Outline

Sequences
- Sequential definitions
- Sequential compactness
- Subsequences and “diagonalization”

Continuity
- Uniform continuity
- Equicontinuity and Arzela-Ascoli

Other constructions
- Fixed point theorems
- Fréchet spaces

Problems

Fall 1989 6 Let $X \subseteq \mathbb{R}^n$ be a closed set and $r$ a fixed positive real number. Let $Y = \{ y \in \mathbb{R}^n : |x - y| = r, \text{some } x \in X \}$. Show that $Y$ is closed.

Fall 1980 19 Let $X$ be a compact metric space and $f : X \to X$ an isometry. Show that $f(X) = X$.

Fall 1989 14 Let $X \subseteq \mathbb{R}^n$ be compact and let $f : X \to \mathbb{R}$ be continuous. Given $\epsilon > 0$, show there is an $M$ such that for all $x, y \in X$,

$$|f(x) - f(y)| \leq M|x - y| + \epsilon.$$

Spring 1987 16 Let $\mathcal{F}$ be a uniformly bounded, equicontinuous family of real valued functions on the metric space $(X, d)$. Prove that the function

$$g(x) = \sup\{ f(x) : f \in \mathcal{F} \}$$

is continuous.

Fall 1982 18 Let $K$ be a continuous function on the unit square $0 \leq x, y \leq 1$ satisfying $|K(x, y)| < 1$ for all $x$ and $y$. Show that there is a continuous function $f(x)$ on $[0, 1]$ such that we have

$$f(x) + \int_0^1 K(x, y)f(y) \, dy = e^{x^2}.$$

Can there be more than one such function $f$?