E on \mathbf{C} , a U -reduction of $E|_0$, a point $z \in \mathbf{C}$, and a trivialization of $E|_{\mathbf{C}-\{0,z\}}$. It suffices to prove the following two claims.

- (i) $Z(A) * B \simeq \psi(A \boxtimes B \boxtimes k_{\mathbf{C}^\times})$ such that monodromy acts by $\mu_A * \text{id}_B$.
- (ii) $B * Z(A) \simeq \psi(A \boxtimes B \boxtimes k_{\mathbf{C}^\times})$ such that monodromy acts by $\text{id}_B * \mu_A$.

We now prove (i) using the following family. Let $\text{Fl}_{\mathbf{C}} \tilde{\times} \text{Fl}$ be the moduli of: G -bundles E, E' on \mathbf{C} , U reductions of $E|_0$ and $E'|_0$, a point $z \in \mathbf{C}$, a trivialization of $E'|_{\mathbf{C}-z}$, and an isomorphism $E|_{\mathbf{C}^\times} \simeq E'|_{\mathbf{C}^\times}$. Forgetting E' gives a map $m : \text{Fl}_{\mathbf{C}} \tilde{\times} \text{Fl} \rightarrow \text{Fl}'_{\mathbf{C}}$, that is generically a G/U -torsor and induces convolution $\text{Fl} \times \text{Fl} \rightarrow \text{Fl}$ on the special fiber.

For $A \in \text{Perv}_{G[t]}(\text{Gr})$ and $B \in \text{Perv}_{(I)}(\text{Fl})$, we claim that

$$(8) \quad Z(A) \tilde{\boxtimes} B \simeq \psi((A \boxtimes R) \tilde{\boxtimes} B \boxtimes k_{\mathbf{C}^\times}),$$

is obtained by nearby cycles from $\text{Fl}_{\mathbf{C}^\times} \tilde{\times} \text{Fl} \simeq \text{Gr} \times G/U \times \text{Fl} \times \mathbf{C}^\times$ to $\text{Fl} \tilde{\times} \text{Fl}$. Moreover we claim that monodromy acts by $\mu_A \tilde{\boxtimes} \text{id}_B$. Indeed, let $\text{Fl}_{\mathbf{C}}^\infty \times \text{Fl}$ classify the same data as $\text{Fl}_{\mathbf{C}} \tilde{\times} \text{Fl}$, plus a trivialization of E' on the formal disc at 0, compatible with the U -reduction at 0. Then (8) follows because nearby cycles commutes with pro-smooth pullback along the uniformization maps

$$\text{Fl}_{\mathbf{C}} \times \text{Fl} \leftarrow \text{Fl}_{\mathbf{C}}^\infty \times \text{Fl} \rightarrow \text{Fl}_{\mathbf{C}} \tilde{\times} \text{Fl}.^{10}$$

Using that nearby cycles commutes with pushforward along $m : \text{Fl}_{\mathbf{C}} \tilde{\times} \text{Fl} \rightarrow \text{Fl}'_{\mathbf{C}}$, we obtain

$$(9) \quad \psi(A \boxtimes B \boxtimes k_{\mathbf{C}^\times}) \simeq \psi m_*((A \boxtimes R) \tilde{\boxtimes} B \boxtimes k_{\mathbf{C}^\times}) \simeq m_* \psi((A \boxtimes R) \tilde{\boxtimes} B \boxtimes k_{\mathbf{C}^\times}) \simeq Z(A) * B$$

such that monodromy acts by $\mu_A * \text{id}_B$.

In particular $Z(A) * B$ is perverse by exactness of nearby cycles. Therefore central sheaves are convolution exact.

The proof of (ii) is similar to the proof of (i). Note that the proof presented in section 4.3 of [G01] is more involved, because Gaitsgory's argument does not assume that B is weakly I -constructible. \square

6.5. Central sheaves are Wakimoto filtered. The following proposition categorifies Bernstein's description of the center of the affine Hecke algebras, described in equation (8.2) of [Lus83]. The proof follows Proposition 5 of [AB09] and uses the notion of universal perversity.

¹⁰To avoid sheaves with infinite dimensional support, one can argue as in [G01]. Construct a finite dimensional uniformization, by trivializing E' on the n th order disc, for n sufficiently large compared to the support of B .

First we generalize Wakimoto sheaves to all elements of the affine Weyl group, not just the weight lattice. Define $W_v := W_\lambda * \nabla_x$, where we uniquely expressed $v = \lambda x \in W$ as a product of $\lambda \in \Lambda$ and $x \in W^f$. Note that these generalized Wakimoto sheaves can be defined canonically using Remark 4.1.2.

- (i) If $\lambda \in \Lambda^+$ is dominant and $x \in W^f$, then λ is minimal length in its left coset λW^f , hence $W_{\lambda x} = \nabla_{\lambda x}$.
- (ii) If $\lambda \in \Lambda^{--}$ is strictly anti-dominant and $x \in W^f$, then λ is maximal length in its left coset λW^f , hence $W_{\lambda x} = \Delta_{\lambda x}$.

Let $\langle W[\geq 0] \rangle$ and $\langle W[\leq 0] \rangle$ be the subcategories of $\text{Shv}_{(I)}(\text{Fl})$ generated under extensions by $W_v[i]$ for $v \in W$ and respectively $i \geq 0$ and $i \leq 0$.

Proposition 6.5.1. *Central sheaves admit Wakimoto filtrations indexed by the weight lattice, i.e. $Z_\lambda \in \text{Perv}_{(I)}^{\text{waki}}(\text{Fl})$.*

Proof. If $\mu \in \Lambda$ then $\Delta_{-\mu} * Z_\lambda \in \langle \Delta[\geq 0] \rangle$ is universally perverse. If μ is sufficiently dominant, then $\Delta_{-\mu} * Z_\lambda \in \langle W[\geq 0] \rangle$ by Lemma 15 of [AB09]. If $\nu \in \Lambda^+$ is dominant, then $W_{\nu-\mu} * Z_\lambda \simeq \nabla_\nu * \Delta_{-\mu} * Z_\lambda \in \langle W[\geq 0] \rangle$. On the other hand $W_{\nu-\mu} * Z_\lambda \in \langle \nabla[\leq 0] \rangle$ is universally perverse. If ν is sufficiently dominant compared to μ , then $W_{\nu-\mu} * Z_\lambda \in \langle W[\leq 0] \rangle$. Therefore $W_{\nu-\mu} * Z_\lambda \in \langle W[\geq 0] \rangle \cap \langle W[\leq 0] \rangle$ admits a generalized Wakimoto filtration, hence so does Z_λ . Since the left and right monodromy actions coincide, $Z_\lambda \in \text{Perv}_{(I)}^{\text{waki}}(\text{Fl})$ admits a Wakimoto filtration indexed only by the weight lattice. \square

6.6. Centrality. Recall from Section 2 of [B04] that a central structure on a monoidal functor is equivalent to a braided monoidal factorization through the Drinfeld center of the target.

Proposition 6.6.1. *Gaitsgory's functor lifts to a braided monoidal functor $Z : \text{Free}_{\mathbb{C}}(\mathbb{G}) \rightarrow Z(\text{Perv}_{(I)}^{\text{waki}}(\text{Fl}))$.*

Proof. By Proposition 6.4.1 it suffices to check the braiding compatibilities, which follow by the same arguments as in [G04] (see also Proposition 13 of [B16]). \square

7. THE PLÜCKER RELATIONS

To lift Gaitsgory's construction to a functor out of $\text{Rep}(\mathbb{B})$, we need to construct highest weight arrows satisfying the Plücker relations. This is slightly more delicate in the universal monodromic setting, because certain sections of Gaitsgory's family acquire poles at the special fiber. We will use that nearby cycles commutes with open restriction to the highest weight semi-infinite orbit. Therefore it will suffice to study a version of Gaitsgory's family for the Borel.

The results of this section will also be important in Section 8 to prove that associated graded of the central functor is monoidally isomorphic to restriction to the maximal torus.

7.1. Gaitsgory's family for the Borel. First we introduce versions of Gaitsgory's family for the Borel.

Let $\text{Fl}_B := B((t)) \times^{B[[t]]} B/U$, a B/U -torsor over $\text{Gr}_B := B((t))/B[[t]]$.

Let $\text{Fl}_{B,\mathbb{C}}$ be the moduli of: a B -bundle F on \mathbb{C} , a trivialization of $F|_0 \times^B T$, a point $z \in \mathbb{C}$, and a trivialization of $F|_{\mathbb{C}-z}$. The special fiber is Fl_B and the generic fiber is $\text{Gr}_B \times B/U$.

Let $(\text{Fl} \tilde{\times} \text{Fl})_{B,\mathbb{C}}$ be the moduli of: B -bundles F, F' on \mathbb{C} , trivializations of $F|_0$ and $F'|_0$, a point $z \in \mathbb{C}$, a trivialization of $F'|_{\mathbb{C}-z}$, and an isomorphism $F|_{\mathbb{C}-z} \simeq F'|_{\mathbb{C}-z}$. The special fiber is $\text{Fl}_B \tilde{\times} \text{Fl}_B$ and the generic fiber is $\text{Gr}_B \tilde{\times} \text{Gr}_B \times B/U \times B/U$.

Forgetting F' gives the convolution map $m : (\text{Fl} \tilde{\times} \text{Fl})_{B,\mathbb{C}} \rightarrow \text{Fl}_{B,\mathbb{C}}$.

7.2. Two different trivializations. Here we describe two different trivializations away from 0 of Gaitsgory family for the Borel. The first is used to define Gaitsgory's functor, and the second extends through 0.

If $z \neq 0$ then there are two U -reductions at 0 that differ by an element of T . This gives trivializations

$$(10) \quad \mathrm{Fl}_{B, \mathbf{C}^\times} \simeq \mathrm{Gr}_B \times T \times \mathbf{C}^\times \quad \text{and} \quad \mathrm{Fl}_{B, \mathbf{C}^\times} \simeq \mathrm{Gr}_B \widetilde{\times} \mathrm{Gr}_B \times T \times T \times \mathbf{C}^\times.$$

Let F be a B -bundle on \mathbf{C} , equipped with a trivialization of $F|_{\mathbf{C}-z}$. The choice of coordinate determines a specific global trivialization of $F \times^B T$, which can be compared at 0 to a trivialization of $L|_0 \times^B T$. This gives global trivializations

$$(11) \quad \mathrm{Fl}_{B, \mathbf{C}} \simeq \mathrm{Gr}_B \times T \times \mathbf{C} \quad \text{and} \quad (\mathrm{Fl} \widetilde{\times} \mathrm{Fl})_{B, \mathbf{C}} \simeq \mathrm{Gr}_B \widetilde{\times} \mathrm{Gr}_B \times T \times T \times \mathbf{C}.$$

The trivializations (10) and (11) are not compatible. Rather on the λ -connected component Gr_B^λ , they differ by the transition function

$$\mathrm{Gr}_B^\lambda \times T \times \mathbf{C}^\times \rightarrow \mathrm{Gr}_B^\lambda \times T \times \mathbf{C}^\times, \quad (b, 1, z) \mapsto (b, \lambda(z), z).^{11}$$

7.3. Gaitsgory's functor for the Borel. Here we define Gaitsgory's functor for the Borel, identify it with the identity functor, and check a monoidality property. The challenge is that the trivialization (10) does not extend through 0, but this is overcome using the weak T -constructibility.

Define Gaitsgory's functors for the Borel

$$Z : \mathrm{Shv}(\mathrm{Gr}_B \times T) \rightarrow \mathrm{Shv}(\mathrm{Fl}_B) \quad \text{and} \quad Z : \mathrm{Shv}(\mathrm{Gr}_B \widetilde{\times} \mathrm{Gr}_B \times T \times T) \rightarrow \mathrm{Shv}(\mathrm{Fl}_B \widetilde{\times} \mathrm{Fl}_B)$$

by nearby cycles

$$Z(A \boxtimes R) := \psi(A \boxtimes R \boxtimes k_{\mathbf{C}^\times}) \quad \text{and} \quad Z(A \widetilde{\boxtimes} B \boxtimes R \boxtimes R) := \psi(A \widetilde{\boxtimes} B \boxtimes R \boxtimes R \boxtimes k_{\mathbf{C}^\times}).$$

Here $A \boxtimes R \boxtimes k_{\mathbf{C}^\times}$ and $A \widetilde{\boxtimes} B \boxtimes R \boxtimes R \boxtimes k_{\mathbf{C}^\times}$ were defined using the trivialization (10).

Using the choice of coordinate, identify

$$(12) \quad \mathrm{Fl}_B \simeq \mathrm{Gr}_B \times T \quad \text{and} \quad \mathrm{Fl}_B \widetilde{\times} \mathrm{Fl}_B \simeq \mathrm{Gr}_B \widetilde{\times} \mathrm{Gr}_B \times T \times T.$$

With respect to these isomorphisms, we will show that Gaitsgory's functor for the Borel is naturally isomorphic to the identity functor.

Lemma 7.3.1. *For $A, B \in \mathrm{Shv}_{B[\ell]}(\mathrm{Gr}_B)$ there under (12) there are canonical isomorphisms*

$$(13) \quad Z(A \boxtimes R) \simeq A \boxtimes R \quad \text{and} \quad Z(A \widetilde{\boxtimes} B \boxtimes R \boxtimes R) \simeq A \widetilde{\boxtimes} B \boxtimes R \boxtimes R.$$

Proof. Let $\exp : \mathrm{Fl}_{B, \mathbf{C}^\times} \times_{\mathbf{C}^\times} \mathbf{C} \rightarrow \mathrm{Fl}_{B, \mathbf{C}}$ be induced by the exponential map $\mathbf{C} \rightarrow \mathbf{C}^\times$. Recall that nearby cycles is defined

$$Z(A \boxtimes R) := (\exp_* \exp^*(A \boxtimes R \boxtimes k_{\mathbf{C}^\times}))|_{\mathrm{Fl}_{B, 0}}$$

We may assume that A and B are supported on single connected components Gr_B^λ and Gr_B^μ . Write $(A \boxtimes R \boxtimes k_{\mathbf{C}^\times})|_{\mathrm{Fl}_{B, \mathbf{C}^\times}}$ and $(A \widetilde{\boxtimes} B \boxtimes R \boxtimes R \boxtimes k_{\mathbf{C}^\times})|_{(\mathrm{Fl} \widetilde{\times} \mathrm{Fl})_{B, \mathbf{C}^\times}}$ for the external products defined with respect to the global trivialization (11).

Because the universal local system R is equivariant for the universal cover of the image of $\lambda : \mathbf{C}^\times \rightarrow T$, there is a canonical isomorphism

$$(14) \quad \exp^*(A \boxtimes R \boxtimes k_{\mathbf{C}^\times}) \simeq \exp^*((A \boxtimes R \boxtimes k_{\mathbf{C}^\times})|_{\mathrm{Fl}_{B, \mathbf{C}^\times}}).^{12}$$

This gives a canonical isomorphism $Z(A \boxtimes R) \simeq A \boxtimes R$.

¹¹We used the choice of coordinate to write $\lambda(z)$.

¹²Because R is only weakly T -constructible not T -equivariant, $A \boxtimes R \boxtimes k_{\mathbf{C}^\times} \not\simeq (A \boxtimes R \boxtimes k_{\mathbf{C}^\times})|_{\mathrm{Fl}_{B, \mathbf{C}^\times}}$ are not isomorphic.

Because the universal local system $R \boxtimes R$ is equivariant for the universal cover of the image of $(\lambda, \mu) : \mathbf{C}^\times \rightarrow T \times T$, there is a canonical isomorphism

$$(15) \quad \exp^*(A \tilde{\boxtimes} B \boxtimes R \boxtimes R \boxtimes k_{\mathbf{C}^\times}) \simeq \exp^*((A \tilde{\boxtimes} B \boxtimes R \boxtimes R \boxtimes k_{\mathbf{C}})|_{(\mathrm{Fl} \tilde{\times} \mathrm{Fl})_{B, \mathbf{C}^\times}}).$$

This gives a canonical isomorphism $Z(A \tilde{\boxtimes} B \boxtimes R \boxtimes R) \simeq A \tilde{\boxtimes} B \boxtimes R \boxtimes R$. \square

Now we prove a monoidality property of Gaitsgory’s functor for the Borel.

Lemma 7.3.2. *For $A, B \in \mathrm{Shv}_{B[[t]]}(\mathrm{Gr}_B)$ there is a canonical isomorphism*

$$(16) \quad Z((A * B) \boxtimes R) \simeq m_* Z(A \tilde{\boxtimes} B \boxtimes R \boxtimes R),$$

such that the following square commutes

$$(17) \quad \begin{array}{ccc} Z((A * B) \boxtimes R) & \longrightarrow & m_* Z(A \tilde{\boxtimes} B \boxtimes R \boxtimes R) \\ (13) \downarrow & & \downarrow (13) \\ (A * B) \boxtimes R & \longrightarrow & m_*(A \tilde{\boxtimes} B \boxtimes R \boxtimes R). \end{array}$$

Proof. The isomorphism (16) follows because nearby cycles commutes with pushforward along convolution $m : (\mathrm{Fl} \tilde{\times} \mathrm{Fl})_{B, \mathbf{C}} \rightarrow \mathrm{Fl}_{B, \mathbf{C}}$. The square (17) commutes because (14) is the pushforward along m of (15). \square

7.4. Highest weight arrows. Now we construct highest weight arrows and check the Plücker relations.¹³

For dimension reasons, Fl_λ is open in the support of Z_λ . Typically it is not dense because the support contains other irreducible components.

Let $\mathcal{Y}_{\mathbf{C}}^\lambda$ be the closure of $\mathrm{Gr}^\lambda \times T \times \mathbf{C}^\times$ inside $\mathrm{Fl}_{\mathbf{C}}$. Let $\mathcal{U}_{\mathbf{C}}^\lambda := \mathrm{Fl}_{B, \mathbf{C}}^\lambda \cap \mathcal{Y}_{\mathbf{C}}^\lambda$ be its intersection with a semi-infinite orbit.

Let $(\mathcal{Y}^\lambda \tilde{\times} \mathcal{Y}^\mu)_{\mathbf{C}}$ be the closure of $\mathrm{Gr}^\lambda \times \mathrm{Gr}^\mu \times T \times T \times \mathbf{C}^\times$ inside $(\mathrm{Fl} \tilde{\times} \mathrm{Fl})_{\mathbf{C}}$. Let $(\mathcal{U}^\lambda \tilde{\times} \mathcal{U}^\mu)_{\mathbf{C}} := (\mathrm{Fl}^\lambda \tilde{\times} \mathrm{Fl}^\mu)_{B, \mathbf{C}} \cap (\mathcal{Y}^\lambda \tilde{\times} \mathcal{Y}^\mu)_{\mathbf{C}}$ be its intersection with a semi-infinite orbit.

The following claim is similar to Proposition 15 of [B16]. But the section $(t^\lambda, 1, \mathrm{id}) : \mathbf{C}^\times \rightarrow \mathrm{Fl}_{\mathbf{C}^\times} \simeq \mathrm{Gr} \times G/U \times \mathbf{C}^\times$ does not seem to extend to \mathbf{C} .

Proposition 7.4.1. *There is a canonical isomorphism $Z_\lambda|_{\mathrm{Fl}_\lambda} \simeq R_{\mathrm{Fl}_\lambda}$.*

Proof. Nearby cycles commutes with restriction along the open inclusion $\mathcal{U}_{\mathbf{C}}^\lambda \hookrightarrow \mathcal{Y}_{\mathbf{C}}^\lambda$. Therefore Lemma 7.3.1 implies a canonical isomorphism

$$Z_\lambda|_{\mathrm{Fl}_\lambda} \simeq Z(\mathrm{IC}_\lambda|_{\mathrm{Gr}_B^\lambda} \boxtimes R) \simeq \mathrm{IC}_\lambda|_{\mathrm{Gr}_B^\lambda} \boxtimes R \simeq R_{\mathrm{Fl}_\lambda}. \quad \square$$

By adjunction we obtain highest weight arrows $b_\lambda : Z_\lambda \rightarrow W_\lambda$. Below we check the Plücker relations, using crucially that

$$(m^{-1} \mathrm{Fl}_{\lambda+\mu}) \cap (\mathrm{supp}(Z_\lambda \tilde{\boxtimes} Z_\mu)) = \mathrm{Fl}_\lambda \tilde{\times} \mathrm{Fl}_\mu,$$

where $\mathrm{supp}(Z_\lambda \tilde{\boxtimes} Z_\mu) = (\mathcal{Y}^\lambda \tilde{\times} \mathcal{Y}^\mu)_0$.

¹³The proof of Lemma 10(b) of [AB09] needs more care in the universal setting because $\mathrm{Fl}_{\mathbf{C}} \rightarrow \mathrm{Gr} \times \mathbf{C}$ does not lift to a T -equivariant map to $\mathrm{Gr} \times T \times \mathbf{C}$.

Proposition 7.4.2. *The following square commutes*

$$(18) \quad \begin{array}{ccc} Z_{\lambda+\mu} & \xrightarrow{b_{\lambda+\mu}} & W_{\lambda+\mu} \\ \downarrow & & \downarrow \\ Z_\lambda * Z_\mu & \xrightarrow{b_\lambda * b_\mu} & W_\lambda * W_\mu. \end{array}$$

Proof. Construct two Cartesian cubes by pulling back the Cartesian square

$$\begin{array}{ccc} (\mathcal{U}^\lambda \widetilde{\times} \mathcal{U}^\mu)_{\mathbf{C}} & \xrightarrow{j} & (\mathcal{Y}^\lambda \widetilde{\times} \mathcal{Y}^\mu)_{\mathbf{C}} \\ m \downarrow & & \downarrow m \\ \mathcal{U}_{\mathbf{C}}^{\lambda+\mu} & \xrightarrow{j} & \mathcal{Y}_{\mathbf{C}}^{\lambda+\mu} \end{array}$$

to the special fiber $i : \mathcal{Y}_0^{\lambda+\mu} \rightarrow \mathcal{Y}_{\mathbf{C}}^{\lambda+\mu}$ and along the exponential map $\exp : \mathcal{Y}_{\mathbf{C}^\times}^{\lambda+\mu} \times_{\mathbf{C}^\times} \mathbf{C} \rightarrow \mathcal{Y}_{\mathbf{C}}^{\lambda+\mu}$. Pasting three commutative cubes, each obtained by A.3 of [AHR15], we obtain a commutative hexagon

$$\begin{array}{ccccc} j^* i^* \exp_* \exp^* m_* & \longrightarrow & i^* \exp_* \exp^* j^* m_* & \longrightarrow & i^* \exp_* \exp^* m_* j^* \\ \downarrow & & & & \downarrow \\ j^* m_* i^* \exp_* \exp^* & \longrightarrow & m_* j^* i^* \exp_* \exp^* & \longrightarrow & m_* i^* \exp_* \exp^* j^*, \end{array}$$

which identifies with

$$\begin{array}{ccccc} Z_{\lambda+\mu}|_{\mathbb{F}1_{\lambda+\mu}} & \longrightarrow & Z(\mathrm{IC}^{\lambda+\mu}|_{\mathrm{Gr}^{\lambda+\mu}} \boxtimes R) & \longrightarrow & Z((\mathrm{IC}^\lambda|_{\mathrm{Gr}_B^\lambda} * \mathrm{IC}^\mu|_{\mathrm{Gr}_B^\mu}) \boxtimes R) \\ \downarrow & & & & \downarrow \\ (Z_\lambda * Z_\mu)|_{\mathbb{F}1_{\lambda+\mu}} & \longrightarrow & Z_\lambda|_{\mathbb{F}1_\lambda} * Z_\mu|_{\mathbb{F}1_\mu} & \longrightarrow & m_* Z(\mathrm{IC}^\lambda|_{\mathrm{Gr}_B^\lambda} \widetilde{\boxtimes} \mathrm{IC}^\mu|_{\mathrm{Gr}_B^\mu} \boxtimes R \boxtimes R). \end{array}$$

The Plücker relations (18) now follow by Lemma 7.3.2. \square

8. THE FIBER FUNCTOR

Here we prove that the associated graded of the central functor is monoidally isomorphic to restriction to the maximal torus. This will be important in Proposition 9.3.1(c) below, to identify the associated graded of the Arkhipov–Bezrukavnikov functor.

By a Tannakian argument of Arkhipov–Bezrukavnikov, it suffices to identify a certain \mathbf{G} -torsor on \mathbb{T}/\mathbb{T} . Using the Plücker relations, we construct a \mathbf{B} -reduction such that the associated graded \mathbb{T} -torsor is $\mathbb{T} \rightarrow \mathbb{T}/\mathbb{T}$. Then we argue that there is a unique such \mathbf{B} -torsor, thereby identifying the \mathbf{G} -torsor.

8.1. The tensor structure. Define the combined central and Wakimoto functor by

$$(19) \quad Z' : \mathrm{Rep}(\mathbf{G} \times \mathbb{T}) \rightarrow \mathrm{Perv}_{(\mathbb{T})}^{\mathrm{waki}}(\mathbb{F}1), \quad V_\lambda \boxtimes k_\mu \mapsto Z_\lambda * W_\mu.$$

Note that Z' admits a natural monoidal structure by the centrality of Z . The following argument is taken from section 3.6.5 of [AB09].

Lemma 8.1.1. *The monoidal structure on $\mathrm{gr} Z'$ constructed in Propositions 6.3.1 and 5.3.1 makes it a tensor functor.*

Proof. We need to check that the monoidal structure on $\text{gr } Z'$ is compatible with the symmetric monoidal structures on the source and target. Since gr admits a monoidal section, the central structure on Z constructed in Proposition 6.6.1 induces a central structure on $\text{gr } Z$, denoted

$$(20) \quad \sigma_{\lambda, \mu} : \text{gr } Z_\lambda \otimes R_\mu \simeq R_\mu \otimes \text{gr } Z_\lambda.$$

We claim that (20) coincides with the commutativity constraint in the symmetric monoidal category $\text{Free}_\mathbb{T}(\mathbb{T})$. Proposition 7.4.2 implies that the outer square of the following diagram commutes

$$\begin{array}{ccccc} Z_\mu * Z_\lambda & \xrightarrow{\text{id} * b_\lambda} & Z_\mu * W_\lambda & \xrightarrow{b_\mu * \text{id}} & W_\mu * W_\lambda \\ \downarrow & & \downarrow & & \downarrow \\ Z_\lambda * Z_\mu & \xrightarrow{b_\lambda * \text{id}} & W_\lambda * Z_\mu & \xrightarrow{\text{id} * b_\mu} & W_\lambda * W_\mu \end{array}$$

Since $\text{id} * b_\lambda$ and $b_\lambda * \text{id}$ are surjective, the right square also commutes. Applying gr gives commutativity of

$$\begin{array}{ccc} \text{gr } Z_\lambda \otimes R_\mu & \longrightarrow & R_\lambda \otimes R_\mu \\ \sigma_{\lambda, \mu} \downarrow & & \downarrow \\ R_\mu \otimes \text{gr } Z_\lambda & \longrightarrow & R_\mu \otimes R_\lambda, \end{array}$$

where the right side is the commutativity constraint in $\text{Free}_\mathbb{T}(\mathbb{T})$. Lemma 18 of [AB09] says that $\sigma_{\lambda, \mu}$ agrees with the commutativity constraint in $\text{Free}_\mathbb{T}(\mathbb{T})$. Therefore the monoidal structure on $\text{gr } Z'$ intertwines the commutativity constraints in $\text{Rep}(\mathbb{G})$ and $\text{Free}_\mathbb{T}(\mathbb{T})$. \square

8.2. Identifying the torsor. By Lemma 8.1.1 and Tannakian formalism, $\text{gr } Z' \simeq j^*$ is monoidally isomorphic to pullback along some map

$$(21) \quad j : \mathbb{T} \times \text{pt} / \mathbb{T} \rightarrow \text{pt} / \mathbb{G} \times \text{pt} / \mathbb{T},$$

such that the composition $\mathbb{T} \times \text{pt} / \mathbb{T} \rightarrow \text{pt} / \mathbb{G} \times \text{pt} / \mathbb{T} \rightarrow \text{pt} / \mathbb{T}$ is isomorphic to projection. It remains to identify this $(\mathbb{G} \times \mathbb{T})$ -torsor on $\mathbb{T} \times \text{pt} / \mathbb{T}$. This is a non-vacuous problem if k is not algebraically closed, as then there may exist nontrivial \mathbb{G} -bundles already on a point, say for $\mathbb{G} = \text{Spin}(n)$.

To proceed, we first we use highest weight arrows to reduce the $(\mathbb{G} \times \mathbb{T})$ -torsor to a \mathbb{B} -torsor. Let $\beta_\lambda : V_\lambda|_{\mathbb{B}} \rightarrow k_\lambda|_{\mathbb{B}}$ be the highest weight arrow, a map of \mathbb{B} -modules.

Lemma 8.2.1. *The map (21) factors through*

$$j : \mathbb{T} \times \text{pt} / \mathbb{T} \dashrightarrow \text{pt} / \mathbb{B} \rightarrow \text{pt} / \mathbb{G} \times \text{pt} / \mathbb{T},$$

such that the following commutes

$$(22) \quad \begin{array}{ccc} \text{gr } Z_\lambda & \xrightarrow{\text{gr } b_\lambda} & \text{gr } W_\lambda \\ \downarrow & & \downarrow \\ j^* V_\lambda & \xrightarrow{\beta_\lambda|_{\mathbb{T} \times \text{pt} / \mathbb{T}}} & j^* k_\lambda. \end{array}$$

Proof. Let $(\mathbb{G}/\mathbb{U})^{\text{aff}}$ be the affine closure of base affine space. Proposition 7.4.2 supplies highest weight arrows satisfying the Plücker relations, giving a factorization

$$j : \mathbb{T} \times \text{pt} / \mathbb{T} \rightarrow \mathbb{T} \setminus (\mathbb{G}/\mathbb{U})^{\text{aff}} / \mathbb{G} \rightarrow \text{pt} / \mathbb{G} \times \text{pt} / \mathbb{T}.$$

After taking associated graded, the highest weight arrow $\text{gr } b_\lambda : \text{gr } Z_\lambda \rightarrow \text{gr } W_\lambda$ becomes a split surjection in $\text{Free}_\mathbb{T}(\mathbb{T})$. Therefore they define a genuine \mathbb{B} -reduction

$$\mathbb{T} \times \text{pt} / \mathbb{T} \rightarrow \mathbb{T} \setminus (\mathbb{G}/\mathbb{U}) / \mathbb{G} = \text{pt} / \mathbb{B} \rightarrow \text{pt} / \mathbb{G} \times \text{pt} / \mathbb{T}. \quad \square$$

Having constructed a \mathbf{B} -reduction, we now identify the $(\mathbf{G} \times \mathbf{T})$ -torsor.

Proposition 8.2.2. *There is an equivalence of monoidal functors*

$$(23) \quad \mathrm{gr} Z' \simeq j^* : \mathrm{Rep}(\mathbf{G} \times \mathbf{T}) \rightarrow \mathrm{Free}_{\mathbf{T}}(\mathbf{T}),$$

where $j : \mathbf{T} \times \mathrm{pt}/\mathbf{T} \rightarrow \mathrm{pt}/\mathbf{T} \rightarrow \mathrm{pt}/\mathbf{G} \times \mathrm{pt}/\mathbf{T}$ is projection followed by the map induced by the diagonal inclusion $\mathbf{T} \rightarrow \mathbf{G} \times \mathbf{T}$.

Proof. By adjunction, the map $\mathbf{T} \times \mathrm{pt}/\mathbf{T} \rightarrow \mathrm{pt}/\mathbf{B}$ constructed in Lemma 8.2.1 corresponds to a map from \mathbf{T} to the stack classifying maps $\mathrm{pt}/\mathbf{T} \rightarrow \mathrm{pt}/\mathbf{B}$ such that the composition to pt/\mathbf{T} is isomorphic to the identity map, i.e.,

$$\mathbf{T} \rightarrow \mathrm{Hom}(\mathrm{pt}/\mathbf{T}, \mathrm{pt}/\mathbf{B}) \times_{\mathrm{Hom}(\mathrm{pt}/\mathbf{T}, \mathrm{pt}/\mathbf{T})} \mathrm{id}/\mathbf{T}.$$

We will rewrite the target as follows. Using that all \mathbf{B} -bundles on a point are trivializable, it follows that $\mathrm{Hom}(\mathrm{pt}/\mathbf{T}, \mathrm{pt}/\mathbf{B}) = \mathrm{Hom}(\mathbf{T}, \mathbf{B})/\mathbf{B}$, so that in particular

$$\mathrm{Hom}(\mathrm{pt}/\mathbf{T}, \mathrm{pt}/\mathbf{B}) \times_{\mathrm{Hom}(\mathrm{pt}/\mathbf{T}, \mathrm{pt}/\mathbf{T})} \mathrm{id}/\mathbf{T} \simeq (\mathrm{Hom}(\mathbf{T}, \mathbf{B}) \times_{\mathrm{Hom}(\mathbf{T}, \mathbf{T})} \mathrm{id})/\mathbf{B} \simeq \mathrm{pt}/\mathbf{T},$$

using that all maximal tori in \mathbf{B} are conjugate and self centralizing to conclude the final isomorphism. Since R is a PID, all \mathbf{T} -torsors on \mathbf{T} are trivializable. Thus $\mathbf{T} \times \mathrm{pt}/\mathbf{T} \rightarrow \mathrm{pt}/\mathbf{B}$ is identified with projection to pt/\mathbf{T} followed by the map induced by inclusion of the maximal torus. \square

8.3. Alternative approach. An alternative approach to Proposition 8.2.2 is to identify the associated graded functor with the constant term functor. Using that nearby cycles commutes with hyperbolic localization by Braden's theorem [B03] then gives the desired isomorphism of plain functors.

However, it appears more difficult to prove directly that this isomorphism intertwines the monoidal structures, except on the highest weight lines where we checked the Plücker relations above. For example, in the universal monodromic setting, total cohomology computes only the fiber at $1 \in \mathbf{T}$ of constant term, so the approach of Proposition 4.8.2 of [AR24] does not directly adapt.

9. CONSTRUCTION OF THE FUNCTOR

Here we construct the monoidal Arkhipov–Bezrukavnikov functor

$$F : \mathrm{QCoh}_{\mathbf{G}}(\tilde{\mathbf{G}}) \rightarrow \mathrm{Shv}_{(I)}(\mathbf{Fl}).$$

In the course of the argument, we show that the functor of taking the associated graded of the Wakimoto filtration agrees with restriction to the maximal torus, cf. Proposition 9.3.1(c) below. This property is important not only in the construction of the functor itself, but also in Part II for the proof of fully faithfulness.

9.1. Base affine space. Let $(\mathbf{G}/\mathbf{U})^{\mathrm{aff}}$ be the affine closure of base affine space, and

$$\mathbf{G} \times \mathbf{T} \curvearrowright \mathbf{G} \times (\mathbf{G}/\mathbf{U})^{\mathrm{aff}} \times \mathbf{T} \quad \text{by} \quad (a, b)(g, x, t) := (aga^{-1}, axb^{-1}, btb^{-1}).$$

The following inclusion is equivariant for the diagonal $\mathbf{T} \subset \mathbf{G} \times \mathbf{T}$:

$$(24) \quad \mathbf{T} \rightarrow \mathbf{G} \times (\mathbf{G}/\mathbf{U})^{\mathrm{aff}} \times \mathbf{T}, \quad t \mapsto (t, 1, t).$$

Now we construct $\tilde{\mathbf{G}}$ as the quotient by \mathbf{T} of a locally closed subvariety of $\mathbf{G} \times (\mathbf{G}/\mathbf{U})^{\mathrm{aff}} \times \mathbf{T}$. Define $\tilde{\mathbf{G}}^{\mathrm{aff}} := \{(g, x, t) \in \mathbf{G} \times (\mathbf{G}/\mathbf{U})^{\mathrm{aff}} \times \mathbf{T} \text{ satisfying } gx = xt\}$ and $\tilde{\mathbf{G}}^{\mathrm{un}} := \{(g, x, t) \in \tilde{\mathbf{G}}^{\mathrm{aff}} \text{ satisfying } x \in \mathbf{G}/\mathbf{U}\}$. The Grothendieck alteration is the quotient $\tilde{\mathbf{G}} = \tilde{\mathbf{G}}^{\mathrm{un}}/\mathbf{T}$. Let $\tilde{\mathbf{G}}^{\mathrm{bdry}} = \tilde{\mathbf{G}}^{\mathrm{aff}} - \tilde{\mathbf{G}}^{\mathrm{un}}$ be the closed boundary. Beware that $\tilde{\mathbf{G}}^{\mathrm{aff}}$ is reducible, hence larger than the closure of $\tilde{\mathbf{G}}^{\mathrm{un}}$.¹⁴

¹⁴This is apparent already when $\mathbf{G} = \mathrm{SL}(2)$ and $(\mathbf{G}/\mathbf{U})^{\mathrm{aff}} = \mathbf{A}^2$. Then $\tilde{\mathbf{G}}^{\mathrm{aff}} \subset \mathbf{G} \times \mathbf{A}^2 \times \mathbf{T}$ has two irreducible components: $\tilde{\mathbf{G}}^{\mathrm{bdry}} = \mathbf{G} \times 0 \times \mathbf{T}$ and the closure of $\tilde{\mathbf{G}}^{\mathrm{un}}$.

9.2. Monodromy and highest weight arrows. Loop rotation monodromy induces a tensor automorphism of Z' denoted

$$m_{V_\lambda} : Z_\lambda \rightarrow Z_\lambda \quad \text{and} \quad m_{k_\mu} : W_\mu \rightarrow W_\mu.$$

Proposition 7.4.2 gives highest weight arrows $b_\lambda : Z_\lambda \rightarrow W_\lambda$ satisfying the Plücker relations.

Denote the respective coactions induced by $V_\lambda \curvearrowright \mathbf{G}$ and $\mathbf{T} \curvearrowright k_\lambda$ by

$$\Delta : V_\lambda \rightarrow V_\lambda \otimes \mathcal{O}(\mathbf{G}) \quad \text{and} \quad \Delta : k_\lambda \rightarrow \mathcal{O}(\mathbf{T}) \otimes k_\lambda.$$

The \mathbf{G} -coaction extends an $\mathcal{O}(\mathbf{G})$ -linear tensor automorphism

$$M_{V_\lambda} : V_\lambda \otimes \mathcal{O}(\mathbf{G}) \rightarrow V_\lambda \otimes \mathcal{O}(\mathbf{G}),$$

and the \mathbf{T} -coaction extends to an $\mathcal{O}(\mathbf{T})$ -linear tensor automorphism

$$M_{k_\mu} : \mathcal{O}(\mathbf{T}) \otimes k_\mu \rightarrow \mathcal{O}(\mathbf{T}) \otimes k_\mu, \quad x \otimes v \mapsto e^\mu x \otimes v.$$

We fix an identification $\mathcal{O}(\mathbf{G}/\mathbf{U}) \simeq \bigoplus V_\lambda \boxtimes k_{-\lambda}$. Define the highest weight arrows by

$$B_\lambda : V_\lambda \otimes \mathcal{O}(\mathbf{G}/\mathbf{U}) \rightarrow \mathcal{O}(\mathbf{G}/\mathbf{U}) \otimes k_\lambda, \quad v \otimes 1 \mapsto (v \boxtimes 1) \otimes 1.$$

9.3. The initial functor. Monodromy and highest weight arrows give the following functor.

Proposition 9.3.1. *The central and Wakimoto functor factors monoidally*

$$(25) \quad Z' : \text{Rep}(\mathbf{G} \times \mathbf{T}) \xrightarrow{p^*} \text{Free}_{\mathbf{G} \times \mathbf{T}}(\mathbf{G} \times (\mathbf{G}/\mathbf{U})^{\text{aff}} \times \mathbf{T}) \xrightarrow{F} \text{Perv}_{(I)}^{\text{waki}}(\mathbf{Fl}),$$

where p^* is pullback along $p : \mathbf{G} \times (\mathbf{G}/\mathbf{U})^{\text{aff}} \times \mathbf{T} \rightarrow \text{pt}$, satisfying

- (a) $F(M_{V_\lambda}) = m_{V_\lambda}$, $F(M_{k_\lambda}) = m_{k_\lambda}$, and $F(B_\lambda) = b_\lambda$,
- (b) F is R -linear with respect to projection to \mathbf{T} and T -monodromy,
- (c) $\text{gr } F$ is monoidally equivalent to pullback along (24).

Proof. First we construct the functor F satisfying (a). Set $A := \text{Hom}(\Delta_1, Z'(\mathcal{O}(\mathbf{G}) \boxtimes \mathcal{O}(\mathbf{T})))$, naturally a commutative algebra. Proposition 4 of [AB09] gives a factorization

$$(26) \quad \text{Rep}(\mathbf{G} \times \mathbf{T}) \rightarrow \text{Free}_{\mathbf{G} \times \mathbf{T}}(A) \xrightarrow{F'} \text{Perv}_{(I)}^{\text{waki}}(\mathbf{Fl})$$

such that F' is monoidal and faithful. Loop rotation monodromy and highest weight arrows induce a map

$$(27) \quad \mathcal{O}(\mathbf{G} \times \mathbf{G}/\mathbf{U} \times \mathbf{T}) \rightarrow A$$

such that (a) is satisfied by

$$F : \text{Free}_{\mathbf{G} \times \mathbf{T}}(\mathbf{G} \times (\mathbf{G}/\mathbf{U})^{\text{aff}} \times \mathbf{T}) \rightarrow \text{Free}_{\mathbf{G} \times \mathbf{T}}(A) \rightarrow \text{Perv}_{(I)}^{\text{waki}}(\mathbf{Fl}),$$

by Tannakian formalism and Proposition 4 of [AB09].

Now we check (b). Proposition 5.2.1 says that loop rotation monodromy on W_μ coincides with both left and right T -monodromy by e^μ . Therefore F is R -linear with respect to projection to \mathbf{T} and T -monodromy.

Finally we check (c). Proposition 8.2.2 implies that $\text{gr } F p^* \simeq j^*$ is monoidally equivalent to restriction along the diagonal maximal torus $j : \mathbf{T}/\mathbf{T} \rightarrow \text{pt}/\mathbf{G} \times \text{pt}/\mathbf{T}$. Therefore we have a commutative diagram

$$\begin{array}{ccccc} \text{Free}_{\mathbf{G} \times \mathbf{T}}(\mathbf{G} \times \mathbf{G}/\mathbf{U} \times \mathbf{T}) & \xrightarrow{F} & \text{Perv}_{(I)}(\mathbf{Fl}) & \xrightarrow{\text{gr}} & \text{Free}_{\mathbf{T}}(\mathbf{T}) \\ & \swarrow p^* & \uparrow Z' & \searrow j^* & \\ & & \text{Rep}(\mathbf{G} \times \mathbf{T}) & & \end{array}$$

By Tannakian formalism¹⁵ $\text{gr } F \simeq i^*$ is isomorphic to pullback along some \mathbb{T} -equivariant map

$$i : \mathbb{T} \rightarrow \mathbb{G} \times \mathbb{G}/\mathbb{U} \times \mathbb{T},$$

such that the following commutes

$$(28) \quad \begin{array}{ccc} \text{gr } Fp^* & \xrightarrow{\sim} & i^*p^* \\ \sim \downarrow & & \downarrow \sim \\ \text{gr } Z' & \xrightarrow{\sim} & j^*. \end{array}$$

We now prove that i is the map (24) sending $t \mapsto (t, 1, t)$.

Equation (22) implies that that the following commutes

$$(29) \quad \begin{array}{ccccccc} i^*(V_\lambda \otimes \mathcal{O}(\mathbb{G}/\mathbb{U})) & \xrightarrow{\sim} & \text{gr } F(V_\lambda \otimes \mathcal{O}(\mathbb{G}/\mathbb{U})) & \xrightarrow{\sim} & \text{gr } Z_\lambda & \xrightarrow{\sim} & V_\lambda \otimes \mathcal{O}(\mathbb{T}) \\ \downarrow i^*B_\lambda & & \downarrow \text{gr } FB_\lambda & & \downarrow \text{gr } b_\lambda & & \downarrow \beta_\lambda|_{\mathbb{T} \times \text{pt}/\mathbb{T}} \\ i^*(\mathcal{O}(\mathbb{G}/\mathbb{U}) \otimes k_\lambda) & \xrightarrow{\sim} & \text{gr } F(\mathcal{O}(\mathbb{G}/\mathbb{U}) \otimes k_\lambda) & \xrightarrow{\sim} & \text{gr } W_\lambda & \xrightarrow{\sim} & \mathcal{O}(\mathbb{T}) \otimes k_\lambda \end{array}$$

Combining (28) and (29) shows that $i^*B_\lambda = \beta_\lambda|_{\mathbb{T} \times \text{pt}/\mathbb{T}}$ under the identification $i^*p^* \simeq j^*$. The following diagram

$$\begin{array}{ccc} V_\lambda \otimes \mathcal{O}(\mathbb{G}/\mathbb{U}) & \xrightarrow{i^*} & V_\lambda \otimes \mathcal{O}(\mathbb{T}) & & v \otimes 1 & \xrightarrow{\quad} & v \otimes 1 \\ B_\lambda \downarrow & & \downarrow i^*B_\lambda & & \downarrow & & \downarrow \\ \mathcal{O}(\mathbb{G}/\mathbb{U}) \otimes k_\lambda & \xrightarrow{i^*} & \mathcal{O}(\mathbb{T}) \otimes k_\lambda & & (v \boxtimes 1) \otimes 1 & \xrightarrow{\quad} & i^*(v \boxtimes 1) \otimes 1 = 1 \otimes \beta_\lambda(v) \end{array}$$

shows that restriction along i induces projection onto the highest weight line

$$i^* : \mathcal{O}(\mathbb{G}/\mathbb{U}) = \bigoplus V_\lambda \boxtimes k_{-\lambda} \rightarrow \mathcal{O}(\mathbb{T}), \quad v \boxtimes 1 \in V_\lambda \boxtimes k_{-\lambda} \mapsto \beta_\lambda(v) \in k \subset \mathcal{O}(\mathbb{T}).$$

Moreover $\text{gr } m_V$ acts by e^μ on the μ th graded piece. Hence $i^* : \mathcal{O}(\mathbb{G}) \rightarrow \mathcal{O}(\mathbb{T})$ is restriction to the maximal torus, and $i^* : \mathcal{O}(\mathbb{T}) \rightarrow \mathcal{O}(\mathbb{T})$ is the identity. Therefore $\text{gr } F$ is restriction along (24). \square

9.4. Factorization through a closed. Here we factor the Arkhipov–Bezrukavnikov functor through the closed subvariety $\tilde{\mathbb{G}}^{\text{aff}} \subset \mathbb{G} \times (\mathbb{G}/\mathbb{U})^{\text{aff}} \times \mathbb{T}$, by checking certain equations coming from the commutativity of (33).

Proposition 9.4.1. *The central and Wakimoto functor factors monoidally*

$$(30) \quad Z' : \text{Rep}(\mathbb{G} \times \mathbb{T}) \xrightarrow{P^*} \text{Free}_{\mathbb{G} \times \mathbb{T}}(\tilde{\mathbb{G}}^{\text{aff}}) \xrightarrow{F} \text{Perv}_{(I)}^{\text{waki}}(\text{Fl}),$$

where p^* is pullback along $p : \tilde{\mathbb{G}}^{\text{aff}} \rightarrow \text{pt}$, satisfying

- (a) $F(M_V) = m_V$ and $F(B_\lambda) = b_\lambda$,
- (b) F is R -linear with respect to projection to \mathbb{T} and T -monodromy,
- (c) $\text{gr } F$ is monoidally equivalent to pullback along (24).

Proof. We need to show that (27) kills the equations defining $\tilde{\mathbb{G}}^{\text{aff}}$.

$$(31) \quad \text{act}_{\mathbb{G}} : \mathbb{G} \times (\mathbb{G}/\mathbb{U})^{\text{aff}} \rightarrow (\mathbb{G}/\mathbb{U})^{\text{aff}} \quad \text{and} \quad \text{act}_{\mathbb{T}} : (\mathbb{G}/\mathbb{U})^{\text{aff}} \times \mathbb{T} \rightarrow (\mathbb{G}/\mathbb{U})^{\text{aff}}$$

be the left and right actions respectively. For $v \in V_\lambda$ we write $\text{act}_{\mathbb{G}}^*(v \boxtimes 1) \in \mathcal{O}(\mathbb{G} \times \mathbb{G}/\mathbb{U})$ and $\text{act}_{\mathbb{T}}^*(v \boxtimes 1) \in \mathcal{O}(\mathbb{G}/\mathbb{U} \times \mathbb{T})$ for the pullback along (31) of $v \boxtimes 1$ regarded as an element of $\mathcal{O}(\mathbb{G}/\mathbb{U}) = V_\lambda \boxtimes k_{-\lambda}$.

¹⁵See for example section 6.3.2 of [AR24].

The following square does not commute

$$(32) \quad \begin{array}{ccc} V_\lambda \otimes \mathcal{O}(\mathbf{G} \times \mathbf{G}/\mathbf{U} \times \mathbf{T}) & \xrightarrow{M_{V_\lambda}} & V_\lambda \otimes \mathcal{O}(\mathbf{G} \times \mathbf{G}/\mathbf{U} \times \mathbf{T}) \\ B_\lambda \downarrow & & \downarrow B_\lambda \\ \mathcal{O}(\mathbf{G} \times \mathbf{G}/\mathbf{U} \times \mathbf{T}) \otimes k_\lambda & \xrightarrow{M_{k_\lambda}} & \mathcal{O}(\mathbf{G} \times \mathbf{G}/\mathbf{U} \times \mathbf{T}) \otimes k_\lambda. \end{array}$$

Rather for $v \in V_\lambda$, one can check that it sends

$$\begin{array}{ccc} v \otimes 1 \otimes 1 \otimes 1 & \longmapsto & (\Delta v) \otimes 1 \otimes 1 & & v \otimes 1 \otimes 1 \otimes 1 \\ & & \downarrow & & \downarrow \\ & & (\text{act}_\mathbf{G}^*(v \boxtimes 1)) \otimes 1 \otimes 1 & & 1 \otimes v \otimes 1 \otimes 1 \longmapsto 1 \otimes v \otimes e^\lambda \otimes 1 = 1 \otimes (\text{act}_\mathbf{T}^*(v \boxtimes 1)) \otimes 1. \end{array}$$

Since all morphisms in $\text{Perv}_{(I)}(\text{Fl})$ commute with loop rotation monodromy, the following square commutes

$$(33) \quad \begin{array}{ccc} Z_\lambda & \xrightarrow{m_{V_\lambda}} & Z_\lambda \\ b_\lambda \downarrow & & \downarrow b_\lambda \\ W_\lambda & \xrightarrow{m_{k_\lambda}} & W_\lambda. \end{array}$$

Therefore, since F' is faithful, the restriction of (32) along (27) gives a commuting square

$$\begin{array}{ccc} V_\lambda \otimes A & \xrightarrow{M_{V_\lambda}|_A} & V_\lambda \otimes A \\ B_\lambda|_A \downarrow & & \downarrow B_\lambda|_A \\ A \otimes k_\lambda & \xrightarrow{M_{k_\lambda}|_A} & A \otimes k_\lambda. \end{array}$$

Therefore (27) kills all functions of the form

$$(\text{act}_\mathbf{G}^*(v \boxtimes 1)) \otimes 1 - 1 \otimes (\text{act}_\mathbf{T}^*(v \boxtimes 1)) \in \mathcal{O}(\mathbf{G} \times \mathbf{G}/\mathbf{U} \times \mathbf{T}),$$

so it factors through

$$\mathcal{O}(\mathbf{G} \times \mathbf{G}/\mathbf{U} \times \mathbf{T}) \rightarrow \mathcal{O}(\tilde{\mathbf{G}}^{\text{aff}}) \dashrightarrow A.$$

Hence the functor (26) factors as desired through $\text{Free}_{\mathbf{G} \times \mathbf{T}}(\tilde{\mathbf{G}}^{\text{aff}})$. \square

9.5. Factorization through an open. Here we factor the Arkhipov–Bezrukavnikov functor through the open subvariety $\tilde{\mathbf{G}}^{\text{un}} \subset \tilde{\mathbf{G}}^{\text{aff}}$. In other words we show that the highest weight arrows define a genuine \mathbf{B} -reduction. The idea is that the associated graded of the Arkhipov–Bezrukavnikov functor is isomorphic to a certain restriction functor that kills complexes supported on the boundary.

Proposition 9.5.1. *The central and Wakimoto functor factors monoidally*

$$Z' : \text{Rep}(\mathbf{G} \times \mathbf{T}) \xrightarrow{p^*} \text{QCoh}_\mathbf{G}(\tilde{\mathbf{G}}) \xrightarrow{-F} \text{Shv}_{(I)}(\text{Fl}),$$

where p^* is pullback along $\tilde{\mathbf{G}}^{\text{un}} \rightarrow \text{pt}$, satisfying

- (a) F is continuous,
- (b) F is R -linear with respect to projection $\tilde{\mathbf{G}} \rightarrow \mathbf{T}$ and T -monodromy,
- (c) $\text{gr } F$ is monoidally equivalent to pullback along (24).

Proof. Passing to bounded homotopy categories, Proposition 9.4.1 gives a functor

$$(34) \quad \mathrm{Perf}_{\mathbb{G} \times \mathbb{T}}(\tilde{\mathbb{G}}^{\mathrm{aff}}) = \mathrm{KFree}_{\mathbb{G} \times \mathbb{T}}(\tilde{\mathbb{G}}^{\mathrm{aff}}) \rightarrow \mathrm{KPerv}_{(\mathbb{I})}^{\mathrm{waki}}(\mathbb{F}\mathbb{1}) \rightarrow \mathrm{Shv}_{(\mathbb{I})}(\mathbb{F}\mathbb{1})$$

such that the following commutes

$$\begin{array}{ccc} \mathrm{Perf}_{\mathbb{G} \times \mathbb{T}}(\tilde{\mathbb{G}}^{\mathrm{aff}}) & \longrightarrow & \mathrm{KPerv}_{(\mathbb{I})}^{\mathrm{waki}}(\mathbb{F}\mathbb{1}) \\ & \searrow i^* & \swarrow \mathrm{Kgr} \\ & & \mathrm{Coh}_{\mathbb{T}}(\mathbb{T}) \end{array}$$

where i^* is pullback along (24).

We claim that therefore (34) kills the full subcategory $\mathrm{Perf}_{\mathbb{G} \times \mathbb{T}}(\tilde{\mathbb{G}}^{\mathrm{aff}})_{\tilde{\mathbb{G}}^{\mathrm{bdry}}}$ of perfect complexes supported on the boundary. Indeed i^* kills such complexes, because i factors through $\tilde{\mathbb{G}}^{\mathrm{un}}$. Moreover Kgr is manifestly conservative: if an object c in a DG-category has a finite filtration with associated graded isomorphic to zero, then $c \simeq 0$ as well. Therefore, the commutativity of the diagram implies the desired vanishing.

Lemma 9.5.2 below then implies that (34) factors through

$$\mathrm{Perf}_{\mathbb{G} \times \mathbb{T}}(\tilde{\mathbb{G}}^{\mathrm{aff}}) \rightarrow \mathrm{Perf}_{\mathbb{G}}(\tilde{\mathbb{G}}) \dashrightarrow \mathrm{Shv}_{(\mathbb{I})}(\mathbb{F}\mathbb{1}).$$

Ind-extending gives the desired continuous functor F . \square

The following argument of [B16] provides an alternative to the more explicit Koszul complex arguments of [AB09].

Lemma 9.5.2. *The idempotent completion of $\mathrm{Perf}_{\mathbb{G} \times \mathbb{T}}(\tilde{\mathbb{G}}^{\mathrm{aff}})/\mathrm{Perf}_{\mathbb{G} \times \mathbb{T}}(\tilde{\mathbb{G}}^{\mathrm{aff}})_{\tilde{\mathbb{G}}^{\mathrm{bdry}}}$ is $\mathrm{Perf}_{\mathbb{G}}(\tilde{\mathbb{G}})$.*

Proof. Quasi-coherent restriction along $\tilde{\mathbb{G}}^{\mathrm{un}} \hookrightarrow \mathbb{G}^{\mathrm{aff}}$ admits a fully faithful right adjoint, hence

$$\mathrm{QCoh}_{\mathbb{G} \times \mathbb{T}}(\tilde{\mathbb{G}}^{\mathrm{aff}})/\mathrm{QCoh}_{\mathbb{G} \times \mathbb{T}}(\tilde{\mathbb{G}}^{\mathrm{aff}})_{\tilde{\mathbb{G}}^{\mathrm{bdry}}} \simeq \mathrm{QCoh}_{\mathbb{G}}(\tilde{\mathbb{G}}).$$

Since $\mathrm{Perf}_{\mathbb{G} \times \mathbb{T}}(\tilde{\mathbb{G}}^{\mathrm{aff}})$ generates $\mathrm{QCoh}_{\mathbb{G} \times \mathbb{T}}(\tilde{\mathbb{G}}^{\mathrm{aff}})$, its essential image also compactly generates $\mathrm{QCoh}_{\mathbb{G}}(\tilde{\mathbb{G}})$. Theorem 2.1 of [N92] (see also [TT90]) implies the desired result. \square

Part II. Proof of the equivalence

10. OUTLINE OF PART II

Our main theorem says that $F : \mathrm{QCoh}_{\mathbb{G}}(\tilde{\mathbb{G}}) \rightarrow \mathrm{Shv}_{(\mathbb{I})}(\mathbb{F}\mathbb{1})$ becomes an isomorphism after Whittaker averaging. Our proof is by localizing away from certain intersections of walls, to reduce to an order of vanishing calculation in semi-simple rank one.

To prove fully faithfulness it suffices to check for all $\lambda, \nu \in \Lambda^+$, that the following composition is an isomorphism

$$(35) \quad \mathrm{RHom}_{\mathbb{B}/\mathbb{B}}(\mathcal{O}, V_\lambda \otimes \mathcal{O}(\mu)) \rightarrow \mathrm{RHom}(\Delta_1, Z_\lambda * W_\mu) \rightarrow \mathrm{RHom}({}^x\Delta_1, {}^x\Delta_1 * Z_\lambda * W_\mu).$$

Using that Whittaker averaged central sheaves are tilting, both sides are free R -modules concentrated in degree zero. Moreover the second map induces an isomorphism in degree zero. Therefore it suffices to prove that the following is an isomorphism

$$(36) \quad \mathrm{Hom}_{\mathbb{B}/\mathbb{B}}(\mathcal{O}, V_\lambda \otimes \mathcal{O}(\mu)) \rightarrow \mathrm{Hom}(\Delta_1, Z_\lambda * W_\mu).$$

Using faithfulness of the associated graded functor, we compatibly embed both sides of (36) as free submodules of the same free R -module. To check that these two submodules coincide, we characterize both in terms of certain order of vanishing conditions along the walls.

After localizing in \mathbb{T} away from all but one wall, \mathbb{B}/\mathbb{B} becomes isomorphic to the analogous Grothendieck–Springer stack $\mathbb{B}_\alpha/\mathbb{B}_\alpha$ for a certain subgroup $\mathbb{G}_\alpha \subset \mathbb{G}$ of semi-simple rank one. By Hartogs’ lemma we are therefore able to reduce to the semi-simple rank one case, where we calculate the image of the associated graded functor using the weight filtration.

11. SPECTRAL SIDE

Here we exhibit compact generators of $\mathrm{QCoh}(\mathbb{B}/\mathbb{B})$, and prove that certain Hom spaces are free R -modules concentrated in degree 0. Moreover we localize away from all but one wall, and calculate the image of a certain associated graded map, in terms of an order of vanishing condition along the wall.

11.1. Compact generators. The following is similar to Lemma 21 of [AB09].

Proposition 11.1.1. $\mathrm{QCoh}_{\mathbb{G}}(\tilde{\mathbb{G}}) = \mathrm{QCoh}_{\mathbb{B}}(\mathbb{B})$ is compactly generated by each of the following:

- (a) $\mathcal{O}(\eta)$ for $\eta \in \Lambda$,
- (b) $V_\lambda \otimes \mathcal{O}(\mu)$ for $\lambda, \mu \in \Lambda^+$.

Proof. The above sheaves on \mathbb{B}/\mathbb{B} are compact because they are vector bundles.

Direct image along $p : \mathbb{B}/\mathbb{B} \rightarrow \mathrm{pt}/\mathbb{B}$ is conservative because \mathbb{B} is affine. Suppose $F \in \mathrm{QCoh}_{\mathbb{B}}(\mathbb{B})$ satisfies

$$\mathrm{RHom}_{\mathrm{QCoh}_{\mathbb{B}}(\mathbb{B})}(\mathcal{O}(\eta), F) \simeq \mathrm{RHom}_{\mathrm{QCoh}_{\mathbb{B}}(\mathrm{pt})}(k_\eta, p_*F) \simeq 0, \quad \text{for all } \eta \in \Lambda.$$

Then $F \simeq 0$ because k_η generate $\mathrm{QCoh}_{\mathbb{B}}(\mathrm{pt})$. Therefore the objects (a) generate $\mathrm{QCoh}_{\mathbb{B}}(\mathbb{B})$.

Let $\nu \in \Lambda^+$ and $d = \dim V_\nu$. Then $\mathrm{Sym}^d(V_\nu[1] \rightarrow k_\nu)$ is acyclic because the kernel of the highest weight arrow has dimension $d - 1$. Therefore the following Koszul complex is exact

$$(37) \quad 0 \rightarrow \Lambda^d V_\nu \otimes \mathcal{O} \rightarrow \Lambda^{d-1} V_\nu \otimes \mathcal{O}(\nu) \rightarrow \cdots \rightarrow V_\nu \otimes \mathcal{O}((d-1)\nu) \rightarrow \mathcal{O}(d\nu) \rightarrow 0.$$

If $\nu \in \Lambda^+$ is sufficiently dominant compared to $\eta \in \Lambda$, then twisting (37) by $(\Lambda^d V_\nu)^* \otimes \mathcal{O}(\eta)$ shows that $\mathcal{O}(\eta)$ is a colimit of $V_\lambda \otimes \mathcal{O}(\mu)$ for $\lambda \in \Lambda$ and $\mu \in \Lambda^+$. Therefore the objects (b) also generate $\mathrm{QCoh}_{\mathbb{B}}(\mathbb{B})$. \square

11.2. Affinization. Now we calculate the affinization of the Grothendieck alteration. The answer involves the characteristic polynomial map $\mathbb{G} = \mathbb{G}^{\mathrm{sc}}/\mathbb{Z} \rightarrow \mathbb{C} := (\mathbb{G}^{\mathrm{sc}}//\mathbb{G}^{\mathrm{sc}})/\mathbb{Z}$.

Proposition 11.2.1. *Pushforward of the structure sheaf along $p : \tilde{\mathbb{G}} \rightarrow \mathbb{G} \times_{\mathbb{C}} \mathbb{T}$ yields $p_*\mathcal{O}_{\tilde{\mathbb{G}}} \simeq \mathcal{O}_{\mathbb{G} \times_{\mathbb{C}} \mathbb{T}}$.*

Proof. The fiber product $\mathbb{G} \times_{\mathbb{C}} \mathbb{T}$ is

- (i) smooth in codimension 1 because $\mathbb{G}^{\mathrm{reg}} \times_{\mathbb{C}} \mathbb{T} \simeq \tilde{\mathbb{G}}^{\mathrm{reg}}$ by Corollary 3.12 of [S65],
- (ii) a complete intersection because $\mathbb{T} \rightarrow \mathbb{C}$ is flat.

By Serre’s criterion $\mathbb{G} \times_{\mathbb{C}} \mathbb{T}$ is normal.¹⁶ Therefore $p_*\mathcal{O}_{\tilde{\mathbb{G}}} \simeq \mathcal{O}_{\mathbb{G} \times_{\mathbb{C}} \mathbb{T}}$, because p is proper and birational. \square

11.3. Freeness and higher vanishing. The following freeness and higher vanishing will be needed later to invoke Hartogs’ lemma.

Proposition 11.3.1. *If $\lambda, \mu \in \Lambda^+$ then $\mathrm{R}\Gamma(\tilde{\mathbb{G}}/\mathbb{G}, V_\lambda \otimes \mathcal{O}(\mu))$ is*

- (a) perfect as a complex of R -modules,
- (b) concentrated in degree 0,
- (c) free over R .

¹⁶If the derived subgroup of \mathbb{G} is not simply connected, then $\mathbb{G} \times_{\mathbb{G}} \mathbb{T}$ is neither smooth in codimension 1 nor normal.

Proof. First we prove (a). Since $\text{Rep}(\mathbf{G})$ is semi-simple,

$$(38) \quad R^i\Gamma(\tilde{\mathbf{G}}/\mathbf{G}, V_\lambda \otimes \mathcal{O}(\mu)) \simeq (V_\lambda \otimes R^i\Gamma(\tilde{\mathbf{G}}, \mathcal{O}(\mu)))^{\mathbf{G}}.$$

The affinization $\tilde{\mathbf{G}} \rightarrow \mathbf{G} \times_{\mathbf{C}} \mathbf{T}$ is proper, thus $R\Gamma(\tilde{\mathbf{G}}, \mathcal{O}(\mu))$ is bounded coherent as a complex of $\mathcal{O}(\mathbf{G} \times_{\mathbf{C}} \mathbf{T})$ -modules. Hence (38) vanishes for all but finitely many i .

Therefore, for each i , there exists $V \in \text{Rep}(\mathbf{G})$ with a surjection

$$V \otimes \mathcal{O}(\mathbf{G} \times_{\mathbf{C}} \mathbf{T}) \rightarrow V_\lambda \otimes R^i\Gamma(\tilde{\mathbf{G}}, \mathcal{O}(\mu)).$$

Since the regular semi-simple locus $\mathbf{G}^{\text{rs}} \subset \mathbf{G}$ is dense, and every regular semi-simple element is conjugate to an element in \mathbf{T} , we get injections

$$(V \otimes \mathcal{O}(\mathbf{G} \times_{\mathbf{C}} \mathbf{T}))^{\mathbf{G}} \hookrightarrow (V \otimes \mathcal{O}(\mathbf{G}^{\text{rs}} \times_{\mathbf{C}} \mathbf{T}))^{\mathbf{G}} \hookrightarrow (V \otimes \mathcal{O}(\mathbf{T} \otimes_{\mathbf{C}} \mathbf{T})).$$

It follows that $(V \otimes \mathcal{O}(\mathbf{G} \times_{\mathbf{C}} \mathbf{T}))^{\mathbf{G}}$ is finitely generated over $\mathbf{T} \times_{\mathbf{C}} \mathbf{T}$, hence also finitely generated over \mathbf{T} . Therefore (38) is finitely generated over R .

Now we prove (b) and (c). For $\zeta \in \mathbf{T}$, flat base change and Proposition 11.4.1 imply

$$(39) \quad R\Gamma(\mathbf{B}/\mathbf{B}, V_\lambda \otimes \mathcal{O}(\mu)) \otimes_R k_\zeta \simeq R\Gamma(\mathbf{B}_\zeta^\xi/\mathbf{B}_\zeta, V_\lambda \otimes \mathcal{O}(\mu)).$$

Theorem 2 of [KLT99] implies that (39) is concentrated in degree 0. Lemma 11.5.1 implies $R\Gamma(\mathbf{B}/\mathbf{B}, V_\lambda \otimes \mathcal{O}(\mu))$ is concentrated in degree 0 and free over R . \square

11.4. Specializing in the torus. Let $\zeta \in \mathbf{T}$. Let $\mathbf{B}^\zeta := \mathbf{B} \times_{\mathbf{T}} \zeta = \mathbf{U}_\zeta \subset \mathbf{B}$. Let \mathbf{G}_ζ be the centralizer of ζ . Set $\mathbf{B}_\zeta := \mathbf{G}_\zeta \cap \mathbf{B}$, $\mathbf{U}_\zeta := \mathbf{G}_\zeta \cap \mathbf{U}$, and $\mathbf{B}_\zeta^\xi := \mathbf{B}_\zeta \times_{\mathbf{T}} \zeta = \mathbf{U}_\zeta \zeta \subset \mathbf{B}_\zeta$.

For α a finite root, let $\mathbf{T}^{(\alpha)}$ be the localization of \mathbf{T} away from all walls, except for the wall \mathbf{T}_α where α vanishes. Let $\mathbf{B}^{(\alpha)} := \mathbf{B} \times_{\mathbf{T}} \mathbf{T}^{(\alpha)} = \mathbf{U}\mathbf{T}^{(\alpha)} \subset \mathbf{B}$. Let $\mathbf{G}_\alpha := \mathbf{T}\text{SL}(2)_\alpha \subset \mathbf{G}$ be the subgroup generated by the torus and $\pm\alpha$ root subgroups. Set $\mathbf{B}_\alpha := \mathbf{G}_\alpha \cap \mathbf{B}$ and $\mathbf{B}_\alpha^{(\alpha)} := \mathbf{B}_\alpha \times_{\mathbf{T}} \mathbf{T}^{(\alpha)} \subset \mathbf{B}_\alpha$.

Proposition 11.4.1. *There are natural isomorphisms of stacks*

- (a) $\mathbf{B}_\zeta^\xi/\mathbf{B}_\zeta \simeq \mathbf{B}^\zeta/\mathbf{B}$,
- (b) $\mathbf{B}_\alpha^{(\alpha)}/\mathbf{B}_\alpha \simeq \mathbf{B}^{(\alpha)}/\mathbf{B}$.

Proof. Write $\mathbf{B}_\zeta^\xi/\mathbf{B}_\zeta \simeq (\mathbf{B} \times^{\mathbf{B}_\zeta} \mathbf{B}_\zeta^\xi)/\mathbf{B}$, where \mathbf{B}_ζ acts by conjugation on \mathbf{B}_ζ^ξ and by right multiplication on \mathbf{B} . To prove (a), it suffices to show that the \mathbf{B} -equivariant map

$$(40) \quad \mathbf{B} \times^{\mathbf{B}_\zeta} \mathbf{B}_\zeta^\xi \rightarrow \mathbf{B}^\zeta, \quad (a, b) \mapsto aba^{-1}$$

is an isomorphism. Moreover it suffices to check this after base change to the algebraic closure, so we now assume that k is algebraically closed. Then since $\mathbf{B} \times^{\mathbf{B}_\zeta} \mathbf{B}_\zeta^\xi$ is a connected variety and \mathbf{B}^ζ is a normal variety, it suffices to show that (40) is a bijection on closed points.

First we check surjectivity. Let $b \in \mathbf{B}^\zeta$ and write $b = us$ for the Jordan decomposition. Both ζ and s are semisimple with the same image in \mathbf{B}/\mathbf{B} . Therefore ζ and s are conjugate $\zeta = a^{-1}sa$ by some $a \in \mathbf{B}$. Hence $b = a((a^{-1}ua)\zeta)a^{-1}$ where $a^{-1}ua \in \mathbf{B}_\zeta \cap \mathbf{U}$.

Now we check injectivity. Suppose that $aba^{-1} = b'$ for $a \in \mathbf{B}$ and $b, b' \in \mathbf{B}_\zeta^\xi = \mathbf{U}_\zeta \zeta$. Write $b = u\zeta$ and $b' = u'\zeta$ for $u, u' \in \mathbf{U}_\zeta$. These are the Jordan decompositions of b and b' , because both b and b' commute with ζ . By uniqueness of the Jordan decomposition, $a \in \mathbf{B}_\zeta$ commutes with ζ . Hence $(a, b) \sim (1, b')$ are equivalent in $\mathbf{B} \times^{\mathbf{B}_\zeta} \mathbf{B}_\zeta^\xi$.

The same argument also proves (b). Indeed $\mathbf{B} \times^{\mathbf{B}_\alpha} \mathbf{B}_\alpha^{(\alpha)} \simeq \mathbf{B}^{(\alpha)}$, using that $\mathbf{B}_\zeta \subset \mathbf{B}_\alpha$ for all $\zeta \in \mathbf{T}^{(\alpha)}$.¹⁷ \square

¹⁷If \mathbf{G} has disconnected center then in contrast \mathbf{G}_ζ may be disconnected and need not be contained in \mathbf{G}_α .

11.5. Nakayama’s Lemma. The following is similar to Lemma 5.2 of [BR21].

Lemma 11.5.1. *Let $M \in \text{Coh}(R)$ be a perfect complex of R -modules such that $M \otimes_R k_\zeta$ is concentrated in degree 0 for every closed point $\zeta \in \mathbb{T}$. Then M is quasi-isomorphic to a free R -module in degree 0.*

Proof. It suffices to show that for each closed point $\zeta \in \mathbb{T}$, the localization M_ζ is a free R_ζ -module concentrated in degree 0. Represent M_ζ by a finite complex M_ζ^* of free R_ζ -modules.

Let i be the largest integer such that $M_\zeta^i \neq 0$. If $i > 0$ then $M_\zeta^{i-1} \otimes_R k_\zeta \rightarrow M_\zeta^i \otimes_R k_\zeta$ is surjective. Therefore $M_\zeta^{i-1} \rightarrow M_\zeta^i$ is a surjection by Nakayama’s lemma, so we may replace M_ζ^* by a complex in degrees $\leq i-1$. Repeating shows that M_ζ is quasi-isomorphic to a finite complex of free R_ζ -modules in degrees ≤ 0 .

Let i be the smallest integer such that $M_\zeta^i \neq 0$. If $i < 0$ then $M_\zeta^i \otimes_{R_\zeta} k_\zeta \rightarrow M_\zeta^{i+1} \otimes_{R_\zeta} k_\zeta$ is injective. By Nakayama’s Lemma $M_\zeta^i \rightarrow M_\zeta^{i+1}$ is a split injection, so we can replace M_ζ^* by a finite complex of free R_ζ -modules in degrees $\geq i+1$. Repeating shows that M_ζ is quasi-isomorphic to a free R_ζ -module concentrated in degree 0.

Therefore M is quasi-isomorphic to a finitely generated flat R -module in degree 0. The Laurent polynomial Quillen–Suslin theorem [S78] implies M is free. \square

11.6. Order of vanishing in semi-simple rank 1. Here we perform the spectral order of vanishing calculation by restricting to the regular locus and then diagonalizing a regular section. In this subsection only, assume \mathbf{G} has semi-simple rank 1, that is \mathbf{G} has a unique finite simple root α .

Proposition 11.6.1. *Let $\lambda \in \Lambda^+$, set $n = \langle \check{\alpha}, \lambda \rangle$, and define*

$$V_\lambda(i) := V_\lambda \otimes k_{-\lambda+(n-i)\alpha} \in \text{Rep}(\mathbf{B}).$$

If $0 \leq i \leq n$, then restriction to the torus

$$(41) \quad i^* : (V_\lambda(i) \otimes \mathcal{O}(\mathbf{B}))^{\mathbf{B}} \rightarrow (V_\lambda(i) \otimes \mathcal{O}(\mathbb{T}))^{\mathbb{T}} \simeq R$$

is injective with image $(e^\alpha - 1)^i R$.

Proof. Let $V_\lambda(i)^{\leq 0} \subset V_\lambda(i)$ be the subobject in $\text{Rep}(\mathbf{B})$ comprised of the \mathbb{T} -weight spaces of weight ≤ 0 , and consider the tautological short exact sequence

$$0 \rightarrow V_\lambda(i)^{\leq 0} \rightarrow V_\lambda(i) \rightarrow V_\lambda(i)_{>0} \rightarrow 0.$$

Consider the associated short exact sequence of vector bundles on \mathbf{B}/\mathbf{B} , and the left exact sequence of their global sections

$$0 \rightarrow (V_\lambda(i)^{\leq 0} \otimes \mathcal{O}(\mathbf{B}))^{\mathbf{B}} \rightarrow (V_\lambda(i) \otimes \mathcal{O}(\mathbf{B}))^{\mathbf{B}} \rightarrow (V_\lambda(i)_{>0} \otimes \mathcal{O}(\mathbf{B}))^{\mathbf{B}}.$$

We note that as $V_\lambda(i)_{>0}$ and $\mathcal{O}(\mathbf{B})$ have positive weights, we have $(V_\lambda(i)_{>0} \otimes \mathcal{O}(\mathbf{B}))^{\mathbf{B}} \simeq 0$, whence

$$(V_\lambda(i)^{\leq 0} \otimes \mathcal{O}(\mathbf{B}))^{\mathbf{B}} \simeq (V_\lambda(i) \otimes \mathcal{O}(\mathbf{B}))^{\mathbf{B}}.$$

After pullback of this exact sequence of vector bundles along $i : \mathbb{T}/\mathbb{T} \rightarrow \mathbf{B}/\mathbf{B}$, a similar (easier) argument shows that

$$(V_\lambda(i)^{\leq 0} \otimes \mathcal{O}(\mathbb{T}))^{\mathbb{T}} \simeq (V_\lambda(i) \otimes \mathcal{O}(\mathbb{T}))^{\mathbb{T}}.$$

To proceed, define $V_\lambda(i)^{\leq -\alpha} \subset V_\lambda(i)^{\leq 0}$ as the subobject in $\text{Rep}(\mathbf{B})$ comprised of the \mathbb{T} -weight spaces of weight $\leq -\alpha$, so that we have a tautological exact sequence

$$0 \rightarrow V_\lambda(i)^{\leq -\alpha} \rightarrow V_\lambda(i)^{\leq 0} \rightarrow V_\lambda(i)_{=0} \rightarrow 0,$$

where $V_\lambda(i)_{=0}$ is a one dimensional $\text{Rep}(\mathbf{B})$ -module of weight zero. Consider the associated short exact sequence of vector bundles on \mathbf{B}/\mathbf{B} , and the left exact sequence of their global sections

$$0 \rightarrow (V_\lambda(i)^{\leq -\alpha} \otimes \mathcal{O}(\mathbf{B}))^{\mathbf{B}} \rightarrow (V_\lambda(i)^{\leq 0} \otimes \mathcal{O}(\mathbf{B}))^{\mathbf{B}} \rightarrow (V_\lambda(i)_{=0} \otimes \mathcal{O}(\mathbf{B}))^{\mathbf{B}}.$$

Note that after pulling back the vector bundles along $i : \mathbb{T}/\mathbb{T} \rightarrow \mathbb{B}/\mathbb{B}$, the analogous left exact sequence of global sections is tautologically exact, i.e.

$$0 \rightarrow (V_\lambda(i)^{\leq -\alpha} \otimes \mathcal{O}(\mathbb{T}))^\mathbb{T} \rightarrow (V_\lambda(i)^{\leq 0} \otimes \mathcal{O}(\mathbb{T}))^\mathbb{T} \rightarrow (V_\lambda(i)_{=0} \otimes \mathcal{O}(\mathbb{T}))^\mathbb{T} \rightarrow 0,$$

and moreover, as \mathbb{T} acts trivially on $\mathcal{O}(\mathbb{T})$ via the adjoint action, we have $(V_\lambda(i)^{\leq -\alpha} \otimes \mathcal{O}(\mathbb{T}))^\mathbb{T} \simeq 0$.

After choosing a trivialization $V_\lambda(i)_{=0} \simeq k_0$, we obtain a commutative diagram

$$\begin{array}{ccccc} (V_\lambda(i) \otimes \mathcal{O}(\mathbb{B}))^\mathbb{B} & \xleftarrow{\sim} & (V_\lambda(i)^{\leq 0} \otimes \mathcal{O}(\mathbb{B}))^\mathbb{B} & \longrightarrow & (V_\lambda(i)_{=0} \otimes \mathcal{O}(\mathbb{B}))^\mathbb{B} \simeq \mathcal{O}(\mathbb{B})^\mathbb{B} \\ \downarrow (41) & & \downarrow i^* & & \downarrow \sim \\ (V_\lambda(i) \otimes \mathcal{O}(\mathbb{T}))^\mathbb{T} & \xleftarrow{\sim} & (V_\lambda(i)^{\leq 0} \otimes \mathcal{O}(\mathbb{T}))^\mathbb{T} & \xrightarrow{\sim} & (V_\lambda(i)_{=0} \otimes \mathcal{O}(\mathbb{T}))^\mathbb{T} \simeq \mathcal{O}(\mathbb{T})^\mathbb{T}. \end{array}$$

Therefore it suffices to check that $(e^\alpha - 1)^i \mathcal{O}(\mathbb{T})$ is the image of

$$(42) \quad (V_\lambda(i)^{\leq 0} \otimes \mathcal{O}(\mathbb{B}))^\mathbb{B} \rightarrow (V_\lambda(i)_{=0} \otimes \mathcal{O}(\mathbb{B}))^\mathbb{B} \simeq \mathcal{O}(\mathbb{B})^\mathbb{B} \simeq \mathcal{O}(\mathbb{T}).$$

Let $\mathbb{B}^{\text{reg}} \subset \mathbb{B}$ be the regular locus. Then $\mathbb{B}^{\text{reg}}/\mathbb{B} \simeq \mathbb{T}/\mathbb{J}$ where $\mathbb{J} \hookrightarrow \mathbb{B} \times \mathbb{T}$ is the centralizer of a regular section. More precisely \mathbb{J} is the closed group subscheme of the constant group scheme $\mathbb{B} \times \mathbb{T}$ over \mathbb{T} , obtained as the centralizer of a section $s : \mathbb{T} \rightarrow \mathbb{B}^{\text{reg}}$ of the tautological map $\mathbb{B}^{\text{reg}} \hookrightarrow \mathbb{B} \rightarrow \mathbb{T}$. Because $\mathbb{B} - \mathbb{B}^{\text{reg}}$ has codimension ≥ 2 , the global sections of our vector bundles are unchanged by restricting to $\mathbb{B}^{\text{reg}}/\mathbb{B}$, i.e., the map (42) coincides with

$$(43) \quad \text{gr} : (V_\lambda(i)^{\leq 0} \otimes \mathcal{O}(\mathbb{T}))^\mathbb{J} \rightarrow (V_\lambda(i)_{=0} \otimes \mathcal{O}(\mathbb{T}))^\mathbb{J} \simeq \mathcal{O}(\mathbb{T})^\mathbb{J}.$$

We first consider the case when \mathbb{G} has derived subgroup $\text{SL}(2)$. Let $\omega \in \Lambda$ such that $\langle \check{\alpha}, \omega \rangle = 1$. Since $V_\lambda = V_{n\omega} \otimes k_{\lambda - n\omega}$, it suffices to consider the case $\lambda = n\omega$. Choose an ordered weight basis $x, y \in V_\omega$ scaled so that the regular section acts by

$$\mathbb{T} \xrightarrow{s} \mathbb{B} \rightarrow \text{End}(V_\omega), \quad \zeta \mapsto \begin{pmatrix} \omega(\zeta) & \\ & s\omega(\zeta) \end{pmatrix}.$$

This regular section has eigenvectors $x \otimes (e^\omega - e^{s\omega}) + y \otimes 1 \in V_\omega \otimes \mathcal{O}(\mathbb{T})$ and $y \otimes 1 \in V_\omega \otimes \mathcal{O}(\mathbb{T})$. We write \mathcal{L}_x and \mathcal{L}_y for the line subbundles of $V_\omega \otimes \mathcal{O}(\mathbb{T})$ respectively spanned by these eigen-sections. By explicitly computing the centralizers of certain regular elements in $\text{GL}(2)$, one can check that \mathcal{L}_x and \mathcal{L}_y are equivariant for the entire action of \mathbb{J} , not just the regular section s . Their respective eigenvalues with respect to \mathbb{J} are the characters e^ω and $e^{s\omega}$, restricted from \mathbb{B} to \mathbb{J} . That is, we have a tautological \mathbb{J} -equivariant map

$$\mathcal{L}_x \oplus \mathcal{L}_y \rightarrow V_\omega \otimes \mathcal{O}(\mathbb{T}),$$

which is an isomorphism over the regular semi-simple locus.

Passing to n th symmetric powers, we obtain a map

$$\mathcal{L}_x^{\otimes n} \oplus (\mathcal{L}_x^{\otimes(n-1)} \otimes \mathcal{L}_y) \oplus \cdots \oplus (\mathcal{L}_x \otimes \mathcal{L}_y^{\otimes(n-1)}) \oplus \mathcal{L}_y^{\otimes n} \rightarrow V_\lambda \otimes \mathcal{O}(\mathbb{T}).$$

With this, note that $(V_\lambda(i)^{\leq 0} \otimes \mathcal{O}(\mathbb{T}))^\mathbb{J}$ identifies with the sections on \mathbb{T} of the line subbundle

$$\mathcal{L}_x^{\otimes i} \otimes \mathcal{L}_y^{\otimes(n-i)} \hookrightarrow V_\lambda \otimes \mathcal{O}(\mathbb{T}),$$

which is explicitly the subbundle generated by the section given in evident notation by

$$((e^\omega - e^{s\omega})x + y)^i y^{n-i} = ((e^\omega - e^{s\omega})^i x^i y^{n-i} + i(e^\omega - e^{s\omega})^{i-1} x^{i-1} y^{n-i+1} + \cdots + y^n).$$

The map (43) kills all but the leading term, because the lower terms $i(e^\omega - e^{s\omega})^{i-1} x^{i-1} y^{n-i+1}, \dots, y^n \in V_\lambda(i)^{\leq -\alpha} \otimes \mathcal{O}(\mathbb{T})$ lie in lower weight spaces. Therefore (43) is injective with the desired image $(e^\omega - e^{s\omega})^i \mathcal{O}(\mathbb{T}) = (e^\alpha - 1)^i \mathcal{O}(\mathbb{T})$.

If \mathbb{G} has derived subgroup $\text{PGL}(2)$, then there exists a reductive group \mathbb{G}^{sc} with derived subgroup $\text{SL}(2)$, such that $\mathbb{G} = \mathbb{G}^{\text{sc}}/\pm 1$. If $\mathbb{B}^{\text{sc}} \subset \mathbb{G}^{\text{sc}}$ is the Borel, then

$$(44) \quad i^* : (V_\lambda(i) \otimes \mathcal{O}(\mathbb{B}))^\mathbb{B} \rightarrow (V_\lambda(i) \otimes \mathcal{O}(\mathbb{T}))^\mathbb{T}$$

identifies with the ± 1 -invariants invariants of

$$i^* : (V_\lambda(i) \otimes \mathcal{O}(\mathbf{B}^{\text{sc}}))^{\mathbf{B}^{\text{sc}}} \rightarrow (V_\lambda(i) \otimes \mathcal{O}(\mathbf{T}^{\text{sc}}))^{\mathbf{T}^{\text{sc}}},$$

where ± 1 acts by multiplication on \mathbf{T}^{sc} and \mathbf{B}^{sc} . Therefore the image of (44) is the desired $((e^\alpha - 1)^i \mathcal{O}(\mathbf{T}^{\text{sc}}))^{\pm 1} = (e^\alpha - 1)^i \mathcal{O}(\mathbf{T})$. \square

12. IWAHORI–WHITTAKER SHEAVES

Here we define the Iwahori–Whittaker category, the automorphic side of the universal monodromic Arkhipov–Bezrukavnikov equivalence. First we define the finite Whittaker category as the module on which the finite Hecke category acts through a universal monodromic version of Soergel’s functor. Then we define the Iwahori–Whittaker category by tensoring over the finite Hecke category with the finite Whittaker module.

12.1. The finite Whittaker category. Let $\Xi \in \text{Perv}_{(I)}(\text{Fl})$ be the universal monodromic big tilting sheaf [T23] supported on $\overline{\text{Fl}}_{w_0} = G/U$.

Definition 12.1.1. Define Soergel’s functor

$$\mathbf{V} - := \text{Hom}(\Xi, -) : \text{Shv}_{(B)}(G/U) \rightarrow \text{Bim}(R).$$

Soergel’s functor is lax monoidal according to [LNY24] and strict monoidal by [T23].

Definition 12.1.2. Define the finite Whittaker category

$$\text{Shv}_{(B, \chi)}(G/U) := \text{QCoh}(\mathbf{T}) \in \text{mod-}\text{Shv}_{(B)}(G/U),$$

a right module category on which the finite Hecke category acts via

$$\text{Shv}_{(B)}(G/U) \xrightarrow{\mathbf{V}} \text{Bim}(R) \curvearrowright \text{Mod}(R).$$

Let ${}^x\Delta_1 \in \text{Shv}_{(B, \chi)}(G/U)$ correspond to the structure sheaf $R \in \text{QCoh}(\mathbf{T})$.

Definition 12.1.3. Define Whittaker averaging, a $\text{Shv}_{(B)}(G/U)$ -module functor

$$(45) \quad {}^x\Delta_1 * - : \text{Shv}_{(B)}(G/U) \rightarrow \text{Shv}_{(B, \chi)}(G/U), \quad A \mapsto \mathbf{V}(A)|_{1 \otimes R}.$$

Lemma 12.1.4. *Whittaker averaging admits a continuous left adjoint $\text{Shv}_{(B)}(G/U)$ -module functor*

$$(46) \quad \Delta_1^X * - : \text{Shv}_{(B, \chi)}(G/U) \rightarrow \text{Shv}_{(B)}(G/U), \quad M \mapsto \Xi \otimes_R M.$$

Proof. The left adjoint $\Delta_1^X * -$ is continuous because Ξ is compact. Moreover $\Delta_1^X * -$ is a strict, not just lax, $\text{Shv}_{(B)}(G/U)$ -module functor by the rigidity of $\text{Shv}_{(B)}(G/U)$, proved in Lemma 12.3.1 \square

Lemma 12.1.5. *Whittaker averaging satisfies the following properties.*

- (a) ${}^x\Delta_1 * \nabla_w \simeq {}^x\Delta_1 * \Delta_w \simeq {}^x\Delta_1 \in \text{Shv}_{(B, \chi)}(G/U)$ for all $w \in W^f$.
- (b) *The counit of adjunction $\eta : \Xi := \Delta_1^X * {}^x\Delta_1 \rightarrow \Delta_1$ is surjective for the perverse t -structure, and $\ker \eta$ admits a filtration with graded pieces Δ_w for $w \neq 1 \in W^f$.*

Proof. First we prove part (a). It is shown in [T23] that $\mathbf{V}(\Delta_w) = \mathbf{V}(\nabla_w) = R_w$, and restricting this bimodule to $1 \otimes R$ gives ${}^x\Delta_1 * \Delta_w \simeq {}^x\Delta_1 * \nabla_w \simeq R$.

Now we prove part (b). It is shown in [T23] that Ξ admits a universal standard filtration with each Δ_w for $w \in W^f$ appearing once. The counit η generates the R -module $\text{Hom}(\Xi, \Delta_1) = \text{Hom}({}^x\Delta_1, {}^x\Delta_1) = R$, therefore part (b) follows by the Cousin filtration. \square

12.2. The Iwahori–Whittaker category. Define the Iwahori–Whittaker category

$$\mathrm{Shv}_{(I,\chi)}(\mathrm{Fl}) := \mathrm{Shv}_{(B,\chi)}(G/U) \otimes_{\mathrm{Shv}_{(B)}(G/U)} \mathrm{Shv}_{(I)}(\mathrm{Fl}) \in \mathrm{mod}\text{-}\mathrm{Shv}_{(I)}(\mathrm{Fl}).$$

Remark 12.2.1. The use of Lurie’s tensor product was not essential, alternatively this is the category of left comodules for the coalgebra object $\Xi \in \mathrm{Shv}_{(I)}(\mathrm{Fl})$.

The functors (45) and (46) induce a pair of adjoint $\mathrm{Shv}_{(I)}(\mathrm{Fl})$ -module functors

$${}^x\Delta_1 * - : \mathrm{Shv}_{(I)}(\mathrm{Fl}) \rightleftarrows \mathrm{Shv}_{(I,\chi)}(\mathrm{Fl}) : \Delta_1^x * -$$

such that there is an isomorphism of comonads $\Delta_1^x * {}^x\Delta_1 * - \simeq \Xi * -$.

12.3. Compact generators. Here we exhibit compact generators of the Iwahori–Whittaker category.

Lemma 12.3.1. *For $A \in \mathrm{Shv}_{(I)}(\mathrm{Fl})$ the following are equivalent:*

- (a) A is compact,
- (b) A is a finite colimit of Δ_w for $w \in W$,
- (c) the stalks $A|_{\mathrm{Fl}_w}$ are perfect over R , and vanish except for finitely many $w \in W$,
- (d) A is dualizable.

Moreover $\mathrm{Shv}_{(I)}(\mathrm{Fl})$ is rigid monoidal.

Proof. Conditions (a), (b), (c) are equivalent by arguments similar to in [T23]. We now show that they are also equivalent to (d).

Universal standards Δ_w are invertible by Proposition 4.1.1 and hence also dualizable. Therefore finite colimits of Δ_w are dualizable. Conversely, all dualizable objects are compact because the monoidal unit Δ_1 is compact.

To prove rigidity, it remains to check that $\mathrm{Shv}_{(I)}(\mathrm{Fl})$ is compactly generated by Δ_w . Indeed every sheaf in $\mathrm{Shv}_{(I)}(\mathrm{Fl})$ admits a Cousin filtration whose w th graded piece is a colimit of Δ_w . \square

Proposition 12.3.2. $\mathrm{Shv}_{(I,\chi)}(\mathrm{Fl})$ is compactly generated by each of the following,

- (a) ${}^x\Delta_1 * W_\eta$ for $\eta \in \Lambda$,
- (b) ${}^x\Delta_1 * Z_\lambda * W_\mu$ for $\lambda, \mu \in \Lambda^+$.

Proof. For $w \in W^f$ and $\eta \in \Lambda$, the sheaves $\nabla_w * W_\eta$ compactly generate $\mathrm{Shv}_{(I)}(\mathrm{Fl})$ for support reasons. Therefore ${}^x\Delta_1 * \nabla_w * W_\eta$ compactly generate $\mathrm{Shv}_{(I,\chi)}(\mathrm{Fl})$. Lemma 12.1.5(a) says ${}^x\Delta_1 * \nabla_w * W_\eta \simeq {}^x\Delta_1 * W_\eta$, therefore the objects (a) compactly generate.

The objects (b) also compactly generate using the Koszul complex for highest weight arrows, as in Proposition 11.1.1. \square

13. WHITTAKER AVERAGED CENTRAL SHEAVES ARE TILTING

Here we prove that universal central sheaves convolved with Ξ admit universal monodromic standard and costandard filtrations. This will be used to show that certain Hom spaces are free R -modules concentrated in degree 0.

13.1. Classical groups. Suppose for this subsection that G is classical, i.e., its Lie algebra does not contain an exceptional simple Lie algebra as a direct factor. Following Section 4.4 of [AB09], let us give a short logically self-contained proof that Whittaker averaged central sheaves are tilting, using minuscule weights.

Proposition 13.1.1. *Assume G is classical and let $\lambda \in \Lambda^+$. Then $\Xi * Z_\lambda \in \text{Shv}_{(I)}(\text{Fl})$ is universally tilting.*

Proof. First suppose that G is the product of a torus and an adjoint classical group. Then every representation of G is a summand of a tensor product of minuscule representations. The following Lemmas 13.1.2 and 13.1.3 together imply that all $\Xi * Z_\lambda$ are tilting.

In general write $G/Z = G^{\text{ad}}$, where Z is the center of the derived subgroup, and G^{ad} is the product of a torus and an adjoint classical group. Let $\pi : \text{Fl} \rightarrow \text{Fl}/Z$ denote the quotient map. Note that Fl/Z is naturally the union of some connected components inside the enhanced affine flag variety of G^{ad} . The following observations imply that $\Xi * Z_\lambda$ is universally tilting.

- (i) $\pi_*(\Xi * Z_\lambda)$ is universally tilting, because $\pi_* Z_\lambda$ is a central sheaf for G^{ad} and $\pi_* \Xi$ is the big tilting sheaf for G^{ad} .
- (ii) $\Xi * Z_\lambda$ is a summand of $\pi^* \pi_*(\Xi * Z_\lambda)$.
- (iii) π^* sends universally tilting sheaves to universally tilting sheaves. □

In the previous proof, we used the following universal versions of Lemmas 25 and 26 of [AB09].

Lemma 13.1.2. *If $\Xi * Z_\lambda$ and $\Xi * Z_\mu$ are universally tilting, then so is $\Xi * Z_\lambda * Z_\mu$.*

Proof. The product of universally tilting sheaves is universally tilting by Proposition 4.2.7. Therefore by centrality $\Xi * \Xi * Z_\lambda * Z_\mu \simeq (\Xi * Z_\lambda) * (\Xi * Z_\mu)$ is universally tilting.

However $\Xi * \Xi = \bigoplus_W \Xi$ is a direct sum of copies of Ξ . Indeed since Ξ admits a standard filtration, $\Xi * \Xi$ admits a filtration whose graded pieces are $\Delta_w * \Xi \simeq \Xi$ indexed by $w \in W^f$, which splits because $\text{Ext}^1(\Xi, \Xi) = 0$. Since

$$\Xi * \Xi * Z_\lambda * Z_\mu \simeq \bigoplus_{W^f} \Xi * Z_\lambda * Z_\mu$$

is universally tilting, so is the summand $\Xi * Z_\lambda * Z_\mu$. □

Lemma 13.1.3. *If λ is minuscule then $\Xi * Z_\lambda$ is tilting.*

Proof. To see that $\Xi * Z_\lambda$ admits a costandard filtration, it is equivalent to check that $\text{RHom}(\Delta_\mu, \Xi * Z_\lambda)$ is a free R -module in degree 0, for all $\mu = w\lambda w^{-1}$ appearing as a weight of Z_λ . The complex $\text{RHom}(\Delta_y, \Xi * Z_\lambda)$ is constant for y in each W^f -double coset, because

- (i) $\text{RHom}(\Delta_s * \Delta_y, \nabla_s * \Xi * Z_\lambda) \simeq \text{RHom}(\Delta_y, \Xi * Z_\lambda)$,
- (ii) $\text{RHom}(\nabla_s * \Delta_y, \Delta_s * \Xi * Z_\lambda) \simeq \text{RHom}(\Delta_y, \Xi * Z_\lambda)$,
- (iii) $\text{RHom}(\Delta_y * \Delta_s, \Xi * Z_\lambda) \simeq \text{RHom}(\Delta_y, \Xi * Z_\lambda * \nabla_s) \simeq \text{RHom}(\Delta_y, \Xi * \nabla_s * Z_\lambda) \simeq \text{RHom}(\Delta_y, \Xi * Z_\lambda)$,
- (iv) $\text{RHom}(\Delta_y * \nabla_s, \Xi * Z_\lambda) \simeq \text{RHom}(\Delta_y, \Xi * Z_\lambda * \Delta_s) \simeq \text{RHom}(\Delta_y, \Xi * \Delta_s * Z_\lambda) \simeq \text{RHom}(\Delta_y, \Xi * Z_\lambda)$.

Because $\text{Fl}_{w_0\lambda}$ is open in the support of $\Xi * Z_\lambda$, it follows that

$$\text{RHom}(\Delta_\mu, \Xi * Z_\lambda) \simeq \text{RHom}(\Delta_{w_0\lambda}, \Xi * Z_\lambda)$$

is indeed a free R -module concentrated in degree 0. □

13.2. General case. Some exceptional groups do not have minuscule weights, so we resort to an argument that logically depends on [AB09, BFO09].

Proposition 13.2.1. *For any $\lambda \in \Lambda^+$ the sheaf $\Xi * Z_\lambda \in \text{Shv}_{(T)}(\text{Fl})$ is universally tilting.*

Proof. For any closed point $\zeta \in T$, Section 2.6 of [BFO09] says that $\Xi * Z_\lambda \otimes_R k_\zeta$ is tilting. Therefore $\text{RHom}(\Delta_w, \Xi * Z_\lambda) \otimes_R k_\zeta \simeq \text{RHom}(\Delta_w, \Xi * Z_\lambda \otimes_R k_\zeta)$ is concentrated in degree 0. Lemma 11.5.1 says $\text{RHom}(\Delta_w, \Xi * Z_\lambda)$ is a free R -module in degree 0. Hence $\Xi * Z_\lambda$ admits a universal costandard filtration. A similar argument gives a universal costandard filtration. \square

13.3. Freeness and higher vanishing. Using that Whittaker averaging central sheaves are tilting, we now prove the following freeness and higher vanishing. This will be needed later to invoke Hartogs' lemma.

Proposition 13.3.1. *If $\lambda, \mu \in \Lambda^+$ then*

- (a) $\text{RHom}({}^x\Delta_1, {}^x\Delta_1 * Z_\lambda * W_\mu)$ is concentrated in degree 0 and free over R ,
- (b) Whittaker averaging induces an isomorphism $\text{Hom}(\Delta_1, Z_\lambda * W_\mu) \rightarrow \text{Hom}({}^x\Delta_1, {}^x\Delta_1 * Z_\lambda * W_\mu)$.

Proof. First we prove (a). By adjunction

$$(47) \quad \text{RHom}({}^x\Delta_1, {}^x\Delta_1 * Z_\lambda * W_\mu) \simeq \text{RHom}(\Delta_1^x * {}^x\Delta_1, Z_\lambda * W_\mu) \simeq \text{RHom}(\Xi * Z_{-w_0(\lambda)}, \nabla_\mu).$$

Proposition 13.2.1 implies $\Xi * Z_{-w_0(\lambda)}$ admits a universal standard filtration. Hence (47) is concentrated in degree 0 and free over R , by equation (1).

For (b), if $w \neq 1 \in W^f$ and $\nu \in \Lambda$ then $\text{Hom}(\Delta_w, W_\nu) = 0$ by Lemma 4.3.1. Lemma 12.1.5(b) then implies $\text{Hom}(\ker \eta, Z_\lambda * W_\mu) = 0$. Therefore part (b) follows by the exact sequence

$$0 \rightarrow \text{Hom}(\Delta_1, Z_\lambda * W_\mu) \rightarrow \text{Hom}(\Xi, Z_\lambda * W_\mu) \rightarrow \text{Hom}(\ker \eta, Z_\lambda * W_\mu) \rightarrow \cdots . \quad \square$$

14. ORDER OF VANISHING CALCULATION

Here we prove that all endomorphisms of ∇_w that factor through Δ_w must satisfy certain order of vanishing conditions on the walls. This will be used to calculate the image of the associated graded functor.

14.1. Integral Weyl groups. After specializing the right monodromy, the affine Hecke category splits as a direct sum of blocks, each of which is governed by a certain integral Weyl group.

Let R_ζ be the set of affine coroots $\beta \in R$ satisfying $\beta(\zeta) = 1$. Write $W_\zeta \subset W$ for the subgroup generated by reflections corresponding to coroots in R_ζ . Then W_ζ is the Weyl group for the root system R_ζ . Let ℓ_ζ denote its length function.

Proposition 14.1.1. *Each left coset $wW_\zeta \subset W$ contains a unique minimal length element w^ζ , which may be characterized as the unique coset element satisfying $w^\zeta(R_\zeta^+) \subset R^+$.*

Proof. Let w^ζ be a minimal length coset representative. Then $w^\zeta(R_\zeta^+) \subset R^+$ by Proposition 5.7 of [H90]. Moreover if $w_\zeta \neq 1 \in W_\zeta$ then $w^\zeta w_\zeta(R_\zeta^+) \not\subset R^+$. Therefore w^ζ is the *unique* minimal length representative of its coset. See also Lemma 1.9 of [Lus84]. \square

Let $W^\zeta \subset W$ be the set of minimal length representatives of the left W_ζ -cosets. Proposition 14.1.1 above implies that any affine Weyl group element can be written uniquely $w = w^\zeta w_\zeta$ for $w^\zeta \in W^\zeta$ and $w_\zeta \in W_\zeta$.

Lemma 14.1.2. *Write $w = vs$ such that $\ell(w) = \ell(v) + 1$ and s is a simple reflection.*

- (a) If $s \in W_\zeta$ then $w^\zeta = v^\zeta$, $w_\zeta = v_\zeta s$, and $\ell_\zeta(w_\zeta) = \ell_\zeta(v_\zeta) + 1$.
- (b) If $s \notin W_\zeta$ then $w^\zeta = v^{s\zeta} s$, $w_\zeta = sv_{s\zeta} s$, and $\ell_\zeta(w_\zeta) = \ell_{s\zeta}(v_{s\zeta})$.

Proof. If $s \in W_\zeta$ then it is clear that $w^\zeta = v^\zeta$ and $w_\zeta = v_\zeta s$. Moreover $v\alpha \in \mathbb{R}^+$ and $v^\zeta(\mathbb{R}_\zeta^+) \subset \mathbb{R}^+$ implies that $v_\zeta \alpha \in \mathbb{R}_\zeta^+$. Therefore $\ell_\zeta(w_\zeta) = \ell_\zeta(v_\zeta) + 1$.

If $s \notin W_\zeta$ then $w^\zeta = v^{s\zeta}s$ because both are contained in $wW_\zeta = vW_{s\zeta}s$ and both $w^\zeta \mathbb{R}_\zeta^+ \subset \mathbb{R}^+$ and $v^{s\zeta}s\mathbb{R}_\zeta^+ = v^{s\zeta}\mathbb{R}_{s\zeta}^+ \subset \mathbb{R}^+$. It follows that $w_\zeta = sv_{s\zeta}s$. Since $s(\mathbb{R}_\zeta^+) = \mathbb{R}_{s\zeta}^+$, we get $\ell_\zeta(w_\zeta) = \ell_\zeta(sv_{s\zeta}s) = \ell_{s\zeta}(v_{s\zeta})$. \square

Let $\mathbb{R}_{(\alpha)}$ be the set of $\beta \in \mathbb{R}$ satisfying $\langle \alpha, \beta \rangle = 0$. Write $W_{(\alpha)} \subset W$ for the subgroup generated by reflections corresponding to roots in $\mathbb{R}_{(\alpha)}$. Then $W_{(\alpha)}$ is the Weyl group for the root system $\mathbb{R}_{(\alpha)}$. Let $\ell_{(\alpha)}$ denote its length function.

Let $W^{(\alpha)}$ be the set of minimal length representatives of the left $W_{(\alpha)}$ -cosets. Any affine Weyl group element can be written uniquely $w = w^{(\alpha)}w_{(\alpha)}$ for $w^{(\alpha)} \in W^{(\alpha)}$ and $w_{(\alpha)} \in W_{(\alpha)}$.

14.2. Mixed standards and costandards at a fixed parameter. If $\zeta \in \mathbb{T}$, let $k^\zeta \in \text{QCoh}(\mathbb{T})$ be the augmentation module. Let $\text{Shv}_{(I)}(\text{Fl})^\zeta := \text{Shv}_{(I)}(\text{Fl}) \otimes_{\text{QCoh}(\mathbb{T})} \text{QCoh}(k^\zeta)$ be the category of sheaves whose right monodromy is ζ . Having specialized the monodromy, the objects are now all ind-constructible. Therefore $\text{Shv}_{(I)}(\text{Fl})^\zeta$ admits a graded lift, the derived category of mixed Hodge modules $\text{Shv}_{(I)}^{\text{mix}}(\text{Fl})^\zeta$.

Let $k_{\mathbb{F}1_w}^\zeta \in \text{Perv}_{(I)}(\text{Fl}_w)^\zeta$ be the mixed local system with right monodromy ζ , shifted and Tate twisted to be perverse and pure of weight 0. Let

$$\Delta_w^\zeta := j_{w!}k_{\mathbb{F}1_w}^\zeta, \quad \text{IC}_w^\zeta := j_{w!*}k_{\mathbb{F}1_w}^\zeta, \quad \nabla_w^\zeta := j_{w*}k_{\mathbb{F}1_w}^\zeta \in \text{Perv}_{(I)}^{\text{mix}}(\text{Fl})^\zeta.$$

The following is a monodromic version of Lemma 4.4.7 of [BY13].

Proposition 14.2.1. *At fixed monodromy $\zeta \in \mathbb{T}$, the (co)socle of (co)standard perverse sheaves are given as follows.*

- (a) *The socle $\text{soc } \Delta_w^\zeta \simeq \text{IC}_{w\zeta}(\ell_\zeta(w_\zeta)/2)$ and no further Tate twists of $\text{IC}_{w\zeta}$ appear as Jordan-Holder factors.*
- (b) *The cosocle $\text{cos } \nabla_w^\zeta \simeq \text{IC}_{w\zeta}(-\ell_\zeta(w_\zeta)/2)$ and no further Tate twists of $\text{IC}_{w\zeta}$ appear as Jordan-Holder factors.*

Proof. The proof is by induction on the length of w . Write $w = vs$ such that $\ell(w) = \ell(v) + 1$ and s is a simple reflection.

First suppose that $s \in W_\zeta$. By [LY20] there is a short exact sequence

$$0 \rightarrow \text{IC}_1^\zeta(1/2) \rightarrow \Delta_s^\zeta \rightarrow \text{IC}_s^\zeta \rightarrow 0.$$

We now use that the character local system ζ extends from I to $P_s := I \sqcup I\dot{s}I$, see Section 2.6 of [LY20]. Convolving with Δ_v^ζ gives a short exact sequence

$$0 \rightarrow \Delta_v^\zeta(1/2) \rightarrow \Delta_w^\zeta \rightarrow \Delta_v^\zeta \star \text{IC}_s^\zeta \simeq p^* \Delta_v^\zeta[1](1/2) \rightarrow 0.$$

The quotient sheaf is the pull back along $p : G((t))/(I, \zeta) \rightarrow G((t))/(P_s, \zeta)$ of the standard extension $\Delta_v^\zeta \in \text{Shv}_{(I)}(G((t))/(P_s, \zeta))$. By induction and Lemma 14.1.2(a), there is an injection $\text{IC}_{w\zeta}(\ell_\zeta(w_\zeta)/2) \hookrightarrow \Delta_w^\zeta$ and no further Tate twists of $\text{IC}_{w\zeta}$ appear in the Jordan-Holder filtration.

It remains to check that $\text{Hom}(\text{IC}_x^\zeta, \Delta_w^\zeta) \simeq 0$ for all $x \neq w^\zeta \in W$.

- (i) If $x < xs$, then IC_x^ζ does not appear as a Jordan-Holder factor of $\Delta_v^\zeta \star \text{IC}_s^\zeta$, hence $\text{Hom}(\text{IC}_x^\zeta, \Delta_w^\zeta) \simeq 0$. Therefore $\text{Hom}(\text{IC}_x^\zeta, \Delta_w^\zeta) \simeq \text{Hom}(\text{IC}_x^\zeta, \Delta_v^\zeta(1/2)) \simeq 0$ vanishes by induction.

(ii) If $xs < x$, then

$$\begin{aligned} \mathrm{Hom}(\mathrm{IC}_x^\zeta, \Delta_w^\zeta) &\simeq \mathrm{Hom}(p^* \mathrm{IC}_x^\zeta[1](1/2), \Delta_w^\zeta) \\ &\simeq \mathrm{Hom}(\mathrm{IC}_x^\zeta[1](1/2), p_* \Delta_w^\zeta) \\ &\simeq \mathrm{Hom}(\mathrm{IC}_x^\zeta[1](1/2), \Delta_w^\zeta[-1](-1/2)) \\ &\simeq 0 \end{aligned}$$

vanishes for perverse cohomological degree reasons.

Now suppose that $s \notin W_\zeta$. There are isomorphisms

$$\Delta_s^\zeta \simeq \mathrm{IC}_s^\zeta \simeq \nabla_s^\zeta.$$

Therefore $-\star \mathrm{IC}_s^\zeta$ is exact and takes simples to simples. Hence Lemma 14.1.2(b) and induction on length implies

$$\mathrm{soc} \Delta_w^\zeta \simeq (\mathrm{soc} \Delta_v^{s\zeta}) \star \mathrm{IC}_s^\zeta \simeq \mathrm{IC}_{v^{s\zeta}}^{s\zeta}(\ell_{s\zeta}(v_{s\zeta})/2) \star \mathrm{IC}_s^\zeta \simeq \mathrm{IC}_{w^\zeta}(\ell_\zeta(w_\zeta)/2). \quad \square$$

14.3. Invoking the weight filtration. Let $I \subset R$ be the maximal ideal of functions vanishing at a closed point $\zeta \in \mathbb{T}$. Let $\mathrm{Shv}_{(I)}(\mathrm{Fl})^{n\zeta} := \mathrm{Shv}_{(I)}(\mathrm{Fl}) \otimes_{\mathrm{QCoh}(\mathbb{T})} \mathrm{QCoh}(R/I^n)$. It admits a graded lift, the derived category of mixed Hodge modules $\mathrm{Shv}_{(I)}^{\mathrm{mix}}(\mathrm{Fl})^{n\zeta}$. Let $(R/I^n)_{\mathrm{Fl}_w}$ be the perverse mixed local system on Fl_w , such that the cosocle $(R/I)_{\mathrm{Fl}_w} \simeq k_{\mathrm{Fl}_w}^\zeta$ is pure of weight 0. Setting $I^0 := R$, the subquotient $(I^i/I^{i+1})_{\mathrm{Fl}_w}$ is pure of weight $-2i$. Let

$$\Delta_w^{i\zeta} := j_{w!}(R/I^i)_{\mathrm{Fl}_w}, \quad \mathrm{IC}_w^{i\zeta} := j_{w!*}(R/I^i)_{\mathrm{Fl}_w}, \quad \nabla_w^{i\zeta} := j_{w*}(R/I^i)_{\mathrm{Fl}_w} \in \mathrm{Perv}_{(I)}^{\mathrm{mix}}(\mathrm{Fl})^{n\zeta}.$$

Proposition 14.3.1. *Let $\zeta \in \mathbb{T}$ and let $i = \ell_\zeta(w_\zeta)$. Then the following map vanishes*

$$(48) \quad \mathrm{Hom}(\nabla_w^{i\zeta}, \Delta_w^{i\zeta}) \otimes \mathrm{Hom}(\Delta_w^{i\zeta}, \nabla_w^{i\zeta}) \rightarrow \mathrm{Hom}(\nabla_w^{i\zeta}, \nabla_w^{i\zeta}) = R/I^i.$$

Proof. If $\phi \in \mathrm{Hom}(\nabla_w^{i\zeta}, \Delta_w^{i\zeta})$ is nonzero of weight $-2m$, then for some j it factors through a map $\nabla_w^{i\zeta} \rightarrow I^j \Delta_w^{i\zeta}$ such that the composition

$$(49) \quad \nabla_w^{i\zeta} \rightarrow I^j \Delta_w^{i\zeta} \rightarrow I^j \Delta_w^{i\zeta} / I^{j+1} \Delta_w^{i\zeta}$$

is nonzero. Moreover (49) factors through a nonzero map

$$\nabla_w^{i\zeta} = \nabla_w^{i\zeta} / I \nabla_w^{i\zeta} \rightarrow I^j \Delta_w^{i\zeta} / I^{j+1} \Delta_w^{i\zeta}.$$

Proposition 14.2.1(b) says that $\mathrm{cos} \nabla_w^{i\zeta} = \mathrm{IC}_{w^\zeta}^{i\zeta}(-i/2)$. Therefore $\mathrm{IC}_{w^\zeta}(-i/2+m)$ is a Jordan-Holder factor of $I^j \Delta_w^{i\zeta} / I^{j+1} \Delta_w^{i\zeta}$. But $I^j \Delta_w^{i\zeta} / I^{j+1} \Delta_w^{i\zeta} \simeq \Delta_w^{i\zeta} \otimes I^j / I^{j+1}$ is a direct sum of copies of $\Delta_w^{i\zeta}(j)$. Thus Proposition 14.2.1(a) implies that ϕ has weight $-2m = -2(i+j)$.

Therefore $\mathrm{Hom}(\nabla_w^{i\zeta}, \Delta_w^{i\zeta})$ is concentrated in weights $\leq -2i$. However $\mathrm{Hom}(\Delta_w^{i\zeta}, \nabla_w^{i\zeta}) = R/I^i$ is concentrated in weight ≤ 0 , and $\mathrm{Hom}(\nabla_w^{i\zeta}, \nabla_w^{i\zeta}) = R/I^i$ is concentrated in weights $> -2i$. Therefore the pairing (48) vanishes. \square

15. LOCALIZED CENTRAL SHEAVES

Here we explain that, after localizing away from all but one wall, the central functor factors through restriction to a reductive subgroup of semi-simple rank 1. Then we calculate the image of a certain associated graded map in terms of an order of vanishing condition on the wall.

15.1. Localizing the central functor. If α is a root, let $\mathbb{T}^{(\alpha)} = \text{Spec}(R^{(\alpha)})$ be the localization of \mathbb{T} away from all walls except for \mathbb{T}_α . Write $\text{Shv}_{(I)}(\mathbb{F}\mathbb{1})^{(\alpha)} := \text{Shv}_{(I)}(\mathbb{F}\mathbb{1}) \otimes_{\text{QCoh}(\mathbb{T})} \text{QCoh}(\mathbb{T}^{(\alpha)})$ for the localization of the affine Hecke category. If $A \in \text{Shv}_{(I)}(\mathbb{F}\mathbb{1})^{(\alpha)}$ write $A^{(\alpha)} := A \otimes R^{(\alpha)} \in \text{Shv}_{(I)}(\mathbb{F}\mathbb{1})^{(\alpha)}$ for its localization.

Lemma 15.1.1. *After localizing away from all but one wall, the central functor*

$$(50) \quad Z^{(\alpha)} : \text{Rep}(\mathbb{G}) \rightarrow \text{Rep}(\mathbb{G}_\alpha) \rightarrow \text{Perv}_{(I)}(\mathbb{F}\mathbb{1})^{(\alpha)}$$

factors through restriction to $\mathbb{G}_\alpha := \mathbb{T}\text{SL}(2)_\alpha$.

Proof. Recall that the Arkhipov–Bezrukavnikov functor $\text{QCoh}(\mathbb{B}/\mathbb{B}) \rightarrow \text{Shv}_{(I)}(\mathbb{F}\mathbb{1})$ is R -linear with respect to the projection $\mathbb{B}/\mathbb{B} \rightarrow \mathbb{T}$ and the right T -monodromy action on $\text{Shv}_{(I)}(\mathbb{F}\mathbb{1})$. By Proposition 11.4.1, there is a commutative diagram

$$\begin{array}{ccccc} \text{Rep}(\mathbb{G}) & \longrightarrow & \text{QCoh}(\mathbb{B}/\mathbb{B}) & \longrightarrow & \text{Shv}_{(I)}(\mathbb{F}\mathbb{1}) \\ \downarrow & & \downarrow & & \downarrow \\ \text{Rep}(\mathbb{G}_\alpha) & \longrightarrow & \text{QCoh}(\mathbb{B}_\alpha^{(\alpha)}/\mathbb{B}_\alpha) & \longrightarrow & \text{Shv}_{(I)}(\mathbb{F}\mathbb{1})^{(\alpha)}. \end{array}$$

□

Write $\underline{Z}_\lambda^{(\alpha)} \in \text{Perv}_{(I)}(\mathbb{F}\mathbb{1})$ for the image of the irreducible \mathbb{G}_α -module of highest weight λ . This is an indecomposable summand of $Z_\lambda^{(\alpha)}$.

15.2. Truncation. The following is an inductive formula for truncations of central sheaves.

Proposition 15.2.1. *There is an isomorphism $\underline{Z}_\lambda^{(\alpha)} * W_{s\omega}^{(\alpha)} \simeq (\underline{Z}_{\lambda+\omega}^{(\alpha)})^{\leq \lambda+s\omega}$, the $\leq \lambda+s\omega$ filtered piece of the Wakimoto filtration.*

Proof. By monoidality $\underline{Z}_\lambda^{(\alpha)} * \underline{Z}_\omega^{(\alpha)} \simeq \underline{Z}_{\lambda+\omega}^{(\alpha)} \oplus \underline{Z}_{\lambda+s\omega}^{(\alpha)}$, and there is a commutative diagram

$$(51) \quad \begin{array}{ccc} \underline{Z}_\lambda^{(\alpha)} * W_{s\omega}^{(\alpha)} & \xrightarrow{a} & (\underline{Z}_{\lambda+\omega}^{(\alpha)})^{\leq \lambda+s\omega} \\ \downarrow & & \downarrow \\ \underline{Z}_\lambda^{(\alpha)} * \underline{Z}_\omega^{(\alpha)} & \longrightarrow & \underline{Z}_{\lambda+\omega}^{(\alpha)}. \end{array}$$

Since $\text{gr } Z : \text{Rep}(\mathbb{G}) \rightarrow \text{Free}_{\mathbb{T}}(\mathbb{T})$ is isomorphic to restriction, gr sends (51) to

$$\begin{array}{ccc} V_\lambda|_{\mathbb{T}} \otimes k_{s\omega} \otimes R & \xrightarrow{\text{gr } a} & V_{\lambda+\omega}|_{\mathbb{T}}^{\leq \lambda+s\omega} \otimes R \\ \downarrow & & \downarrow \\ V_\lambda|_{\mathbb{T}} \otimes V_\omega|_{\mathbb{T}} \otimes R & \longrightarrow & V_{\lambda+\omega}|_{\mathbb{T}} \otimes R, \end{array}$$

in which $\text{gr } a$ is an isomorphism. Therefore a is also an isomorphism. □

15.3. Composition pairing. The following proposition says that all endomorphisms of $\nabla_w^{(\alpha)}$ that factor through $\Delta_w^{(\alpha)}$ must vanish to order $\ell_{(\alpha)}(w_{(\alpha)})$ along the wall \mathbb{T}_α .

Proposition 15.3.1. *The image of*

$$B_w^{(\alpha)} : \text{Hom}(\nabla_w^{(\alpha)}, \Delta_w^{(\alpha)}) \otimes \text{Hom}(\Delta_w^{(\alpha)}, \nabla_w^{(\alpha)}) \rightarrow \text{Hom}(\nabla_w^{(\alpha)}, \nabla_w^{(\alpha)}) = R^{(\alpha)}$$

is the ideal $(e^\alpha - 1)^{\ell_{(\alpha)}(w_{(\alpha)})} R^{(\alpha)}$.

Proof. Proposition 14.3.1 implies that

$$\mathrm{Hom}(\nabla_w^{(\alpha)}, \Delta_w^{(\alpha)}) \otimes \mathrm{Hom}(\Delta_w^{(\alpha)}, \nabla_w^{(\alpha)}) \xrightarrow{b} R^{(\alpha)} \rightarrow R^{(\alpha)} / I^{\ell_{(\alpha)}(w_{(\alpha)})}$$

vanishes for every maximal ideal I containing $(e^\alpha - 1)$. Hence $(\mathrm{im} B_w^{(\alpha)}) \subset (e^\alpha - 1)^{\ell_{(\alpha)}(w_{(\alpha)})} R^{(\alpha)}$.

Conversely, we claim by induction on $\ell(w)$ that $(e^\alpha - 1)^{\ell_{(\alpha)}(w_{(\alpha)})} R^{(\alpha)} \subset (\mathrm{im} B_w^{(\alpha)})$. Write $w = vs$ such that $\ell(w) = \ell(v) + 1$ and s is a simple reflection.

- (i) Suppose that s is the finite simple reflection associated to α . Recall that there is a short exact sequence

$$0 \rightarrow \Delta_s \rightarrow \nabla_s \rightarrow \nabla_1 / (e^\alpha - 1) \rightarrow 0$$

where the first map generates $\mathrm{Hom}(\Delta_s, \nabla_s) = R$. This gives a short exact sequence

$$0 \rightarrow \mathrm{Hom}(\nabla_s, \Delta_s) \rightarrow \mathrm{Hom}(\nabla_s, \nabla_s) \rightarrow \mathrm{Hom}(\nabla_s, \nabla_1 / (e^\alpha - 1)) = R / (e^\alpha - 1) \rightarrow 0,$$

such that the image of the first map is $(e^\alpha - 1)R^{(\alpha)} \simeq (\mathrm{im} B_s^{(\alpha)})$. Therefore by induction

$$(e^\alpha - 1)^{\ell_{(\alpha)}(w_{(\alpha)})} R^{(\alpha)} \simeq (e^\alpha - 1)^{\ell_{(\alpha)}(v_{(\alpha)})} (e^\alpha - 1) R^{(\alpha)}$$

as a submodule of $(\mathrm{im} B_v^{(\alpha)})(\mathrm{im} B_s^{(\alpha)}) \subset (\mathrm{im} B_w^{(\alpha)})$.

- (ii) Suppose that s is not the finite simple reflection associated to α . Then $\Delta_s^{(\alpha)} \simeq \nabla_s^{(\alpha)}$ is clean. Hence $\Delta_w^{(\alpha)} \simeq \Delta_v^{(s\alpha)} * \Delta_s^{(\alpha)}$ and $\nabla_w^{(\alpha)} \simeq \nabla_v^{(s\alpha)} * \Delta_s^{(\alpha)}$. Therefore by induction

$$(e^\alpha - 1)^{\ell_{(\alpha)}(w_{(\alpha)})} R^{(\alpha)} \simeq (e^{s\alpha} - 1)^{\ell_{(s\alpha)}(v_{(s\alpha)})} R^{(s\alpha)} \otimes_{R^{(s\alpha)}} R_s^{(\alpha)}$$

as a submodule of $(\mathrm{im} B_v^{(s\alpha)}) \otimes_{R^{(s\alpha)}} R_s^{(\alpha)} \simeq (\mathrm{im} B_w^{(\alpha)})$. \square

15.4. Image of associated graded. The following is the automorphic counterpart to Proposition 11.6.1.

Proposition 15.4.1. *Let $\lambda \in \Lambda^+$ and set $n = \langle \tilde{\alpha}, \lambda \rangle$. If $0 \leq i \leq n$, then*

$$\mathrm{gr} : \mathrm{Hom}(W_{\lambda - (n-i)\alpha}^{(\alpha)}, \underline{Z}_\lambda^{(\alpha)}) \rightarrow \mathrm{Hom}(\mathrm{gr} W_{\lambda - (n-i)\alpha}^{(\alpha)}, \mathrm{gr} \underline{Z}_\lambda^{(\alpha)}) = R^{(\alpha)}$$

has image $(e^\alpha - 1)^i R^{(\alpha)}$.

Proof. The proof is by induction on n .

First consider the case $i < n$. Then Proposition 15.2.1 implies

$$\begin{aligned} \mathrm{Hom}(W_{\lambda - (n-i)\alpha}^{(\alpha)}, \underline{Z}_\lambda^{(\alpha)}) &= \mathrm{Hom}(W_{\lambda - (n-i)\alpha}^{(\alpha)}, (\underline{Z}_\lambda^{(\alpha)})^{\leq \lambda - \alpha}) \\ &= \mathrm{Hom}(W_{\lambda - (n-i)\alpha}^{(\alpha)}, \underline{Z}_{\lambda - \omega}^\alpha * W_{s\omega}^{(\alpha)}) \\ &= \mathrm{Hom}(W_{\lambda - (n-i)\alpha - s\omega}^{(\alpha)}, \underline{Z}_{\lambda - \omega}^{(\alpha)}). \end{aligned}$$

By induction $\mathrm{gr} \mathrm{Hom}(W_{\lambda - (n-i)\alpha - s\omega}^{(\alpha)}, \underline{Z}_{\lambda - \omega}^{(\alpha)}) = (e^\alpha - 1)^i R^{(\alpha)}$.

Now consider the case $i = n$. Using the Wakimoto filtration for the positive Borel, Lemma 4.3.1 implies $\mathrm{Hom}(\nabla_\lambda^{(\alpha)}, \Delta_\lambda^{(\alpha)}) \simeq \mathrm{Hom}(\nabla_\lambda^{(\alpha)}, \underline{Z}_\lambda^{(\alpha)})$. The map

$$(52) \quad \mathrm{Hom}(\nabla_\lambda^{(\alpha)}, \Delta_\lambda^{(\alpha)}) \xrightarrow{\sim} \mathrm{Hom}(\nabla_\lambda^{(\alpha)}, \underline{Z}_\lambda^{(\alpha)}) \xrightarrow{\mathrm{gr}_\lambda} \mathrm{Hom}(\nabla_\lambda^{(\alpha)}, \nabla_\lambda^{(\alpha)}) = R^{(\alpha)}$$

is induced by the natural maps $\Delta_\lambda^{(\alpha)} \rightarrow \underline{Z}_\lambda^{(\alpha)} \rightarrow \nabla_\lambda^{(\alpha)}$. Proposition 15.3.1 implies that the image of (52) is the ideal $(e^\alpha - 1)^n R^{(\alpha)}$. \square

16. THE WHITTAKER EQUIVALENCE

Here we prove the universal monodromic enhancement of the Arkhipov–Bezrukavnikov equivalence [AB09].

16.1. Fully faithfulness by localization to semi-simple rank 1. For fully faithfulness, we used the following Proposition. Its proof uses Hartogs’ Lemma to reduce to semi-simple rank 1. In the semi-simple rank 1 case, we use the order of vanishing calculations from Sections 11.6 and 14.

Proposition 16.1.1. *If $\lambda, \mu \in \Lambda^+$ then*

$$(53) \quad \mathrm{Hom}_{\mathbb{B}/\mathbb{B}}(\mathcal{O}, V_\lambda \otimes \mathcal{O}(\mu)) \xrightarrow{F} \mathrm{Hom}_{\mathrm{Shv}_{(I)}(\mathrm{Fl})}(W_0, Z_\lambda \otimes W_\mu)$$

is an isomorphism.

Proof. Proposition 9.5.1(c) gives a commuting diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{B}/\mathbb{B}}(\mathcal{O}, V_\lambda \otimes \mathcal{O}(\mu)) & \xrightarrow{F} & \mathrm{Hom}(W_0, Z_\lambda \otimes W_\mu) \\ i^* \downarrow & & \downarrow \mathrm{gr} \\ \mathrm{Hom}_{\mathbb{T}/\mathbb{T}}(i^*\mathcal{O}, i^*(V_\lambda \otimes \mathcal{O}(\mu))) & \xrightarrow{\sim} & \mathrm{Hom}(\mathrm{gr} W_0, \mathrm{gr}(Z_\lambda \otimes W_\mu)). \end{array}$$

Proposition 5.3.1 says gr is injective. By a similar argument i^* is also injective. Therefore it suffices to check that gr and i^* have the same image.

Both sides of (53) are free finite rank R -modules by Propositions 13.3.1 and 11.3.1. By Hartogs’ Lemma it suffices to check, for each root α , that gr and i^* have the same image after localizing away from all walls except \mathbb{T}_α . This follows by Propositions 15.4.1 and 11.6.1. \square

16.2. Proof of the Whittaker equivalence. Let F be the Arkhipov-Bezrukavnikov functor, constructed in Proposition 9.5.1. We now show that it becomes an equivalence after post composition with Whittaker averaging.

Theorem 16.2.1. *The following functor is an equivalence of categories*

$$\chi F : \mathrm{QCoh}_{\mathbb{G}}(\tilde{\mathbb{G}}) \xrightarrow{F} \mathrm{Shv}_{(I)}(\mathrm{Fl}) \xrightarrow{{}^x\Delta_1 * -} \mathrm{Shv}_{(I, \chi)}(\mathrm{Fl}).$$

Proof. The essential image contains $\chi F(\mathcal{O}(\eta)) \simeq {}^x\Delta_1 * W_\eta$ for all $\eta \in \Lambda$. Therefore χF is essentially surjective by Proposition 12.3.2.

Now we check fully faithfulness. Propositions 16.1.1 and 11.3.1(b) imply that F and ${}^x\Delta_1 * -$ induce isomorphisms

$$\mathrm{RHom}(\mathcal{O}, V_\lambda \otimes \mathcal{O}(\mu)) \xrightarrow{\sim} \mathrm{Hom}(W_0, Z_\lambda \otimes W_\mu) \xrightarrow{\sim} \mathrm{RHom}({}^x\Delta_1, {}^x\Delta_1 * Z_\lambda \otimes W_\mu) \quad \text{for all } \lambda, \mu \in \Lambda^+.$$

Therefore χF is fully faithful by Proposition 11.1.1 and rigidity. \square

17. THE BI-WHITTAKER EQUIVALENCE

Here we prove that the universal affine bi-Whittaker category is equivalent to quasi-coherent sheaves on \mathbb{G}/\mathbb{G} . This is a universal monodromic enhancement of a results of [B09, CD23], that are Koszul dual to derived Satake [BF07].

17.1. Notation. In this section, write $\mathcal{H}^f := \mathrm{Shv}_{(B)}(G/U)$ for the finite Hecke category. Let ${}_\chi\mathcal{H}^f$ and \mathcal{H}_χ^f denote the right and left module categories on which \mathcal{H}^f acts via Soergel’s functor as in Section 12.1. Define the finite bi-Whittaker category

$${}_\chi\mathcal{H}_\chi^f := {}_\chi\mathcal{H}^f \otimes_{\mathcal{H}^f} \mathcal{H}_\chi^f \simeq \mathrm{End}_{\mathcal{H}^f}({}_\chi\mathcal{H}^f),$$

where we used the rigidity of \mathcal{H}^f and canonical duality $\mathcal{H}_\chi^f \simeq ({}_\chi\mathcal{H}^f)^\vee$.

In this section, write $\mathcal{H} := \mathrm{Shv}_{(I)}(\mathrm{Fl})$ for the affine Hecke category. Let ${}_{\chi}\mathcal{H} := {}_{\chi}\mathcal{H}^{\mathrm{f}} \otimes_{\mathcal{H}^{\mathrm{f}}} \mathcal{H}$ and $\mathcal{H}_{\chi} := \mathcal{H} \otimes_{\mathcal{H}^{\mathrm{f}}} \mathcal{H}_{\chi}^{\mathrm{f}}$ denote the affine Whittaker categories. Define the bi-Whittaker category

$${}_{\chi}\mathcal{H}_{\chi} := {}_{\chi}\mathcal{H}^{\mathrm{f}} \otimes_{\mathcal{H}^{\mathrm{f}}} \mathcal{H} \otimes_{\mathcal{H}^{\mathrm{f}}} \mathcal{H}_{\chi}^{\mathrm{f}} \simeq \mathrm{End}_{\mathcal{H}^{\mathrm{f}}}({}_{\chi}\mathcal{H}).$$

Recall that there exists \mathbf{G}^{sc} , a product of a torus and a simply connected reductive group, such that $\mathbf{G} = \mathbf{G}^{\mathrm{sc}}/\mathbf{Z}$ is the quotient by a finite central subgroup. We defined $\mathbf{C} := (\mathbf{G}^{\mathrm{sc}}//\mathbf{G}^{\mathrm{sc}})/\mathbf{Z}$.

Let $\Xi \in \mathrm{Perv}_{(B)}(G/U)$ denote the universal monodromic big tilting sheaf. Recall from [T23] that endomorphisms of the big tilting are

$$(54) \quad \mathrm{End}(\Xi) \simeq \mathcal{O}(\mathbf{T} \times_{\mathbf{C}} \mathbf{T}).$$

17.2. The finite bi-Whittaker category. The following is a Betti version of results of [Gin18, Lon18, Gan22] in the de Rham setting and of [BD23] in the ℓ -adic setting. In our setup it follows from the universal monodromic Endomorphismensatz [T23].

Lemma 17.2.1. *There is a monoidal equivalence*

$$\mathrm{QCoh}(\mathbf{C}) \simeq {}_{\chi}\mathcal{H}_{\chi}^{\mathrm{f}},$$

such that the following square commutes

$$(55) \quad \begin{array}{ccc} \mathrm{QCoh}(\mathbf{C}) & \xleftarrow{\sim} & {}_{\chi}\mathcal{H}_{\chi}^{\mathrm{f}} \\ \pi^* \downarrow & & \downarrow -{}^{\chi}\Delta_1 \\ \mathrm{QCoh}(\mathbf{T}) & \xleftarrow{\sim} & {}_{\chi}\mathcal{H}^{\mathrm{f}} \end{array}$$

Proof. The actions ${}_{\chi}\mathcal{H}^{\mathrm{f}} \curvearrowright \mathcal{H}^{\mathrm{f}} \curvearrowright \mathcal{H}_{\chi}^{\mathrm{f}}$ each factor through a monoidal functor

$$\mathcal{H}^{\mathrm{f}} \rightarrow \mathrm{QCoh}(\mathbf{T} \times_{\mathbf{C}} \mathbf{T}),$$

that admits a fully faithful left adjoint by (54). This induces a monoidal colocalization functor

$$(56) \quad {}_{\chi}\mathcal{H}_{\chi}^{\mathrm{f}} \simeq {}_{\chi}\mathcal{H}^{\mathrm{f}} \otimes_{\mathcal{H}^{\mathrm{f}}} \mathcal{H}_{\chi}^{\mathrm{f}} \rightarrow \mathrm{QCoh}(\mathbf{C}) \simeq \mathrm{QCoh}(\mathbf{T}) \otimes_{\mathrm{QCoh}(\mathbf{T} \times_{\mathbf{C}} \mathbf{T})} \mathrm{QCoh}(\mathbf{T}),$$

making (55) commute. The last equivalence is by [BZFN12], using that $\mathbf{T} \rightarrow \mathbf{C}$ is proper and surjective. The left adjoint to (56) is essentially surjective, because it sends $R \mapsto {}^{\chi}\Delta_1 * \Delta_1^{\chi}$. Therefore (56) is an equivalence. \square

17.3. Centrality on the derived level. Proposition 6.6.1 implies that Gaitsgory's functor through a monoidal functor $\mathrm{Free}_{\mathbf{G}}(\mathbf{G}) \rightarrow \mathbf{Z}(\mathrm{Perv}_{(I)}^{\mathrm{waki}}(\mathrm{Fl}))$. Here we extend this central structure to the derived level.

Lemma 17.3.1. *Gaitsgory's functor lifts to a monoidal functor to the center of the affine Hecke category*

$$(57) \quad \mathbf{Z} : \mathrm{QCoh}_{\mathbf{G}}(\mathbf{G}) \rightarrow \mathbf{Z}(\mathcal{H}).$$

Proof. Let $\mathcal{J} \subset \mathrm{Perv}_{(I)}(\mathrm{Fl})$ be the full additive subcategory generated by finite direct sums and summands of $Z_{\lambda} * \Xi_w$, where Z_{λ} is a universal central sheaf, and Ξ_w is universally tilting. Then \mathcal{J} inherits a monoidal structure, compatible with its inclusion into \mathcal{H} . The centrality isomorphism yields a lift

$$\mathbf{Z} : \mathrm{Free}_{\mathbf{G}}(\mathbf{G}) \rightarrow \mathbf{Z}(\mathcal{J}).$$

Passing to the bounded homotopy category gives a natural monoidal functor $\mathbf{K}(\mathcal{J}) \rightarrow \mathrm{Shv}_{(I)}(\mathrm{Fl})$, and a compatible map

$$\mathbf{Z} : \mathbf{K}(\mathrm{Free}_{\mathbf{G}}(\mathbf{G})) \simeq \mathrm{Perf}_{\mathbf{G}}(\mathbf{G}) \rightarrow \mathbf{Z}(\mathbf{K}(\mathcal{J})).$$

Write $A(\mathcal{T}) \subset K(\mathcal{T})$ for the full subcategory of acyclic complexes. The Verdier quotient $K(\mathcal{T})/A(\mathcal{T})$ inherits a monoidal structure, a fully faithful monoidal functor $K(\mathcal{T})/A(\mathcal{T}) \rightarrow \text{Shv}_{(I)}(\text{Fl})$, and a compatible map

$$Z : \text{Perf}_{\mathbb{G}}(\mathbb{G}) \rightarrow Z(K(\mathcal{T})/A(\mathcal{T})).$$

Ind-completing $\text{Ind}(K(\mathcal{T})/A(\mathcal{T})) \simeq \mathcal{H}$, giving the desired functor (57). \square

17.4. Construction of the functor. Now we construct the bi-Whittaker functor, using that the center of an algebra maps to the endomorphisms of any module for that algebra.

The commuting actions $\mathcal{H}^f \curvearrowright \mathcal{H}_\chi^f \simeq \text{QCoh}(\mathbb{T}) \curvearrowright \text{QCoh}(\mathbb{C})$ endow ${}_\chi\mathcal{H}_\chi \simeq {}_\chi\mathcal{H} \otimes_{\mathcal{H}^f} \mathcal{H}_\chi^f$ with a $\text{QCoh}(\mathbb{C})$ -linear structure.

Proposition 17.4.1. *There exists monoidal $\text{QCoh}(\mathbb{C})$ -linear functor*

$$(58) \quad {}_\chi F_\chi : \text{QCoh}(\mathbb{G}/\mathbb{G}) \rightarrow {}_\chi\mathcal{H}_\chi,$$

such that the following commutes

$$(59) \quad \begin{array}{ccc} \text{QCoh}(\mathbb{G}/\mathbb{G}) & \xrightarrow{{}_\chi F_\chi} & {}_\chi\mathcal{H}_\chi \\ \downarrow Z & & \downarrow -*{}^\chi\Delta_1 \\ \mathcal{H} & \xrightarrow{{}_\chi\Delta_1*-} & {}_\chi\mathcal{H}. \end{array}$$

Proof. The action of ${}_\chi\mathcal{H} \curvearrowright \mathcal{H}$ induces a monoidal functor

$${}_\chi F_\chi : \text{QCoh}(\mathbb{G}/\mathbb{G}) \xrightarrow{(57)} Z(\mathcal{H}) \rightarrow \text{End}_{\mathcal{H}}({}_\chi\mathcal{H}) \simeq {}_\chi\mathcal{H}_\chi.$$

Moreover (59) commutes because there is an isomorphism

$${}_\chi F_\chi(-) * {}^\chi\Delta_1 * - \simeq {}^\chi\Delta_1 * - * Z(-) \simeq {}^\chi\Delta_1 * Z(-) * -,$$

of bi-functors $\text{QCoh}(\mathbb{G}/\mathbb{G}) \times \mathcal{H} \rightarrow {}_\chi\mathcal{H}$.

Proposition 9.5.1(b) implies that Z factors through a $\text{QCoh}(\mathbb{T})$ -linear functor $\text{QCoh}(\mathbb{G}/\mathbb{G} \times_{\mathbb{C}} \mathbb{T}) \rightarrow \mathcal{H}$. Therefore ${}_\chi F_\chi(-) * {}^\chi\Delta_1$ is $\text{QCoh}(\mathbb{C})$ -linear by (59). Thus ${}_\chi F_\chi(-) * {}^\chi\Delta_1 * \Delta_1^\chi$ is also $\text{QCoh}(\mathbb{C})$ -linear. But ${}_\chi F_\chi(-) * {}^\chi\Delta_1 * \Delta_1^\chi \simeq \bigoplus_{W^f} {}_\chi F_\chi$ is a direct sum of copies of ${}_\chi F_\chi$. Hence ${}_\chi F_\chi$ is also $\text{QCoh}(\mathbb{C})$ -linear. \square

17.5. Compact generators. Now we show that the bi-Whittaker category is compactly generated by the images of vector bundles on \mathbb{G}/\mathbb{G} .¹⁸

Proposition 17.5.1. *The bi-Whittaker category ${}_\chi\mathcal{H}_\chi$ is compactly generated by ${}_\chi F_\chi(V_\lambda \otimes \mathcal{O})$ for $\lambda \in \Lambda^+$.*

Proof. For support reasons, \mathcal{H} is compactly generated by $\Delta_w * \Delta_\lambda * \Delta_v$ for $w, v \in W^f$ and $\lambda \in \Lambda^+$. By Lemma 12.1.5, bi-Whittaker averaging identifies ${}^\chi\Delta_1 * \Delta_w * \Delta_\lambda * \Delta_v * \Delta_1^\chi \simeq {}^\chi\Delta_1 * \Delta_\lambda * \Delta_1^\chi$. Therefore ${}_\chi\mathcal{H}_\chi \simeq {}_\chi\mathcal{H}^f \otimes_{\mathcal{H}^f} \mathcal{H} \otimes_{\mathcal{H}^f} \mathcal{H}_\chi^f$ is compactly generated by ${}^\chi\Delta_1 * \Delta_\lambda * \Delta_1^\chi$ for $\lambda \in \Lambda^+$. Let ${}_\chi\mathcal{H}_\chi^{\leq \lambda}$ and ${}_\chi\mathcal{H}_\chi^{< \lambda}$ be the full subcategories generated by ${}^\chi\Delta_1 * \Delta_\mu * \Delta_1^\chi$ for $\mu \leq \lambda$ and $\mu < \lambda$. Then ${}_\chi F_\chi(V_\lambda \otimes \mathcal{O})$ generates ${}_\chi\mathcal{H}_\chi^{\leq \lambda} / {}_\chi\mathcal{H}_\chi^{< \lambda}$. Therefore ${}_\chi\mathcal{H}_\chi$ is compactly generated by ${}_\chi F_\chi(V_\lambda \otimes \mathcal{O})$. \square

¹⁸Beware that ${}_\chi F_\chi(V_\lambda \otimes \mathcal{O})$ is different from ${}^\chi\Delta_1 * Z_\lambda * \Delta_1^\chi$, obtained by bi-Whittaker averaging the central sheaf.

17.6. Proof of the bi-Whittaker equivalence. Finally we prove that the bi-Whittaker functor is an equivalence by pulling back along $\mathbb{T} \rightarrow \mathbb{C}$.

Theorem 17.6.1. *The monoidal functor ${}_{\chi}F_{\chi} : \mathrm{QCoh}(\mathbb{G}/\mathbb{G}) \rightarrow {}_{\chi}\mathcal{H}_{\chi}$ is an equivalence.*

Proof. By the above Lemma 17.5.1 it remains to check that ${}_{\chi}F_{\chi}$ is fully faithful. Let

$$\phi : {}_{\chi}\mathcal{H}_{\chi} \otimes_{\mathrm{QCoh}(\mathbb{C})} \mathrm{QCoh}(\mathbb{T}) \simeq {}_{\chi}\mathcal{H} \otimes_{\mathcal{H}^{\mathrm{f}}} \mathrm{QCoh}(\mathbb{T}) \otimes_{\mathrm{QCoh}(\mathbb{C})} \mathrm{QCoh}(\mathbb{T}) \rightarrow {}_{\chi}\mathcal{H} \simeq {}_{\chi}\mathcal{H} \otimes_{\mathcal{H}^{\mathrm{f}}} \mathcal{H}^{\mathrm{f}}$$

be induced by $\Xi \otimes - : \mathrm{QCoh}(\mathbb{T} \times_{\mathbb{C}} \mathbb{T}) \rightarrow \mathcal{H}^{\mathrm{f}}$, the left adjoint to Soergel's functor.

By Proposition 17.4.1, there is a commutative diagram

$$\begin{array}{ccccc} \mathrm{QCoh}(\mathbb{G}/\mathbb{G}) & \longrightarrow & \mathrm{QCoh}(\mathbb{G}/\mathbb{G} \times_{\mathbb{C}} \mathbb{T}) & \xrightarrow{p^*} & \mathrm{QCoh}(\mathbb{B}/\mathbb{B}) \\ \downarrow {}_{\chi}F_{\chi} & & \downarrow {}_{\chi}F_{\chi} \otimes \mathrm{id}_{\mathbb{T}} & & \downarrow {}_{\chi}F \\ {}_{\chi}\mathcal{H}_{\chi} & \longrightarrow & {}_{\chi}\mathcal{H}_{\chi} \otimes_{\mathrm{QCoh}(\mathbb{C})} \mathrm{QCoh}(\mathbb{T}) & \xrightarrow{\phi} & {}_{\chi}\mathcal{H}, \end{array}$$

in which

- (i) p^* is fully faithful by Proposition 11.2.1,¹⁹
- (ii) ${}_{\chi}F$ is fully faithful by Theorem 16.2.1,
- (iii) and ϕ is fully faithful by Equation (54).

Therefore ${}_{\chi}F_{\chi} \otimes \mathrm{id}_{\mathbb{T}}$ is fully faithful. Hence ${}_{\chi}F_{\chi}$ is fully faithful, by faithful flatness of $\mathbb{T} \rightarrow \mathbb{C}$. \square

Remark 17.6.2. As written, our proof of the bi-Whittaker equivalence uses the Whittaker equivalence as an input. However, one should view the bi-Whittaker equivalence as an easier assertion, for the basic reason that \mathbb{G} is affine, unlike $\tilde{\mathbb{G}}$, and one already has the central sheaves. For this reason, we note that one can indeed prove Theorem 17.6.1 directly using only the central functor and the order of vanishing calculations, but without constructing the functor $\mathrm{QCoh}(\mathbb{B}/\mathbb{B}) \rightarrow {}_{\chi}\mathcal{H}$. In particular, one can bypass the arguments concerning the Drinfeld-Plücker formalism.

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¹⁹Strictly speaking, we only checked that the underived pullback functor is fully faithful on the abelian category of coherent sheaves. This suffices for the present argument because $\mathbb{G}/\mathbb{G} \times_{\mathbb{C}} \mathbb{T}$ is the quotient of an affine variety by a reductive group, and maps between Whittaker averaged central sheaves are concentrated in degree 0.

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