Outline

Continuous, real-valued functions on subsets of $\mathbb{R}$
- Intermediate value theorem
- Compactness properties

Inequalities
- Cauchy-Schwarz
- Jensen’s inequality and convexity

Sequences of real numbers
- $\limsup$ and $\liminf$
- Monotone sequences

Taylor expansion
- Taylor’s theorem with remainder

Problems

1.1.10 – Fall 1982 11
1. Prove that there is no continuous map from the closed interval $[0, 1]$ onto the open interval $(0, 1)$.
2. Find a continuous surjective map from the open interval $(0, 1)$ onto the closed interval $[0, 1]$.
3. Prove that no map in Part 2 can be bijective.

1.5.3 – Fall 1990 4
Suppose $f$ is a continuous real valued function. Show that
\[ \int_0^1 f(x)x^2\,dx = \frac{1}{3}f(\xi) \]
for some $\xi \in [0, 1]$.

1.5.9 – Fall 1985 15
Let $0 \leq a \leq 1$ be given. Determine all nonnegative continuous functions $f$ on $[0, 1]$ which satisfy the following three conditions:
\[ \int_0^1 f(x)\,dx = 1, \quad \int_0^1 xf(x)\,dx = a, \quad \int_0^1 x^2f(x)\,dx = a^2. \]

1.3.8 – Spring 2003 6A
Let $x_n$ be a sequence of real numbers so that $\lim_{n \to \infty} 2x_{n+1} - x_n = x$. Show that $\lim_{n \to \infty} x_n = x$.

1.3.9 – Spring 2000 5
Let $a$ and $x_0$ be positive numbers, and define the sequence $(x_n)_{n=1}^{\infty}$...
recursively by
\[ x_n = \frac{1}{2} \left( x_{n-1} + \frac{a}{x_{n-1}} \right). \]
Prove that this sequence converges, and find its limit.

**1.1.17 – Fall 1997 11** Let \( f \) be a \( C^2 \) function on the real line. Assume \( f \) is bounded with bounded second derivative. Let
\[ A = \sup_{x \in \mathbb{R}} |f(x)|, \quad B = \sup_{x \in \mathbb{R}} |f''(x)|. \]
Prove that
\[ \sup_{x \in \mathbb{R}} |f'(x)| \leq 2\sqrt{AB}. \]