

TAME LOCAL BETTI GEOMETRIC LANGLANDS

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ABSTRACT. We prove a monoidal equivalence between spectral and automorphic realizations of the universal affine Hecke category, thereby proving the tamely ramified local Betti geometric Langlands correspondence, as conjectured by Ben-Zvi–Nadler [BZN07, BZN18]. Specializing to the case of unipotent monodromy, this provides another argument for a fundamental theorem of Bezrukavnikov [B16].

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1. INTRODUCTION AND STATEMENT OF RESULTS

1.1. Introduction.

1.1.1. In this paper we establish the tame local Betti geometric Langlands correspondence, confirming a conjecture of Ben-Zvi–Nadler [BZN07, BZN18]. In the remainder of this subsection, we recall some context; the reader may skip it and proceed directly to its sequel for a precise formulation of our results, particularly Theorem 1.2.5.

1.1.2. Let G be a connected reductive group and X a complex algebraic curve, not necessarily projective. To each of the three standard sheaf theories on X , namely

- (i) sheaves of complex vector spaces in the analytic topology,
- (ii) algebraic D-modules, and
- (iii) ℓ -adic sheaves,

one has an associated groupoid of tamely ramified G -connections on X . For example, if X is projective, these are simply the groupoids of symmetric monoidal functors from $\text{Rep}(G)$ to the corresponding category of sheaves on X .

1.1.3. These three groupoids are naturally the complex points of derived algebraic stacks, commonly called the moduli spaces of (i) Betti, (ii) de Rham, and (iii) restricted variation G -connections on X with tame ramification, respectively. While the analytifications of the moduli spaces (i) and (ii) agree, by the Riemann–Hilbert correspondence, and the moduli space (iii) may be identified with a certain formal substack of both (i) and (ii), the algebraic structures of all three are markedly different, and to each one associates a corresponding version of the geometric Langlands correspondence.

1.1.4. Let us specialize to a cylinder C , i.e., a genus zero curve with two points removed. In this case, gluing of cylinders endows the category of sheaves on the moduli spaces of tame local systems with a monoidal structure, and this naturally acts on sheaves on tame moduli spaces for arbitrary X , i.e., stores the tame Hecke operators. The arising 2-category of modules for this monoidal category is the subject of the tame local geometric Langlands correspondence.

1.1.5. If we only consider local systems on C with unipotent monodromy around the two punctures, the Betti, de Rham, and restricted variation moduli spaces are canonically isomorphic, even before analytification, and local geometric Langlands in this case was essentially proven by Bezrukavnikov in the celebrated work [B16].

1.1.6. By contrast, if we allow the monodromy to vary around the punctures, we get three genuinely different moduli spaces, and studying the resulting categories of sheaves introduces new complications not present in the case of a fixed semisimplified monodromy, particularly in the Betti and de Rham cases.

1.1.7. As far as we know, for non-abelian \mathbf{G} , no instances of geometric Langlands, local or global, in such families have been proven in the literature. The contribution of this paper is to settle the local case for tame Betti families.

1.2. Statement of results.

1.2.1. To describe our results, we need some standard notation. Let k denote a coefficient field of characteristic zero.

1.2.2. Fix a split pinned connected reductive group \mathbf{G} over k . Write $\mathbf{B} \subset \mathbf{G}$ for the corresponding Borel, and denote the multiplicative Grothendieck–Springer resolution by

$$\tilde{\mathbf{G}} \simeq \mathbf{G} \times_{\mathbf{B}} \mathbf{B} \rightarrow \mathbf{G}.$$

On the spectral side, we consider the ∞ -category of ind-coherent sheaves on the Steinberg stack

$$\mathrm{IndCoh}_{\mathbf{G}}(\tilde{\mathbf{G}} \times_{\mathbf{G}} \tilde{\mathbf{G}}) \simeq \mathrm{IndCoh}(\mathbf{B}/\mathbf{B} \times_{\mathbf{G}/\mathbf{G}} \mathbf{B}/\mathbf{B}),$$

which is naturally monoidal under convolution.

1.2.3. Denote the Langlands dual pinned complex reductive group to \mathbf{G} by G , with corresponding maximal unipotent subgroup $N \subset G$. Denote the complex arc and loop groups of G by

$$L^+G := G(\mathbf{C}[[t]]) \hookrightarrow LG \simeq G(\mathbf{C}((t))),$$

and denote by \mathring{I} the prounipotent radical of the Iwahori subgroup, i.e., \mathring{I} is the preimage of N under the evaluation map $L^+G \rightarrow G$. We may form the associated universal monodromic Hecke stack $\mathring{I} \backslash LG / \mathring{I}$.

On the automorphic side, we consider the ∞ -category of Betti sheaves of k -vector spaces with nilpotent singular support on the Hecke stack

$$\mathrm{Shv}_{\mathrm{nilp}}(\mathring{I} \backslash LG / \mathring{I}),$$

which is naturally monoidal under convolution.

1.2.4. With this notation in hand, we may state the following.

Theorem 1.2.5. *There is an equivalence of monoidal ∞ -categories*

$$\mathrm{IndCoh}_{\mathbf{G}}(\tilde{\mathbf{G}} \times_{\mathbf{G}} \tilde{\mathbf{G}}) \simeq \mathrm{Shv}_{\mathrm{nilp}}(\mathring{I} \backslash LG / \mathring{I}).$$

We again emphasize that if we pass on both sides to the subcategories with unipotent monodromy, this recovers a fundamental theorem of Bezrukavnikov [B16].

Remark 1.2.6. Let us write $2\text{-IndCoh}_{\text{nilp}}(\mathbf{G}/\mathbf{G})$ for the 2-category of ind-coherent sheaves of categories over \mathbf{G}/\mathbf{G} with nilpotent singular support, and $\text{Shv}_{\text{nilp}}(\mathring{I}\backslash LG/\mathring{I})\text{-mod}$ for the 2-category of k -linear stable, presentable categories equipped with an action of $\text{Shv}_{\text{nilp}}(\mathring{I}\backslash LG/\mathring{I})\text{-mod}$. Then an immediate consequence of Theorem 1.2.5 is that one has an equivalence of $(\infty, 2)$ -categories

$$2\text{-IndCoh}_{\text{nilp}}(\mathbf{G}/\mathbf{G}) \simeq \text{Shv}_{\text{nilp}}(\mathring{I}\backslash LG/\mathring{I})\text{-mod}.$$

However, as we are unaware of a written reference for the theory of 2-IndCoh with singular supports, as developed by Arinkin, we omit a formal deduction; briefly, it follows from the fact that $\tilde{\mathbf{G}}/\mathbf{G} \rightarrow \mathbf{G}/\mathbf{G}$ is a proper map between smooth stacks, surjective on geometric points, and the codirections in \mathbf{G}/\mathbf{G} orthogonal to the tangent complex of $\tilde{\mathbf{G}}/\mathbf{G}$ are precisely the nilpotent codirections.

1.2.7. Theorem 1.2.5 and Remark 1.2.6 establish tame local Betti geometric Langlands, as conjectured by Ben-Zvi–Nadler. Namely, the statement of Theorem 1.2.5 appears in their work [BZN07], and recurs implicitly in their work with Francis [BZFN12]. It was further discussed in their work [BZN18] introducing the Betti Langlands correspondence. The precise formulation proven in the present text appears in the recent very interesting work of Nadler–Li–Yun [LNY24] on functions on commuting stacks, where it is used as a formal input in their analysis.

1.3. Idea of the proof.

1.3.1. At a high level, we essentially follow the beautiful strategy of proof due to Kazhdan–Lusztig [KL87] at the function-theoretic level and Arkhipov–Bezrukavnikov at the sheaf-theoretic level for unipotent monodromy [AB09, B16]. However, in terms of the details, using some tools from higher algebra, we are able to give a proof which perhaps comes closer to directly lifting the function-theoretic argument of Kazhdan–Lusztig to the sheaf-theoretic level than seems possible to do at the triangulated level.

We should mention that there are many further variants, known and expected, of Theorem 1.2.5, e.g. in the de Rham setting, with modular coefficients [BRR20, BR22, BR24], or in mixed characteristic [AGLR22, ALWY23]. We expect that the present approach can be useful in those settings as well.

In the remainder of this subsection, let us describe the ingredients of the argument, old and new, in slightly more detail.

1.3.2. As in the work of Kazhdan–Lusztig and Arkhipov–Bezrukavnikov, we prove Theorem 1.2.5 by matching the tautological module on the spectral side

$$\text{QCoh}_{\mathbf{G}}(\tilde{\mathbf{G}}) \circ \text{IndCoh}_{\mathbf{G}}(\tilde{\mathbf{G}} \times_{\mathbf{G}} \tilde{\mathbf{G}})$$

with the appropriately defined Iwahori–Whittaker module on the automorphic side

$$\text{Shv}_{\text{nilp}}(\mathring{I}, \chi \backslash LG/\mathring{I}) \circ \text{Shv}_{\text{nilp}}(\mathring{I} \backslash LG/\mathring{I}).$$

Here, the underlying equivalence of categories

$$\text{QCoh}_{\mathbf{G}}(\tilde{\mathbf{G}}) \simeq \text{Shv}_{\text{nilp}}(\mathring{I}, \chi \backslash LG/\mathring{I})$$

was established in our previous article [DT24]. We recall that, again in broad strokes, this followed the cue of Kazhdan–Lusztig and Arkhipov–Bezrukavnikov, where the latter established the unipotent analogue

$$\text{QCoh}_{\mathbf{B}}(\tilde{\mathcal{N}}) \simeq \text{Shv}(\mathring{I}, \chi \backslash LG/I),$$

where $\tilde{\mathcal{N}}$ denotes the Springer resolution of the unipotent cone, and $I \subset L^+G$ the Iwahori subgroup.

1.3.3. In the setting of Kazhdan–Lusztig, i.e., for affine Hecke algebras rather than affine Hecke categories, the underlying vector space of the modules on the spectral and automorphic sides are both essentially the Grothendieck group of $\text{Rep}(\mathbf{T})$, and the action of both algebras on it are faithful. Therefore, one has to match the action of generators, and the isomorphism of algebras follows. I.e, one realizes both algebras at the ‘same’ set of operators acting on the ‘same’ vector space.

1.3.4. In formulating a version of this strategy in the categorical setting, Bezrukavnikov deploys some fundamental observations, notably an intrinsic characterization of bounded coherent sheaves as an enlargement of perfect complexes. This in particular has been highly influential on many subsequent works in this area, including the argument given below.

Even with these observations in hand, however, it seems difficult to straightforwardly adapt the original argument of Kazhdan–Lusztig when working at the level of triangulated categories. To circumvent this, Bezrukavnikov uses many further interesting maneuvers; we highlight in particular his use of the pointwise tensor product, rather than convolution, monoidal structure on the spectral side to set up the underlying equivalence of categories.

As he writes in [B16],

In this text we follow the original plan conceived more than a decade ago and treat the issues of homological algebra by ad hoc methods, using explicit DG models for triangulated categories of constructible sheaves based on generalized tilting sheaves. While the properties of tilting sheaves established in the course of the argument are (in the author’s opinion) of an independent interest, it is likely that recent advances in homotopy algebra can be used to develop an alternative approach.

1.3.5. Below we use exactly such advances in homotopy algebra to directly obtain an isomorphism of monoidal categories from matching the modules. The basic idea of the present paper is the following.

Naïvely, one would like to approximately say that the action, say for definiteness on the spectral side, yields a fully faithful monoidal embedding

$$(1) \quad \mathrm{IndCoh}_{\mathbb{G}}(\tilde{\mathbb{G}} \times_{\mathbb{G}} \tilde{\mathbb{G}}) \rightarrow \mathrm{End}(\mathrm{QCoh}_{\mathbb{G}}(\tilde{\mathbb{G}})).$$

If this held, as well as its automorphic analogue, then one could finish by matching essential images, as in the work of Kazhdan–Lusztig. I.e., one would have realized both monoidal categories as the ‘same’ category of endofunctors acting on the ‘same’ category.

This does not quite work, as, up to renormalization issues, the map (1) identifies with the pushforward of quasicoherent sheaves along the map

$$(\tilde{\mathbb{G}} \times_{\mathbb{G}} \tilde{\mathbb{G}})/\mathbb{G} \rightarrow \tilde{\mathbb{G}}/\mathbb{G} \times \tilde{\mathbb{G}}/\mathbb{G},$$

and in particular is not fully faithful, already for a torus. To address this, we note that we instead have

$$\mathrm{QCoh}_{\mathbb{G}}(\tilde{\mathbb{G}} \times_{\mathbb{G}} \tilde{\mathbb{G}}) \simeq \mathrm{End}_{\mathrm{QCoh}_{\mathbb{G}}(\mathbb{G})}(\mathrm{QCoh}_{\mathbb{G}}(\tilde{\mathbb{G}})),$$

i.e., we need to consider the commuting action of $\mathrm{QCoh}_{\mathbb{G}}(\mathbb{G})$ on $\mathrm{QCoh}_{\mathbb{G}}(\tilde{\mathbb{G}})$ as well.

For this reason, in our previous article [DT24], we provided a monoidal equivalence between $\mathrm{QCoh}_{\mathbb{G}}(\mathbb{G})$ and an appropriately defined bi-Whittaker category $\mathrm{Shv}_{\mathrm{nilp}}(\mathring{I}, \chi \backslash G_F / \mathring{I}, \chi)$; we note that in the unipotent case analogous results were obtained in [B09], [CD23]. With this in hand, the above naïve argument can be adapted to prove Theorem 1.2.5. Namely, the commuting actions

$$\mathrm{Shv}_{\mathrm{nilp}}(\mathring{I}, \chi \backslash LG / \mathring{I}, \chi) \circ \mathrm{Shv}_{\mathrm{nilp}}(\mathring{I}, \chi \backslash LG / \mathring{I}) \circ \mathrm{Shv}_{\mathrm{nilp}}(\mathring{I} \backslash LG / \mathring{I})$$

then directly give rise to a monoidal functor

$$(2) \quad \mathrm{Shv}_{\mathrm{nilp}}(\mathring{I} \backslash LG / \mathring{I}) \rightarrow \mathrm{QCoh}_{\mathbb{G}}(\tilde{\mathbb{G}} \times_{\mathbb{G}} \tilde{\mathbb{G}}),$$

which up to renormalization issues is the sought-for equivalence. That is, we realize the two sides of Theorem 1.2.5 as the ‘same’ category of equivariant endofunctors of the ‘same’ category, which is the promised lift of the strategy of Kazhdan–Lusztig.

1.3.6. As far as we are aware, the proof given in this paper is new, even in the unipotent case. However, we would like to mention some antecedents.

First, we should note our approach to constructing the functor was influenced by unpublished strategies of Ben-Zvi–Nadler, Gaitsgory–Lurie, and Arinkin–Bezrukavnikov for the spherical case, which reprove the derived Satake equivalence of Bezrukavnikov–Finkelberg [BF07] via acting on the spherical Whittaker category; we highlight the approach of Gaitsgory–Lurie was recently successfully realized by Campbell–Raskin [CR23].

Second, after constructing the functor, one must handle the arising renormalization issues to obtain an equivalence. Compared to the analysis found in [B16], we have opted to give ‘soft analysis’ proofs throughout, e.g. showing the t -boundedness of certain functors without explicitly bounding the amplitude. Moreover, the proofs of the t -boundedness essentially reduce to the fact that Wakimoto sheaves, which correspond to the lattice in the affine Weyl group, lie in the heart of both the automorphic and spectral t -structures, and the remaining finite Weyl group is, true to its name, finite. In particular, Lusztig’s a -function does not appear in our analysis below. With that said, we should emphasize that our approach nonetheless was greatly influenced by the arguments in [B16], particularly the striking Lemma 38.

1.4. **Acknowledgments.** It is a pleasure to thank Pramod Achar, David Ben-Zvi, Jens Eberhardt, Arnaud Eteve, Joakim Færgeman, Dennis Gaitsgory, Yau Wing Li, Ivan Losev, David Nadler, Sam Raskin, Simon Riche, David Yang, Zhiwei Yun, and Xinwen Zhu for useful correspondence and discussions. We especially thank Harrison Chen, with whom many of the basic ideas of the present paper were jointly formulated, and Roman Bezrukavnikov, for several conversations which greatly influenced our understanding of this subject. G.D. was supported by an NSF Postdoctoral Fellowship under grant No. 2103387. J.T. was partially supported by NSF grant DMS-1646385.

2. PRELIMINARIES

In this section, we recall some basic properties of the two sides of the equivalence, as well as the main results of [DT24]. These are then applied in Section 3 to prove Theorem 1.2.5.

2.1. Generalities.

2.1.1. By a category \mathcal{C} , we will by default mean a k -linear stable, presentable ∞ -category, and by a functor we will mean one commuting with arbitrary colimits. Given two objects c and d of a category \mathcal{C} , we will denote by $\mathrm{Hom}_{\mathcal{C}}(c, d)$ the corresponding complex of k -vector spaces.

In order to discuss compact objects or variants thereof, by a non-cocomplete category we mean a k -linear stable ∞ -category, and by functors between such we mean those commuting with finite colimits.

As a basic example, for a k -algebra A by $A\text{-mod}$ we mean the standard ∞ -category whose underlying homotopy category is the unbounded derived category of all A -modules, not necessarily finitely generated.

The Lurie tensor product endows the totality of categories with the structure of a symmetric monoidal ∞ -category with unit $\mathrm{Vect} := k\text{-mod}$. We denote the underlying binary product of two categories \mathcal{C} and \mathcal{D} by $\mathcal{C} \otimes \mathcal{D}$, and given objects c of \mathcal{C} and d of \mathcal{D} , we denote the corresponding external tensor product object of $\mathcal{C} \otimes \mathcal{D}$ by $c \boxtimes d$. Below, when we speak of monoidal categories, their modules, equivariant maps, etc., it is with respect to this structure.

2.1.2. Given a monoidal category \mathcal{A} , let us denote the binary product on its objects by

$$- \star - : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}, \quad a \boxtimes b \mapsto a \star b.$$

We denote by $\mathcal{A}^{\mathrm{rev}}$ the monoidal category obtained by passing to the reversed multiplication; in particular as categories we have $\mathrm{id} : \mathcal{A} \simeq \mathcal{A}^{\mathrm{rev}}$.

If we denote the monoidal unit of \mathcal{M} by 1 , we recall that an object a of \mathcal{A} is left dualizable if there exists another object a^{\vee} and maps $1 \rightarrow a \star a^{\vee}$ and $a^{\vee} \star a \rightarrow 1$ satisfying the usual identities.

2.1.3. For a category \mathcal{C} , we denote by $\mathcal{C}^c \subset \mathcal{C}$ the full subcategory of compact objects.

2.1.4. For a category \mathcal{C} equipped with a t -structure, and an integer n , let us write $\mathcal{C}^{\leq n}$ and $\mathcal{C}^{\geq n}$ for the corresponding full subcategories of \mathcal{C} , and $\tau^{\leq n}$ and $\tau^{\geq n}$ for the corresponding truncation functors, so that we have a distinguished triangle of endofunctors

$$\tau^{< n} \rightarrow \text{id} \rightarrow \tau^{\geq n} \xrightarrow{+1},$$

where $\tau^{< n} := \tau^{\leq (n-1)}$. It will be very convenient to also adopt the following nonstandard shorthand: *for an object c of \mathcal{C} , we will write $c \leq n$ or $c \geq n$ if c belongs to $\mathcal{C}^{\leq n}$ or $\mathcal{C}^{\geq n}$, respectively.*

Given a category \mathcal{C} equipped with a t -structure, we may consider in particular the full subcategory of infinitely connective objects

$$\mathcal{C}^{\leq -\infty} := \bigcap_n \mathcal{C}^{\leq n}.$$

2.1.5. Given a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ between categories equipped with t -structures, and integers $a, b \in \mathbb{Z}$, recall that F is said to have amplitude at most $[a, b]$ if $F(\mathcal{C}^{\leq 0}) \subset \mathcal{D}^{\leq b}$ and $F(\mathcal{C}^{\geq 0}) \subset \mathcal{D}^{\geq a}$.

2.2. Spectral preliminaries.

2.2.1. We now collect some basic ingredients on the spectral side. Recall the tautological line bundles on $\tilde{\mathbb{G}}$, i.e., the monoidal functor

$$\text{Rep}(\mathbb{T}) \rightarrow \text{Rep}(\mathbb{B}) \rightarrow \text{QCoh}_{\mathbb{G}}(\tilde{\mathbb{G}}), \quad k_\lambda \mapsto \mathcal{O}(\lambda).$$

We follow the convention that the bundles indexed with strictly dominant weights, i.e., $\mathcal{O}(\lambda)$, for $\lambda \in \Lambda^{++}$, are relatively ample for the projection $\tilde{\mathbb{G}} \rightarrow \mathbb{G}$. Tensoring by these bundles in turn gives rise to a monoidal functor

$$\text{Rep}(\mathbb{T}) \rightarrow \text{Rep}(\mathbb{B}) \rightarrow \text{QCoh}_{\mathbb{G}}(\tilde{\mathbb{G}}) \rightarrow \text{End}_{\text{QCoh}_{\mathbb{G}}(\mathbb{G})}(\text{QCoh}_{\mathbb{G}}(\tilde{\mathbb{G}})) \simeq \text{QCoh}_{\mathbb{G}}(\tilde{\mathbb{G}} \times_{\mathbb{G}} \tilde{\mathbb{G}}), \quad \lambda \mapsto \Delta_*(\mathcal{O}(\lambda)),$$

where $\Delta : \tilde{\mathbb{G}} \rightarrow \tilde{\mathbb{G}} \times_{\mathbb{G}} \tilde{\mathbb{G}}$ denotes the diagonal embedding. Here, we recall the final equivalence is a case of the basic assertion, due to Ben-Zvi–Francis–Nadler [BZFN10], that for a map of perfect stacks $X \rightarrow Y$, if we view $\text{QCoh}(X)$ as a $\text{QCoh}(Y)$ module, we have a canonical equivalence

$$\text{End}_{\text{QCoh}(Y)}(\text{QCoh}(X)) \simeq \text{QCoh}(X \times_Y X).$$

2.2.2. Consider the tautological line bundles on $\tilde{\mathbb{G}} \times_{\mathbb{G}} \tilde{\mathbb{G}}$, i.e., the (non-monoidal) functor

$$\text{Rep}(\mathbb{T} \times \mathbb{T}) \rightarrow \text{QCoh}_{\mathbb{G}}(\tilde{\mathbb{G}} \times_{\mathbb{G}} \tilde{\mathbb{G}}), \quad k_\lambda \boxtimes k_\mu \mapsto \mathcal{O}(\lambda, \mu) := \mathcal{O}(\lambda) \boxtimes \mathcal{O}(\mu).$$

Note that, for any object \mathcal{E} of $\text{QCoh}_{\mathbb{G}}(\tilde{\mathbb{G}} \times_{\mathbb{G}} \tilde{\mathbb{G}})$, we have a canonical isomorphism

$$\Delta_*(\mathcal{O}(\lambda)) \star \mathcal{E} \star \Delta_*(\mathcal{O}(\mu)) \simeq \mathcal{E} \otimes \mathcal{O}(\lambda, \mu) =: \mathcal{E}(\lambda, \mu).$$

Similarly, we have the action of tensoring with tautological vector bundles, i.e., the monoidal functor

$$\text{Rep}(\mathbb{G}) \rightarrow \text{Rep}(\mathbb{B}) \rightarrow \text{QCoh}_{\mathbb{G}}(\tilde{\mathbb{G}} \times_{\mathbb{G}} \tilde{\mathbb{G}}), \quad V \mapsto V \otimes \Delta_*(\mathcal{O}).$$

2.2.3. If we write $\sigma : \tilde{\mathbb{G}} \times_{\mathbb{G}} \tilde{\mathbb{G}} \simeq \tilde{\mathbb{G}} \times_{\mathbb{G}} \tilde{\mathbb{G}}$ for the involution swapping the two factors of the fiber product, we recall that the associated involution of its category of quasicoherent sheaves underlies a monoidal equivalence

$$(3) \quad \sigma_* : \text{QCoh}_{\mathbb{G}}(\tilde{\mathbb{G}} \times_{\mathbb{G}} \tilde{\mathbb{G}}) \simeq \text{QCoh}_{\mathbb{G}}(\tilde{\mathbb{G}} \times_{\mathbb{G}} \tilde{\mathbb{G}})^{\text{rev}}.$$

2.2.4. Let us denote the full subcategories of perfect and bounded coherent complexes by

$$\text{Perf}_{\mathbb{G}}(\tilde{\mathbb{G}} \times_{\mathbb{G}} \tilde{\mathbb{G}}) \hookrightarrow \text{Coh}_{\mathbb{G}}(\tilde{\mathbb{G}} \times_{\mathbb{G}} \tilde{\mathbb{G}}) \hookrightarrow \text{QCoh}_{\mathbb{G}}(\tilde{\mathbb{G}} \times_{\mathbb{G}} \tilde{\mathbb{G}}).$$

We recall that $\text{QCoh}_{\mathbb{G}}(\tilde{\mathbb{G}} \times_{\mathbb{G}} \tilde{\mathbb{G}})$ is compactly generated by $\text{Perf}_{\mathbb{G}}(\tilde{\mathbb{G}} \times_{\mathbb{G}} \tilde{\mathbb{G}})$. Moreover, the t -structure on $\text{QCoh}_{\mathbb{G}}(\tilde{\mathbb{G}} \times_{\mathbb{G}} \tilde{\mathbb{G}})$ may be characterized by

$$\xi \geq 0 \quad \text{if and only if} \quad \text{Hom}(\mathcal{O}(\lambda, \mu), \xi) \geq 0, \quad \text{for all } \lambda, \mu \in \Lambda.$$

2.2.5. Let us denote the functor of Serre duality by

$$\mathbb{D}_S : \mathrm{Coh}_G(\tilde{G} \times_G \tilde{G}) \simeq \mathrm{Coh}_G(\tilde{G} \times_G \tilde{G})^{\mathrm{op}}.$$

Recall this functor has amplitude $[0, \dim G]$.

We recall that $\mathrm{Coh}_G(\tilde{G} \times_G \tilde{G})$ is exactly the (left) dualizable objects of $\mathrm{QCoh}_G(\tilde{G} \times_G \tilde{G})$, and moreover one has explicitly for any \mathcal{E} in $\mathrm{Coh}_G(\tilde{G} \times_G \tilde{G})$ a canonical isomorphism

$$\mathcal{E}^\vee \simeq \sigma_* \circ \mathbb{D}_S(\mathcal{E}).$$

Moreover, as $(\tilde{G} \times_G \tilde{G})/G$ is Calabi-Yau, i.e., $\mathcal{O}_{(\tilde{G} \times_G \tilde{G})/G} \simeq \omega_{(\tilde{G} \times_G \tilde{G})/G}$, it follows that duality preserves the full subcategory of perfect complexes.

2.3. Automorphic preliminaries I: the perverse t -structure.

2.3.1. Let us consider the category of Betti sheaves with nilpotent singular support on $\mathring{I} \backslash LG / \mathring{I}$, which we will denote by

$$\mathcal{H} := \mathrm{Shv}_{\mathrm{nilp}}(\mathring{I} \backslash LG / \mathring{I}).$$

We will presently recall its definition and some of its basic properties.

2.3.2. Let us denote the abstract Cartan by $T \simeq I/\mathring{I}$, and its cocharacter lattice by Λ . Let us write $R := k[\Lambda]$ for the group algebra of the latter, i.e., the algebra of regular functions on T . Let us denote the submonoids of antidominant and dominant weights by $\Lambda^- \subset \Lambda$ and $\Lambda^+ \subset \Lambda$, respectively. Let $\Lambda^{--} \subset \Lambda^-$ and $\Lambda^{++} \subset \Lambda^+$ denote the strictly antidominant and strictly dominant weights, respectively.

Denote by W_f the finite Weyl group, and by W the (extended) affine Weyl group, so that $W \simeq W_f \rtimes \Lambda$. Let us write $\ell : W \rightarrow \mathbb{Z}^{\geq 0}$ for the standard length function on the affine Weyl group, and \leq for the Bruhat order. We recall that for any $w \in W$, the set $\{y : y \leq w\}$ is finite.

2.3.3. Recall that LG is stratified by the double cosets IwI , for $w \in W$. Explicitly, each IwI/\mathring{I} is a smooth complex manifold of dimension $d_w := \ell(w) + \dim T$, hence the same is true of any universal cover

$$\pi_w : \widetilde{IwI/\mathring{I}} \rightarrow IwI/\mathring{I}.$$

Moreover, the object $\pi_{w,!}(k[d_w])$ is a compact generator of the category of local systems $\mathrm{Shv}_{\mathrm{nilp}}(\mathring{I} \backslash IwI / \mathring{I})$, and left convolution with it¹ yields a t -exact equivalence

$$\mathrm{Shv}_{\mathrm{nilp}}(\mathring{I} \backslash IwI / \mathring{I}) \simeq \mathrm{Shv}_{\mathrm{nilp}}(\mathring{I} \backslash I / \mathring{I}) \simeq \mathrm{Shv}_{\mathrm{nilp}}(T) \simeq R\text{-mod};$$

here the final monoidal equivalence is the Mellin transform, cf. Section 2.1 of [BZN18]. That is, $\mathrm{Shv}_{\mathrm{nilp}}(\mathring{I} \backslash IwI / \mathring{I})$ is a free module of rank one over $R\text{-mod}$, and any choice of universal cover affords a trivialization.

2.3.4. Write $j_y : IyI/\mathring{I} \rightarrow LG/\mathring{I}$ for the inclusion of a stratum, and note its closure $i_y : X_y \rightarrow LG/\mathring{I}$ is an \mathring{I} -stable union of finitely many double cosets, namely $IxI/\mathring{I}, x \leq y$. We may consider the associated category

$$\mathrm{Shv}_{\mathrm{nilp}}(\mathring{I} \backslash X_y / \mathring{I})$$

of \mathring{I} -equivariant Betti sheaves on X_y/\mathring{I} with nilpotent singular support, i.e., sheaves of k -vector spaces in the analytic topology whose $*$ -restriction (or equivalently, $!$ -restriction) to each $\mathrm{Shv}(\mathring{I} \backslash IxI / \mathring{I})$ lies in $\mathrm{Shv}_{\mathrm{nilp}}(\mathring{I} \backslash IxI / \mathring{I})$.

We have $LG \simeq \varinjlim X_y$, and similarly

$$(4) \quad \mathcal{H} \simeq \varinjlim \mathrm{Shv}_{\mathrm{nilp}}(\mathring{I} \backslash X_y / \mathring{I}) =: \varinjlim \mathcal{H}_{\leq y},$$

where the transition functions in the appearing colimit are given by pushforward along the closed embeddings $X_y \rightarrow X_z, y \leq z$. In particular, for any object η of \mathcal{H} , we have that

$$\varinjlim i_{y,!} \circ i_y^!(\eta) \xrightarrow{\sim} \eta.$$

¹Of course, one could work equally well with the right convolution action.

Moreover, the perverse t -structure is characterized by $\eta \geq 0$ if and only if each $i_y^!(\eta) \geq 0$ for each $y \in W$, where we equip $\mathrm{Shv}_{\mathrm{nilp}}(X_y/\dot{I})$ with its usual perverse t -structure, cf. Equation (7) below for an equivalent characterization.

2.3.5. For a fixed stratum, consider the associated functors

$$(5) \quad j_{w,!} : \mathrm{Shv}_{\mathrm{nilp}}(\dot{I} \setminus IwI/\dot{I}) \rightarrow \mathcal{H} \quad \text{and} \quad j_{w,*} : \mathrm{Shv}_{\mathrm{nilp}}(\dot{I} \setminus IwI/\dot{I}) \rightarrow \mathcal{H}.$$

We may consider a standard object $\Delta_w := j_{w,!} \circ \pi_!(\underline{k}[d_w])$, which is well-defined up to non-unique isomorphism, and similarly we have a costandard object $\nabla_w := j_{w,*} \circ \pi_!(\underline{k}[d_w])$. More precisely, we have non-contractible groupoids of standard and costandard objects Δ_w, ∇_w , each canonically equivalent to pt/Λ , where we think of Λ as the fundamental group of IwI/\dot{I} , and note it is abelian to get the canonical isomorphism. Note the vanishing

$$(6) \quad \mathrm{Hom}(\Delta_y, \nabla_w) \simeq \begin{cases} \mathcal{L}, & y = w, \\ 0, & y \neq w, \end{cases}$$

where \mathcal{L} is a trivializable R -module of rank one.

2.3.6. The standard objects $\Delta_w, w \in W$, compactly generate \mathcal{H} , and the perverse t -structure on it may also be characterized by

$$(7) \quad \xi \in \mathcal{H} \geq 0 \quad \text{if and only if} \quad \mathrm{Hom}(\Delta_w, \xi) \geq 0, \quad \text{for all } w \in W.$$

As the monoidal unit $1 \simeq \Delta_e$ is compact, and each Δ_w is invertible with inverse $\nabla_{w^{-1}}$, it follows straightforwardly that the costandard objects are again compact generators. Moreover, we have the following basic lemma; a proof is given in Section 4.3 of [DT24].

Lemma 2.3.7. *The functors (5) are t -exact, and in particular the objects Δ_w and ∇_w are perverse.*

2.3.8. As \mathcal{H} is compactly generated by invertible objects, it follows that every compact object of \mathcal{H} is dualizable. If we write $\mathrm{inv} : LG \simeq LG$ for the inversion map $g \mapsto g^{-1}$, let us denote the associated anti-equivalence of \mathcal{H}^c by

$$\mathbb{D}_V : \mathcal{H}^c \simeq \mathcal{H}^{c,op}, \quad \xi \mapsto \mathrm{inv}_*(\xi^\vee);$$

see also the interesting paper [EE24, Section 5.3] for a less *ad hoc* construction of the duality.

Note that, by construction $\mathbb{D}_V(\Delta_w) \simeq \nabla_w$ and $\mathbb{D}_V(\nabla_w) \simeq \Delta_w$, and for two compact objects ξ, ζ one has a canonical equivalence

$$\mathbb{D}_V(\xi \star \zeta) \simeq \mathbb{D}_V(\xi) \star \mathbb{D}_V(\zeta).$$

It will be useful in the sequel to have the following estimate.

Lemma 2.3.9. *The functor \mathbb{D}_V has cohomological amplitude $[0, \dim T]$.*

Note that, as the perverse t -structure restricts to a bounded t -structure on compact objects, the statement of the lemma is equivalent to the assertion that if a compact object ξ is moreover perverse, then $\mathbb{D}_V(\xi)$ has perverse amplitude at most $[0, \dim T]$.

Proof of Lemma 2.3.9. Let us first consider the full subcategory of objects supported on the minimal stratum

$$\mathrm{Shv}_{\mathrm{nilp}}(\dot{I} \setminus I/\dot{I}) \simeq \mathrm{Shv}_{\mathrm{nilp}}(T) \simeq R\text{-mod} \simeq \mathrm{QCoh}(T).$$

Here, note that the functor of dualizing a compact object, $\xi \mapsto \xi^\vee$, is given in $\mathrm{QCoh}(T)$ by naïve duality of perfect complexes, i.e., $\mathcal{E}^\vee \simeq \mathrm{Hom}(\mathcal{E}, \mathcal{O}_T)$. As the Mellin transform exchanges pushforward along inversion on T with pushforward along inversion on T , it follows that \mathbb{D}_V corresponds to the operation

$$\mathrm{QCoh}(T)^c \simeq \mathrm{QCoh}(T)^{c,op}, \quad \mathcal{E} \mapsto \mathrm{inv}_* \mathrm{Hom}(\mathcal{E}, \mathcal{O}_T) =: \mathbb{D}_V(\mathcal{E}),$$

and in particular has cohomological amplitude $[0, \dim T]$.

We claim the lemma now follows for objects $!$ -extended from a single stratum IwI . Indeed, such an object ξ may be uniquely written as $\xi \simeq \Delta_w \star \mathcal{E}$, for $\mathcal{E} \in \mathrm{QCoh}(\mathbb{T})^c$. It follows that

$$\mathbb{D}_V(\xi) \simeq \mathbb{D}_V(\Delta_w \star \mathcal{E}) \simeq \mathbb{D}_V(\Delta_w) \star \mathbb{D}_V(\mathcal{E}) \simeq j_{w,*}(\mathbb{D}_V(\mathcal{E})),$$

whence the cohomological estimate holds by previous analysis and the exactness of $j_{w,*}$. A similar argument yields the lemma for objects $*$ -extended from a single stratum.

Finally let us address the general case. Let ξ be a compact object of \mathcal{H} . We must show that $\xi \leq 0$ implies $\mathbb{D}_V(\xi) \leq \dim T$, and $\xi \geq 0$ implies $\mathbb{D}_V(\xi) \geq 0$. To see the first claim, note that a connective compact object admits a finite filtration with successive quotients connective compact objects $!$ -extended from a single stratum, so the first assertion reduces to a previously shown case. Similarly, a coconnective compact object admits a finite filtration with successive quotients coconnective compact objects $*$ -extended from a single stratum, so the latter estimate again reduces to a previously shown case. \square

2.4. Automorphic preliminaries II: Wakimoto sheaves.

2.4.1. Recall that previously we only defined the (co)standard objects up to non-unique isomorphism, as a general stratum IwI does not have a canonical choice of basepoint. However, for $\lambda \in \Lambda \subset W$, the stratum $I\lambda I$ contains a canonical point associated to our global coordinate t , namely $t^\lambda \in LG$, hence we have a canonical choice of Δ_λ and ∇_λ , which we denote by $\Delta_\lambda^{\mathrm{can}}$ and $\nabla_\lambda^{\mathrm{can}}$, respectively. Namely $\Delta_\lambda^{\mathrm{can}}$ and $\nabla_\lambda^{\mathrm{can}}$ are rigidified by specifying an identification of their $!$ -fibers at t^λ with R .

2.4.2. Recall that if we write $2\tilde{\rho}$ for the sum of the positive roots of G , we have for any $\lambda \in \Lambda$ that $\ell(t^\lambda) = |\langle \lambda, 2\tilde{\rho} \rangle|$. In particular, for $\lambda, \mu \in \Lambda^+$, as $t^\lambda \cdot t^\mu = t^{\lambda+\mu}$ in LG , and their lengths add, we have a canonical isomorphism

$$\alpha_{\lambda,\mu} : \nabla_\lambda^{\mathrm{can}} \star \nabla_\mu^{\mathrm{can}} \simeq \nabla_{\lambda+\mu}^{\mathrm{can}},$$

which is associative in the evident sense for triples $\lambda, \mu, \nu \in \Lambda^+$, and similarly for standard objects $\Delta_\lambda^{\mathrm{can}}, \lambda \in \Lambda$.

2.4.3. Let us write $\mathrm{Rep}(\mathbb{T})^+$ for the full subcategory generated by the objects $k_\lambda, \lambda \in \Lambda^+$; this is a monoidal subcategory. The objects $\nabla_\lambda^{\mathrm{can}}, \lambda \in \Lambda^+$ together with the isomorphisms $\alpha_{\lambda,\mu}$ yield a monoidal functor

$$\mathrm{Rep}(\mathbb{T})^+ \rightarrow \mathcal{H}, \quad k_\lambda \mapsto \nabla_\lambda^{\mathrm{can}},$$

which extends by the invertibility of the costandard objects uniquely to a monoidal functor

$$\mathrm{Rep}(\mathbb{T}) \rightarrow \mathcal{H}, \quad k_\mu \mapsto W_\mu,$$

the resulting objects $W_\lambda, \lambda \in \Lambda$, are the *Wakimoto sheaves*. We in particular have canonical isomorphisms

$$W_\lambda \simeq \nabla_\lambda^{\mathrm{can}}, \quad \lambda \in \Lambda^+, \quad W_\lambda \star W_\mu \simeq W_{\lambda+\mu}, \quad \lambda, \mu \in \Lambda, \quad W_\lambda \simeq \Delta_\lambda^{\mathrm{can}}, \quad \lambda \in \Lambda^-.$$

2.4.4. It will be important for us to recall that one can attach Wakimoto sheaves to every element of W , which is done as follows. Recall that in [LNY24], [IY23], [T23], [DT24], a universal Whittaker right module for the finite Hecke category was constructed, along with a canonical isomorphism

$$\mathrm{Shv}_{\mathrm{nilp}}(N^-, \chi \backslash G/N) \simeq R\text{-mod}.$$

Let us denote the underlying binary product of the right module structure by

$$\mathrm{Shv}_{\mathrm{nilp}}(N^-, \chi \backslash G/N) \otimes \mathrm{Shv}_{\mathrm{nilp}}(N \backslash G/N) \rightarrow \mathrm{Shv}_{\mathrm{nilp}}(N, \chi \backslash G/N), \quad \mathcal{M} \boxtimes \mathcal{N} \mapsto \mathcal{M} \star \mathcal{N}.$$

We also recall that acting on the object R yields an adjunction

$$\mathrm{Av}_\chi^L : \mathrm{Shv}_{\mathrm{nilp}}(N^-, \chi \backslash G/N) \rightleftarrows \mathrm{Shv}_{\mathrm{nilp}}(N \backslash G/N) : \mathrm{Av}_\chi,$$

wherein Av_χ^L is conservative, and identifies the Whittaker category with left comodules for the big tilting coalgebra object Ξ in $\mathrm{Shv}_{\mathrm{nilp}}(N \backslash G/N)$. In particular, it carries a unique t -structure for which Av_χ^L is t -exact; we refer to this as the perverse t -structure in what follows.

2.4.5. Having introduced the universal finite Whittaker model, let us recall its affine analogue. The latter is obtained simply by induction

$${}_{\chi}\mathcal{H} := \mathrm{Shv}_{\mathrm{nilp}}(\mathring{I}^-, \chi \backslash G_F / \mathring{I}) := \mathrm{Shv}_{\mathrm{nilp}}(N^-, \chi \backslash G / N) \otimes_{\mathrm{Shv}_{\mathrm{nilp}}(N \backslash G / N)} \mathrm{Shv}_{\mathrm{nilp}}(\mathring{I} \backslash G / \mathring{I}).$$

In particular, we still have an adjunction

$$\mathrm{Av}_{\chi}^L : {}_{\chi}\mathcal{H} \rightleftarrows \mathcal{H} : \mathrm{Av}_{\chi},$$

which again identifies the Iwahori–Whittaker category with left comodules for Ξ . Again ${}_{\chi}\mathcal{H}$ inherits a unique t -structure for which Av_{χ}^L is t -exact, which we refer to as the perverse t -structure.

The corresponding category of bi-Whittaker sheaves is simply defined as its endomorphisms

$${}_{\chi}\mathcal{H}_{\chi} := \mathrm{Shv}_{\mathrm{nilp}}(\mathring{I}^-, \chi \backslash G_F / \mathring{I}^-, \chi) := \mathrm{End}_{\mathcal{G}^{\mathrm{rev}}\text{-mod}}({}_{\chi}\mathcal{H}).$$

2.5. Recollections from [DT24].

2.5.1. Finally, let us recall the relevant results obtained in [DT24] connecting the spectral and automorphic sides.

2.5.2. For a monoidal category \mathcal{A} , write $Z(\mathcal{A})$ for its center. Recall that Gaitsgory’s nearby cycles construction [G01], together with the monodromy automorphism, gives rise to a monoidal functor

$$Z : \mathrm{QCoh}_{\mathbb{G}}(\mathbb{G}) \rightarrow Z(\mathcal{H}),$$

see [DT24, Section 15.4] for the details in the universal monodromic setting. In particular, we have Gaitsgory’s central sheaves, i.e, the composition

$$\mathrm{Rep}(\mathbb{G}) \rightarrow \mathrm{QCoh}_{\mathbb{G}}(\mathbb{G}) \rightarrow Z(\mathcal{H}), \quad V \mapsto Z_V.$$

2.5.3. The first basic result we need to recall is the following, which is a universal monodromic analogue of [AB09, AR24].

Theorem 2.5.4. (1) *The composite monoidal functor*

$$\mathrm{QCoh}_{\mathbb{G}}(\mathbb{G}) \rightarrow Z(\mathcal{H}) \rightarrow \mathcal{H}$$

factors through a monoidal functor $F : \mathrm{QCoh}_{\mathbb{G}}(\tilde{\mathbb{G}}) \rightarrow \mathcal{H}$. Moreover, the obtained monoidal functor

$$\mathrm{Rep}(\mathbb{T}) \rightarrow \mathrm{Rep}(\mathbb{B}) \rightarrow \mathrm{QCoh}_{\mathbb{B}}(\mathbb{B}) \simeq \mathrm{QCoh}_{\mathbb{G}}(\tilde{\mathbb{G}}) \rightarrow \mathcal{H}$$

agrees with the Wakimoto construction. In particular, we have canonical isomorphisms $F(\Theta(\lambda)) \simeq W_{\lambda}$, for $\lambda \in \Lambda$.

(2) *The composition $\mathrm{QCoh}_{\mathbb{G}}(\tilde{\mathbb{G}}) \xrightarrow{F} \mathcal{H} \xrightarrow{\mathrm{Av}_{\chi}} {}_{\chi}\mathcal{H}$ is an equivalence of categories.*

The second basic result we need to recall is the following, which is a universal monodromic analogue of [B09], [CD23]. Note that for any right module \mathcal{M} for a monoidal category \mathcal{A} , we have a tautological monoidal functor

$$Z(\mathcal{A})^{\mathrm{rev}} \rightarrow \mathrm{End}_{\mathcal{A}}(\mathcal{M}).$$

Theorem 2.5.5. *There is a canonical equivalence of monoidal categories $\mathrm{QCoh}_{\mathbb{G}}(\mathbb{G}) \simeq {}_{\chi}\mathcal{H}_{\chi}$, given by the composition*

$$\mathrm{QCoh}_{\mathbb{G}}(\mathbb{G}) \simeq \mathrm{QCoh}_{\mathbb{G}}(\mathbb{G})^{\mathrm{rev}} \xrightarrow{Z} Z(\mathcal{H})^{\mathrm{rev}} \rightarrow {}_{\chi}\mathcal{H}_{\chi}.$$

3. PROOF OF THEOREM 1.2.5

Having gathered all the necessary ingredients in the previous section, in this section we give the proof of Theorem 1.2.5.

3.1. Construction of the functor.

3.1.1. We will begin by producing a monoidal functor from the automorphic side to the spectral side via the strategy sketched in Section 1.3, and record some of its basic properties which will be useful for later analysis.

Proposition 3.1.2. *There exists a canonical monoidal functor*

$$v^! : \mathcal{H} \rightarrow \mathrm{QCoh}_{\mathbb{G}}(\tilde{\mathbb{G}} \times_{\mathbb{G}} \tilde{\mathbb{G}})$$

Moreover, the composition

$$\mathrm{QCoh}_{\mathbb{G}}(\tilde{\mathbb{G}}) \xrightarrow{F} \mathcal{H} \xrightarrow{v^!} \mathrm{QCoh}_{\mathbb{G}}(\tilde{\mathbb{G}} \times_{\mathbb{G}} \tilde{\mathbb{G}}).$$

is canonically identified as monoidal functors with the the tautological map $\Delta_* : \mathrm{QCoh}_{\mathbb{G}}(\tilde{\mathbb{G}}) \rightarrow \mathrm{QCoh}_{\mathbb{G}}(\tilde{\mathbb{G}} \times_{\mathbb{G}} \tilde{\mathbb{G}})$.

Proof. Consider the tautologically commuting actions

$${}_x\mathcal{H}_\chi \circ {}_x\mathcal{H} \circ \mathcal{H},$$

and in particular the arising monoidal functor

$$(8) \quad \mathcal{H} \rightarrow \mathrm{End}_{{}_x\mathcal{H}_\chi}({}_x\mathcal{H})^{\mathrm{rev}}.$$

By Theorems 2.5.4 and 2.5.5, we may rewrite the target of the functor in spectral terms as

$$\mathrm{End}_{{}_x\mathcal{H}_\chi}({}_x\mathcal{H})^{\mathrm{rev}} \simeq \mathrm{End}_{\mathrm{QCoh}_{\mathbb{G}}(\mathbb{G})}(\mathrm{QCoh}_{\mathbb{G}}(\tilde{\mathbb{G}}))^{\mathrm{rev}} \simeq \mathrm{QCoh}_{\mathbb{G}}(\tilde{\mathbb{G}} \times_{\mathbb{G}} \tilde{\mathbb{G}})^{\mathrm{rev}}.$$

Postcomposing this with the tautological swap map $\sigma : \mathrm{QCoh}_{\mathbb{G}}(\tilde{\mathbb{G}} \times_{\mathbb{G}} \tilde{\mathbb{G}})^{\mathrm{rev}} \simeq \mathrm{QCoh}_{\mathbb{G}}(\tilde{\mathbb{G}} \times_{\mathbb{G}} \tilde{\mathbb{G}})$, cf. Section 2.2.3, yields the first claim of the proposition. The second is immediate from the construction. \square

3.2. Colocalization.

3.2.1. It remains to show that $v^!$ is an equivalence, up to renormalization issues. To this end, we now show the following.

Proposition 3.2.2. *The functor $v^!$ admits a fully faithful left adjoint $u_!$ which is a morphism of \mathcal{H} -bimodules. Moreover $u_!$ sends the structure sheaf of $(\tilde{\mathbb{G}} \times_{\mathbb{G}} \tilde{\mathbb{G}})/\mathbb{G}$ to the big tilting object Ξ , i.e.,*

$$u_!(\mathcal{O}_{(\tilde{\mathbb{G}} \times_{\mathbb{G}} \tilde{\mathbb{G}})/\mathbb{G}}) \simeq \Xi.$$

Proof. Consider the tautological functor associated to the bimodule ${}_x\mathcal{H}$, i.e.,

$$(9) \quad {}_x\mathcal{H}_\chi\text{-mod} \leftarrow \mathcal{H}\text{-mod} : {}_x\mathcal{H} \otimes_{\mathcal{H}} -.$$

By a general result in Morita theory, due to Ben-Zvi–Gunningham–Orem [BZGO20, Proposition 3.2], it suffices to see this bimodule is properly dualizable; i.e., that the functor (9) admits a left adjoint, and that the unit and counit of adjunction admit continuous right adjoints.

If we write $\mathcal{H}_\chi := \mathrm{Shv}_{\mathrm{nilp}}(\mathring{I} \backslash \mathring{L}G / \mathring{I}, \chi)$, then by the rigidity of \mathcal{H} we have

$${}_x\mathcal{H} \otimes_{\mathcal{H}} - \simeq \mathrm{Hom}_{\mathcal{H}\text{-mod}}(\mathcal{H}_\chi, -),$$

hence the left adjoint to (9) is given by tensoring with the bimodule \mathcal{H}_χ . The unit of adjunction

$${}_x\mathcal{H}_\chi \simeq {}_x\mathcal{H} \otimes_{\mathcal{H}} \mathcal{H}_\chi$$

is an equivalence, so tautologically admits a continuous right adjoint.

It remains to see the counit of adjunction, i.e., the convolution map

$$\mathcal{H}_\chi \otimes_{{}_x\mathcal{H}_\chi} {}_x\mathcal{H} \rightarrow \mathcal{H}$$

admits a continuous right adjoint. By the rigidity of ${}_x\mathcal{H}_\chi$, the insertion functor

$$\mathcal{H}_\chi \otimes {}_x\mathcal{H} \rightarrow \mathcal{H}_\chi \otimes_{{}_x\mathcal{H}_\chi} {}_x\mathcal{H}$$

admits a continuous right adjoint, and in particular the latter tensor product is compactly generated by insertions of compact objects from $\mathcal{H}_\chi \otimes_\chi \mathcal{H}$. Therefore, it is enough to see that the natural map $\mathcal{H}_\chi \otimes_\chi \mathcal{H} \rightarrow \mathcal{H}$ preserves compact objects. However, as this is a map of \mathcal{H} -bimodules, it is enough to check that, if we write δ_χ for the standard generator of \mathcal{H}_χ as an \mathcal{H} -module, and similarly for ${}_\chi \delta$, that the image of $\delta_\chi \boxtimes {}_\chi \delta$ is compact. But the latter is the big tilting object Ξ in $\mathrm{Shv}_{\mathrm{nilp}}(N \setminus G/N) \hookrightarrow \mathcal{H}$, and in particular is compact, as desired.

That left adjointness equips $\iota_!$ with a datum of strict \mathcal{H} -biequivariance follows from the rigidity of \mathcal{H} . Finally, note that, in spectral terms, the insertion

$$\mathcal{H}_\chi \otimes_\chi \mathcal{H} \rightarrow \mathcal{H}_\chi \otimes_{\chi \mathcal{H}_\chi} \chi \mathcal{H}$$

corresponds to $*$ -pullback to the fiber product, i.e.,

$$\mathrm{QCoh}_G(\tilde{G}) \otimes \mathrm{QCoh}_G(\tilde{G}) \rightarrow \mathrm{QCoh}_G(\tilde{G} \times_G \tilde{G}).$$

In particular, $\delta_\chi \boxtimes {}_\chi \delta$, which corresponds under Theorem 2.5.4 to $\mathcal{O}_{\tilde{G}/G} \boxtimes \mathcal{O}_{\tilde{G}/G}$, is sent to $\mathcal{O}_{(\tilde{G} \times_G \tilde{G})/G}$, as desired. \square

3.3. Left completion.

3.3.1. Having obtained the left adjoint $\iota_!$, our next task is to understand the kernel of the map $\iota^!$. To state the answer, we refer to Section 2.1.4 for our notation and conventions for t -structures, and a reminder on the notion of infinitely connective objects.

In this paper, when we speak of t -structures on our categories of interest, we always mean the perverse t -structure on the automorphic side, as in Section 2.3, and the usual t -structure on quasicohherent or ind-coherent sheaves on the spectral side.

Theorem 3.3.2. *An object of \mathcal{H} lies in the kernel of $\iota^!$ if and only if it is infinitely connective, i.e.,*

$$\ker(\iota^!) \simeq \mathcal{H}^{\leq -\infty}.$$

Proof. We begin with the ‘only if’ implication. For an arbitrary object c of \mathcal{H} , consider the tautological triangle

$$\iota_! \circ \iota^!(c) \rightarrow c \rightarrow \xi \xrightarrow{+1}.$$

We must show that, for every c , the object ξ is infinitely connective. As all appearing functors, including the perverse truncation functors on \mathcal{H} , commute with filtered colimits, without loss of generality we may take c to be a compact object.

We first claim that $\iota^!(c)$ is bounded coherent, i.e., an object of $\mathrm{Coh}_G(\tilde{G} \times_G \tilde{G}) \hookrightarrow \mathrm{QCoh}_G(\tilde{G} \times_G \tilde{G})$. Equivalently, we claim that its image in $\mathrm{QCoh}(\tilde{G} \times \tilde{G})$ is compact, i.e, perfect. However, this is clear, as the compact objects of \mathcal{H} are generated under extensions by invertible objects with respect to convolution, and the diagonal of $\mathrm{QCoh}(\tilde{G} \times \tilde{G})$ is perfect.

We may therefore write $\iota^!(c)$ as a bounded above complex of vector bundles $\mathcal{E} \simeq \iota^!(c)$. In particular, if for an integer n we write $\mathcal{E}^{\geq n}$ for the stupid truncation of \mathcal{E} , this is a compact object of $\mathrm{QCoh}_G(\tilde{G} \times_G \tilde{G})$, and we have $\mathcal{E} \simeq \varinjlim \mathcal{E}^{\geq n}$, and whence

$$\iota_! \circ \iota^!(c) \simeq \varinjlim \iota_!(\mathcal{E}^{\geq n}).$$

For fixed n , consider the triangle

$$\iota_!(\mathcal{E}^{\geq n}) \rightarrow c \rightarrow \xi_n \xrightarrow{+1}.$$

Note that, as $\iota_!$ preserves compactness, ξ_n is a compact object, which satisfies $\iota^!(\xi_n) \simeq \mathcal{E}/\mathcal{E}^{\geq n}$, and in particular $\iota^!(\xi_n) < n$. For this reason, we show the following.

Lemma 3.3.3. *There exists an integer N with the following property. For any compact object η of \mathcal{H} , if $\iota^!(\eta) \leq 0$, then $\eta \leq N$, and if $\iota^!(\eta) \geq 0$, then $\eta \geq -N$.*

Note that, assuming the lemma, it follows that $\xi_n < n + N$, whence upon passing to the colimit we obtain $\xi \simeq \varinjlim \xi_n$ is infinitely connective, as desired.

Proof of Lemma 3.3.3. For ease of reading, we will break the proof into steps.

Step 1. Suppose we can find an integer N' such that $\iota^!(\eta) \geq 0$ implies $\eta \geq N'$ for every $\eta \in \mathcal{H}^c$. By monoidality, $\iota^!$ intertwines the duality functors on \mathcal{H}^c and $\mathrm{Coh}_{\mathbb{G}}(\widetilde{\mathbb{G}} \times_{\mathbb{G}} \widetilde{\mathbb{G}})$. We have already seen that these duality functors are t -bounded. Hence, the existence of an N' implies the existence of an N satisfying both conditions in the statement of the lemma.

Step 2. We first make the preliminary claim that there exists an integer d such that for all $y \in W_f$, the functors

$$(10) \quad - \star \iota^!(\Delta_y)[-d] : \mathrm{QCoh}_{\mathbb{G}}(\widetilde{\mathbb{G}} \times_{\mathbb{G}} \widetilde{\mathbb{G}}) \xrightarrow{\sim} \mathrm{QCoh}_{\mathbb{G}}(\widetilde{\mathbb{G}} \times_{\mathbb{G}} \widetilde{\mathbb{G}})$$

are left exact. The proof of this claim is given in the remainder of this step, and in Step 3 below we will show that one may take $N' = d$.

To see the claim, let us reduce to the analogous assertion for the corresponding endofunctor of $\mathrm{QCoh}(\widetilde{\mathbb{G}})$, as follows. Consider the closed embedding $\delta : (\widetilde{\mathbb{G}} \times_{\mathbb{G}} \widetilde{\mathbb{G}})/\mathbb{G} \rightarrow (\widetilde{\mathbb{G}} \times \widetilde{\mathbb{G}})/\mathbb{G}$, the tautological projection $\pi : \widetilde{\mathbb{G}} \times \widetilde{\mathbb{G}} \rightarrow (\widetilde{\mathbb{G}} \times \widetilde{\mathbb{G}})/\mathbb{G}$, and the associated sequence of functors

$$\mathrm{QCoh}_{\mathbb{G}}(\widetilde{\mathbb{G}} \times_{\mathbb{G}} \widetilde{\mathbb{G}}) \xrightarrow{\delta_*} \mathrm{QCoh}_{\mathbb{G}}(\widetilde{\mathbb{G}} \times \widetilde{\mathbb{G}}) \xrightarrow{\pi^*} \mathrm{QCoh}(\widetilde{\mathbb{G}} \times \widetilde{\mathbb{G}}).$$

It is clear both functors δ_* and π^* are t -exact and conservative, as δ_* is pushforward along a closed embedding, and π^* is pullback along a faithfully flat map. We also note that δ_* and π^* are both naturally monoidal. Indeed, for δ_* , if we consider the pullback monoidal functor $\mathrm{QCoh}(\mathrm{pt}/\mathbb{G}) \rightarrow \mathrm{QCoh}(\mathbb{G}/\mathbb{G})$, we note that δ_* canonically identifies with the composition

$$\mathrm{QCoh}_{\mathbb{G}}(\widetilde{\mathbb{G}} \times_{\mathbb{G}} \widetilde{\mathbb{G}}) \simeq \mathrm{End}_{\mathrm{QCoh}(\mathbb{G}/\mathbb{G})}(\mathrm{QCoh}(\widetilde{\mathbb{G}}/\mathbb{G})) \xrightarrow{\mathrm{Oblv}} \mathrm{End}_{\mathrm{QCoh}(\mathrm{pt}/\mathbb{G})}(\mathrm{QCoh}(\widetilde{\mathbb{G}}/\mathbb{G})) \simeq \mathrm{QCoh}_{\mathbb{G}}(\widetilde{\mathbb{G}} \times \widetilde{\mathbb{G}}),$$

where Oblv denotes the tautological restriction from $\mathrm{QCoh}(\mathbb{G}/\mathbb{G})$ -equivariant functors to $\mathrm{QCoh}(\mathrm{pt}/\mathbb{G})$ -equivariant functors. Similarly, π^* canonically identifies with the composition

$$\begin{aligned} \mathrm{QCoh}_{\mathbb{G}}(\widetilde{\mathbb{G}} \times \widetilde{\mathbb{G}}) &\simeq \mathrm{End}_{\mathrm{QCoh}(\mathrm{pt}/\mathbb{G})}(\mathrm{QCoh}(\widetilde{\mathbb{G}}/\mathbb{G})) \xrightarrow{- \otimes_{\mathrm{QCoh}(\mathrm{pt}/\mathbb{G})}^{\mathrm{QCoh}(\mathrm{pt})}} \\ &\mathrm{End}(\mathrm{QCoh}(\widetilde{\mathbb{G}}/\mathbb{G}) \otimes_{\mathrm{QCoh}(\mathrm{pt}/\mathbb{G})} \mathrm{QCoh}(\mathrm{pt})) \simeq \mathrm{End}(\mathrm{QCoh}(\widetilde{\mathbb{G}})) \simeq \mathrm{QCoh}(\widetilde{\mathbb{G}} \times \widetilde{\mathbb{G}}), \end{aligned}$$

where $- \otimes_{\mathrm{QCoh}(\mathrm{pt}/\mathbb{G})}^{\mathrm{QCoh}(\mathrm{pt})}$ explicitly is the functor of de-equivariantization.

Using these observations, it is therefore enough to produce an integer d such that for all $y \in W_f$ the functors

$$- \star (\pi^* \circ \delta_* \circ \iota^!(\Delta_y))[-d] : \mathrm{QCoh}(\widetilde{\mathbb{G}} \times \widetilde{\mathbb{G}}) \rightarrow \mathrm{QCoh}(\widetilde{\mathbb{G}} \times \widetilde{\mathbb{G}})$$

are left exact. However, we claim more generally that for any finite set $\Phi_i, i \in \mathcal{J}$, of endofunctors of $\mathrm{QCoh}(\widetilde{\mathbb{G}})$ admitting continuous right adjoints, with corresponding integral kernels $\mathcal{K}_i \in \mathrm{QCoh}(\widetilde{\mathbb{G}} \times \widetilde{\mathbb{G}}), i \in \mathcal{J}$, there exists an integer d such that

$$- \star \mathcal{K}_i[-d] : \mathrm{QCoh}(\widetilde{\mathbb{G}} \times \widetilde{\mathbb{G}}) \rightarrow \mathrm{QCoh}(\widetilde{\mathbb{G}} \times \widetilde{\mathbb{G}})$$

are all left exact. Indeed, under the tautological identification of $- \star \mathcal{K}_i$ with

$$\mathrm{QCoh}(\widetilde{\mathbb{G}} \times \widetilde{\mathbb{G}}) \simeq \mathrm{QCoh}(\widetilde{\mathbb{G}}) \otimes \mathrm{QCoh}(\widetilde{\mathbb{G}}) \xrightarrow{\mathrm{id} \otimes \Phi_i} \mathrm{QCoh}(\widetilde{\mathbb{G}}) \otimes \mathrm{QCoh}(\widetilde{\mathbb{G}}) \simeq \mathrm{QCoh}(\widetilde{\mathbb{G}} \times \widetilde{\mathbb{G}}),$$

it follows that $- \star \mathcal{K}_i$ also admits a continuous right adjoint, and in particular preserves compactness. Noting that the structure sheaf of the diagonal $\Delta_*(\mathcal{O}_{\widetilde{\mathbb{G}}})$ is perfect by the smoothness of $\widetilde{\mathbb{G}}$, and that $- \star \mathcal{K}_i$ sends $\Delta_*(\mathcal{O}_{\widetilde{\mathbb{G}}})$ to \mathcal{K}_i , it therefore follows that \mathcal{K}_i is also a perfect complex. Using this perfectness, it follows that each $- \star \mathcal{K}_i$ is t -bounded, whence the existence of the desired d follows by the finiteness of \mathcal{J} .

Step 3. Having shown the existence of an integer d as in the discussion near Equation (10), let us show we may take $N' = d$, where N' is as in the discussion of Step 1. That is, for any compact object η of \mathcal{H} satisfying $\iota^!(\eta) \geq 0$, we will show that $\eta \geq -d$.

Because $\Delta_{-\mu} \star -$ is left exact, for any $\mu \in \Lambda$, and $\Delta_{-\mu} \star \nabla_{\mu} \star \eta \simeq \eta$, it suffices to show for some $\mu \in \Lambda$ that $\nabla_{\mu} \star \eta \geq -d$, i.e., that for all $w \in W$ we have

$$(11) \quad \text{Hom}(\Delta_w, \nabla_{\mu} \star \eta) \geq -d.$$

We will show the claim for any μ for which $\nabla_{\mu} \star \eta$ is in the full subcategory generated by shifts of the objects ∇_x , for $x \in \Lambda^{++} \cdot W_f \subset W$; note the set of such μ contains all sufficiently dominant elements, thanks to the compactness of η , and is in particular nonempty.

For such a μ , to verify (11), we may assume that $w \in \Lambda^{++} \cdot W_f$, as otherwise $\text{Hom}(\nabla_w, \nabla_{\mu} \star \eta)$ vanishes. Note that any $x \in \Lambda^{++} \cdot W_f$ is of maximal length in its right coset $W_f \cdot x$. As $\Xi \star \Delta_w$ admits a standard filtration with successive quotients $\Delta_{z \cdot w}$, for $z \in W_f$, of which Δ_w is tautologically the only graded piece of maximal length in its right W_f -coset, it follows that we have

$$\text{Hom}(\Delta_w, \nabla_{\mu} \star \eta) \simeq \text{Hom}(\Xi \star \Delta_w, \nabla_{\mu} \star \eta).$$

To make contact with the spectral side, let us rewrite $\Xi \star \Delta_w$ using Wakimoto sheaves, as follows. Writing w_{\circ} for the longest element of W_f , note this is equivalently given by

$$\simeq \text{Hom}(\Xi \star \Delta_{w_{\circ} w}, \nabla_{\mu} \star \eta).$$

Since $w \in \Lambda^{++} \cdot W_f$, we have $w_{\circ} w \in \Lambda^{--} \cdot W_f$; write $w_{\circ} w = (-\lambda) \cdot y$ for the corresponding unique factorization, with $-\lambda \in \Lambda^{--}$ and $y \in W_f$. Noting that $-\lambda$ is of maximal length in $-\lambda \cdot W_f$, we have

$$\simeq \text{Hom}(\Xi \star \Delta_{-\lambda} \star \nabla_y, \nabla_{\mu} \star \eta)$$

which after rearranging terms yields the expression

$$\simeq \text{Hom}(\Delta_{-\mu} \star \Xi \star \Delta_{-\lambda}, \eta \star \Delta_{y^{-1}}).$$

To proceed, note that $\Delta_{-\mu}$ and $\Delta_{-\lambda}$ are Wakimoto sheaves, as $-\mu, -\lambda \in \Lambda^{-}$, i.e., we have

$$\simeq \text{Hom}(W_{-\mu} \star \Xi \star W_{-\lambda}, \eta \star \Delta_{y^{-1}}).$$

Recalling that $\iota_!$ is an equivariant functor between \mathcal{H} -bimodules, which satisfies $\iota_!(\mathcal{O}) \simeq \Xi$, we obtain

$$\begin{aligned} &\simeq \text{Hom}(\iota_!(\mathcal{O}(-\mu, -\lambda)), \eta \star \Delta_{y^{-1}}) \\ &\simeq \text{Hom}(\mathcal{O}(-\mu, -\lambda), \iota^!(\eta) \star \iota^!(\Delta_{y^{-1}})). \end{aligned}$$

By the assumption that $\iota^!(\eta) \geq 0$, and the definition of d , it follows that $\iota^!(\eta) \star \iota^!(\Delta_{y^{-1}}) \geq -d$. Therefore, we obtain the inequality (11), as desired. \square

It remains to prove the ‘if’ direction, i.e., that every infinitely connective object lies in the kernel of $\iota^!$. So, fix an infinitely connective object $\xi \in \mathcal{H}^{\leq -\infty}$. We need to show that ξ acts by zero on ${}_{\chi}\mathcal{H}$, i.e., that ${}_{\chi}h \star \xi \simeq 0$ for every object ${}_{\chi}h$ of ${}_{\chi}\mathcal{H}$. It is enough to check this for ${}_{\chi}h$ running through a set of compact generators of ${}_{\chi}\mathcal{H}$, and hence for objects of the form ${}_{\chi}\delta \star c$, for $c \in \mathcal{H}^c$. Noting that $({}_{\chi}\delta \star c) \star \xi \simeq {}_{\chi}\delta \star (c \star \xi)$, and that $\xi' := c \star \xi$ is again infinitely connective, it therefore remains to show that ${}_{\chi}\delta \star \xi' \simeq \text{Av}_{\chi}(\xi')$ vanishes. We claim it is enough to show the following.

Lemma 3.3.4. *The equivalence ${}_{\chi}\mathcal{H} \simeq \text{QCoh}_{\mathbb{G}}(\tilde{\mathbb{G}})$ of Theorem 2.5.4 is t -bounded.*

Indeed, as Av_{χ} is exact, as can be seen by considering ${}_{\chi}\mathcal{H}$ as left comodules for the tilting object Ξ , it follows that $\text{Av}_{\chi}(\xi')$ is again infinitely connective. But by the lemma, ${}_{\chi}\mathcal{H}$ contains no nonzero infinitely connective objects, as this is clear on the spectral side.

Proof of Lemma 3.3.4. Let us write $F : {}_x\mathcal{H} \simeq \mathrm{QCoh}_{\mathbb{G}}(\tilde{\mathbb{G}})$ for the equivalence of Theorem 2.5.4.

Let us first show that F is left exact. Suppose $\xi \in {}_x\mathcal{H}$ and $\xi \geq 0$. Then in particular, recalling that for $V \in \mathrm{Rep}(\mathbb{G})^\heartsuit$ and $\lambda \in \Lambda$ the object ${}_x\delta \star Z_V \star W_\lambda$ is perverse, it follows that

$$\mathrm{Hom}_{\mathrm{QCoh}(\tilde{\mathbb{G}}/\mathbb{G})}(V \otimes \mathcal{O}(\lambda), F(\xi)) \simeq \mathrm{Hom}_{{}_x\mathcal{H}}({}_x\delta \star Z_V \star W_\lambda, \xi) \geq 0,$$

whence $F(\xi) \geq 0$ as well.

It remains to show that F^{-1} is also left exact up to a shift. To do so, let us write ${}^fW \subset W$ for the set of elements $x \in W$ which are minimal in $W_f \cdot x$. Note first that the objects

$$\mathrm{Av}_\chi \Delta_x, \quad x \in {}^fW,$$

are exactly the standard objects of ${}_x\mathcal{H}$, and in particular the perverse t -structure on ${}_x\mathcal{H}$ may be characterized by

$$\zeta \geq 0 \quad \text{if and only if} \quad \mathrm{Hom}_{{}_x\mathcal{H}}(\mathrm{Av}_\chi \Delta_x, \zeta) \geq 0, \quad \text{for all } x \in {}^fW.$$

Note next that, if we write $x \in W$ as $x = \lambda \cdot y$, we have that $x \in {}^fW$ if and only if (i) $\lambda \in \Lambda^-$ and (ii) for each (finite) positive root α such that $\langle \lambda, \alpha \rangle = 0$, we have that $y(\alpha) > 0$. Using this, it is straightforward to see that $\ell(x) = \ell(\lambda) - \ell(y)$, and hence we may write

$$(12) \quad \Delta_x \simeq \Delta_\lambda \star \nabla_y = W_\lambda \star \nabla_y,$$

and in particular $\mathrm{Av}_\chi \Delta_x \simeq (\mathrm{Av}_\chi \Delta_\lambda) \star \nabla_y$.

As in the proof of Step 2 of Lemma 3.3.3, we may choose an integer \mathfrak{d} such that the autoequivalences

$$- \star \iota^!(\nabla_z)[\mathfrak{d}] : \mathrm{QCoh}_{\mathbb{G}}(\tilde{\mathbb{G}}) \simeq \mathrm{QCoh}_{\mathbb{G}}(\tilde{\mathbb{G}})$$

are all right exact. With this, we claim that if an object \mathcal{E} of $\mathrm{QCoh}_{\mathbb{G}}(\tilde{\mathbb{G}})$ satisfies $\mathcal{E} \geq \mathfrak{d}$, then $F^{-1}(\mathcal{E}) \geq 0$. Indeed, for any x, λ , and y as in Equation (12) we have

$$\begin{aligned} \mathrm{Hom}_{{}_x\mathcal{H}}(\mathrm{Av}_\chi \Delta_x, F^{-1}\mathcal{E}) &\simeq \mathrm{Hom}_{{}_x\mathcal{H}}((\mathrm{Av}_\chi W_\lambda) \star \nabla_y, F^{-1}\mathcal{E}) \\ &\simeq \mathrm{Hom}_{\mathrm{QCoh}_{\mathbb{G}}(\tilde{\mathbb{G}})}(\mathcal{O}(\lambda) \star \iota^!(\nabla_y), \mathcal{E}), \end{aligned}$$

which is indeed coconnective as $\mathcal{O}(\lambda) \star \iota^!(\nabla_y) \leq \mathfrak{d}$ and $\mathcal{E} \geq \mathfrak{d}$. □

□

Remark 3.3.5. The semi-infinite Hecke category $\mathrm{Shv}_{\mathrm{nilp}}(N_F \mathring{T}_O \backslash G_F / N_F \mathring{T}_O)$, with its natural t -structure, is from certain perspectives a more natural choice of category on the automorphic side, though it is derived equivalent to \mathcal{H} by a standard argument of Borel–Casselman–Bushnell–Kutzko–Raskin. Nonetheless, its presence may be felt in the proof of Lemma 3.3.3 and Lemma 3.3.4, in a manner directly inspired by its similar presence in the striking proof of Lemma 38 in [B16].

3.3.6. To state the following corollary, note that the Verdier quotient

$$\mathfrak{q} : \mathcal{H} \rightarrow \mathcal{H}/\mathcal{H}^{\leq -\infty}$$

inherits a monoidal structure, as $\mathcal{H}^{\leq -\infty}$ is a two sided ideal in \mathcal{H} , by the t -boundedness of convolution with compact objects in \mathcal{H} . Additionally, the Verdier quotient inherits a unique t -structure with the property that \mathfrak{q} is t -exact, and this induces an equivalence on eventually coconnective parts

$$(13) \quad \mathfrak{q} : \mathcal{H}^+ \simeq (\mathcal{H}/\mathcal{H}^{\leq -\infty})^+.$$

Corollary 3.3.7. *The monoidal functor $\iota^!$ factors uniquely through a t -bounded monoidal equivalence*

$$\mathcal{H}/\mathcal{H}^{\leq -\infty} \simeq \mathrm{QCoh}_{\mathbb{G}}(\tilde{\mathbb{G}} \times_{\mathbb{G}} \tilde{\mathbb{G}}).$$

Proof. The assertion at the level of monoidal categories is immediate from the statement of Theorem 3.3.2. For the assertion regarding t -boundedness, it is enough to see this on compact objects of $\mathrm{QCoh}_{\mathbb{G}}(\widetilde{\mathbb{G}} \times_{\mathbb{G}} \widetilde{\mathbb{G}})$, but this is a special case of Lemma 3.3.3. \square

3.4. Anticompletion.

3.4.1. To prove Theorem 1.2.5, it remains to show the following.

Proposition 3.4.2. *The monoidal functor $v^!$ restricts to an equivalence of small monoidal categories*

$$(14) \quad \mathcal{H}^c \simeq \mathrm{Coh}_{\mathbb{G}}(\widetilde{\mathbb{G}} \times_{\mathbb{G}} \widetilde{\mathbb{G}}).$$

Indeed, Theorem 1.2.5 then follows by passing to the ind-completions of both sides of (14).

3.4.3. For the proof of Proposition 3.4.2, we first recall some convenient terminology from [CD23].

Given a category \mathcal{C} with a t -structure compatible with filtered colimits, we call an object c of \mathcal{C} *pseudocompact* if it is eventually coconnective and $\mathrm{Hom}(c, -)$ commutes with uniformly bounded below filtered colimits. Let us denote the full subcategory of pseudocompact objects by

$$\mathrm{Psc}(\mathcal{C}) \hookrightarrow \mathcal{C}.$$

Note that it only depends on the t -structure up to t -bounded equivalence.

Proof of Proposition 3.4.2. It follows from Corollary 3.3.7 that we have an equivalence

$$\mathrm{Psc}(\mathcal{H}/\mathcal{H}^{\leq -\infty}) \simeq \mathrm{Psc}(\mathrm{QCoh}_{\mathbb{G}}(\widetilde{\mathbb{G}} \times_{\mathbb{G}} \widetilde{\mathbb{G}})).$$

On the one hand, it is standard that the pseudocompact objects in $\mathrm{QCoh}_{\mathbb{G}}(\widetilde{\mathbb{G}} \times_{\mathbb{G}} \widetilde{\mathbb{G}})$ are exactly $\mathrm{Coh}_{\mathbb{G}}(\widetilde{\mathbb{G}} \times_{\mathbb{G}} \widetilde{\mathbb{G}})$. On the other hand, it is straightforward from Equation (13) that \mathfrak{q} restricts to an equivalence

$$\mathfrak{q} : \mathrm{Psc}(\mathcal{H}) \simeq \mathrm{Psc}(\mathcal{H}/\mathcal{H}^{\leq -\infty}).$$

It therefore remains to check that the tautological inclusion $\mathcal{H}^c \hookrightarrow \mathrm{Psc}(\mathcal{H})$ is essentially surjective. So, suppose ξ is a pseudocompact object of \mathcal{H} , and in particular eventually coconnective. By writing

$$\xi \simeq \varinjlim i_{y,!} \circ i_y^!(\xi),$$

cf. Equation (4), we deduce by pseudocompactness that ξ is $!$ -extended from a closed, finite union of strata. Pick a stratum $j_y : IyI \hookrightarrow LG$ maximal in the support of ξ . It is straightforward that $j_y^!(\xi) \simeq j_y^*(\xi) \in R\text{-mod}$ is again pseudocompact, whence compact by the smoothness of R . By considering the triangle

$$j_{y,!} \circ j_y^!(\xi) \rightarrow \xi \rightarrow \xi' \xrightarrow{+1},$$

and noting that ξ' is again pseudocompact and is supported on a strictly smaller closed, finite union of strata, we obtain the compactness of ξ by induction on the number of strata in its support. \square

3.4.4. Let us conclude with a couple remarks.

Remark 3.4.5. Note that as the map $\widetilde{\mathbb{G}}/\mathbb{G} \rightarrow \mathbb{G}/\mathbb{G}$ is proper and relatively Calabi–Yau of dimension zero, the monoidal structures on $\mathrm{Coh}_{\mathbb{G}}(\widetilde{\mathbb{G}} \times_{\mathbb{G}} \widetilde{\mathbb{G}})$ and hence $\mathrm{IndCoh}_{\mathbb{G}}(\widetilde{\mathbb{G}} \times_{\mathbb{G}} \widetilde{\mathbb{G}})$ induced by $*$ -convolution and $!$ -convolution canonically coincide.

In particular, the monoidal categories in Theorem 1.2.5 admit a natural central functor from the symmetric monoidal category $\mathrm{QCoh}_{\mathbb{G}}(\mathbb{G})$, equipped with the $!$ -tensor product, and in particular further from the invariant theory quotient $\mathrm{QCoh}(\mathbb{G}/\mathbb{G})$. Consider within the invariant theory quotient the formal subscheme

$$\mathbb{G}/\mathbb{G}^{\mathrm{res.var.}} \hookrightarrow \mathbb{G}/\mathbb{G}$$

obtained as the disjoint union of its formal completions along all its closed points. We have a tautological adjunction

$$i_* : \mathrm{IndCoh}(\mathbb{G}/\mathbb{G})^{\mathrm{res.var.}} \rightleftarrows \mathrm{QCoh}(\mathbb{G}/\mathbb{G}) : i^!$$

wherein $i^!$ is a monoidal colocalization. In particular, i_* identifies the former with the full subcategory of the latter generated by all skyscraper sheaves, and our use of $!$ -tensor product, rather than $*$ -tensor product, endows this with a monoidal unit, namely the dualizing sheaf.

If we apply the corresponding base change

$$- \otimes_{\mathrm{QCoh}(\mathbb{G}/\mathbb{G})} \mathrm{QCoh}(\mathbb{G}/\mathbb{G})^{\mathrm{res.var.}}$$

to both sides of Theorem 1.2.5, we obtain its analogue with restricted variation, and deduce the corresponding version of Corollary 1.2.6. In particular, by passing to the appropriate direct summand monoidal categories, these results confirm a conjecture of Bezrukavnikov, namely [B16, Conjecture 58] over the complex numbers. This provides an alternative to the endoscopic arguments of [DLYZ24].

Remark 3.4.6. In this paper, we have addressed the case of tame ramification in local Betti geometric Langlands. It is extremely natural to expect the existence of a wildly ramified local Betti correspondence, involving on the spectral side moduli spaces of Stokes data. To our knowledge no precise formulation of such a correspondence is as of yet available.

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