Math 53 Summer 2018 Homework Assignment 20
Solutions to selected problems

James Rowan

August 4, 2018

Assigned: Wednesday, July 18
Due: Monday, July 23

0 Reading

Stewart Section 15.6

1 Textbook Exercises

Graded for completion. This portion of the homework is meant to help you practice important computational skills.

- **Evaluating triple integrals**: Stewart Section 15.6: exercises 8, 10, 12
- **Interperetations of triple integrals**: Stewart Section 15.6: exercise 54
- **Triple integrals and volumes**: Stewart Section 15.6: exercises 20, 22
- **Changing bounds for triple integrals**: Stewart Section 15.6: exercises 32, 34, 36

2 True/false questions

Graded for correctness (+1 for correct, +0 for blank or incorrect). While we will not grade your reasoning, I recommend writing it down for yourself.

1. In the case of triple integrals, we know because of Fubini's theorem that the six different orders in which we can write down a triple integral as an iterated integral will all give the same result.

   $\boxed{\text{True}}$. This is precisely the statement of Fubini's theorem, that we can write the triple integral as an iterated integral and that it does not matter which of the six orders (six orders because there are three choices for the outermost variable, and then two choices for the middle variable once the outermost one is selected, with the inner one determined by the outer two).

2. The only physically meaningful interpretations of triple integrals are that triple integrals of various kinds of densities give total mass, total charge, etc.

   False. We can also take triple integrals to find average values (the average value of f on E is $\iiint_E f(x,y,z)dV$ divided by the volume of E) as well as volumes (the volume of E is $\iiint_E 1dV$), which are both interesting and meaningful interpretations of triple integrals.

3. A type I solid region is a solid region of the form $a \leq x \leq b$, $c \leq y \leq d$, $u_1(x,y) \leq z \leq u_2(x,y)$ for some constants a,b,c,d and some functions of two variables u_1 and u_2.

   False. For a type I solid region, the bounds on x and y do not have to be constants (i.e. we do not have to have a rectangle in the xy-plane as the "shadow" of our region). The shadow in the xy-plane can be a type I or type II region in the plane, for example.

4. The two triple integrals $\int_0^1 \int_x^1 \int_y^1 f(x,y,z)dzdydx$ and $\int_0^1 \int_0^y \int_0^z f(x,y,z)dydxdz$ are the same for every function f, as the bounds of these integrals represent two ways to look at the same region.

   True. The two regions of integration are the same. An algebraic argument is as follows: The first iterated integral is over the region where $0 \leq x \leq 1$, $x \leq y \leq 1$, and $y \leq z \leq 1$. Combining the first two inequalities, we see that on this region, $0 \leq x \leq y$. Combining the first, second, and third inequalities, we see that on this region, $0 \leq z \leq 1$ and that $0 \leq y \leq z$. The iterated integral over the region described in this way is precisely the second iterated integral.

5. After we have done of the three integrals when we set up a triple integral as an iterated integral, we are left with a double integral, which sometimes we can convert to polar coordinates to evaluate more easily.

   True. For example, if we are doing a triple integral over a circular cylinder with radius 1, center along the z-axis, and height 4, after doing the inner integral in z, we are left with an integral over a disk of radius 1 in the xy-plane, which might be easier to evaluate in polar coordinates.

6. If $f(x,y,z)$ is a function that factors as a product $f(x,y,z) = g(x)h(y)k(z)$, then

$$\int_a^b \int_c^d \int_r^s f(x,y,z)dzdydx = \int_a^b g(x)dx \int_c^d h(y)dy \int_r^s k(z)dz.$$

   True. This is exactly analogous to the result we proved for double integrals. We could prove it in basically the same way, using the two-variable result we already proved in the last step:

$$\int_a^b \int_c^d \int_r^s g(x)h(y)k(z)dzdydx = \int_a^b \int_c^d g(x)h(y) \left(\int_r^s k(z)dz \right) dydx$$
$$= \left(\int_r^s k(z)dz \right) \int_a^b \int_c^d g(x)h(y)dydx$$
$$= \int_a^b g(x)dx \int_c^d h(y)dy \int_r^s k(z)dz$$

3 Problems

Justify your answer, clearly explaining your reasoning. Correct answers with incorrect or unclear explanations will not receive full credit.

1. **(volumes of spheres and hyperspheres)** In this problem, we use triple and quadruple integrals to find formulas for the volume inside a three-dimensional sphere and the "hyerpvolume" inside a four dimensional "hypersphere."

   (a) Set up a triple integral (in Cartesian coordinates) that represents the volume of a sphere of radius R.

   The sphere consists of those points in space lying between the surfaces $z = -\sqrt{R^2 - x^2 - y^2}$ and $z = \sqrt{R^2 - x^2 - y^2}$ with (x, y) lying in a circle of radius R centered at the origin. We can set up this triple integral as

   $$\int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \int_{-\sqrt{R^2-x^2-y^2}}^{\sqrt{R^2-x^2-y^2}} dzdydz.$$

   (b) Evaluate your triple integral from part a). *Hint: a trigonometric substitution might be helpful if you see something that looks like an integral of* $\sqrt{u^2 - v^2}$.

   The innermost integral above simply gives $2\sqrt{R^2 - x^2 - y^2}$, as we are integrating 1 over an interval of that length. We can now change to polar coordinates in the xy-plane to make this integral a bit easier, getting

$$\begin{aligned}\int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\int_{-\sqrt{R^2-x^2-y^2}}^{\sqrt{R^2-x^2-y^2}} dzdydz &= \int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} 2\sqrt{R^2-x^2-y^2}dydx \\ &= \int_{0}^{2\pi}\int_{0}^{R} 2\sqrt{R^2-r^2}rdrd\theta \\ &= 2\pi\int_{R^2}^{0} -\sqrt{u}du \\ &= 2\pi\int_{0}^{R^2} \sqrt{u}du \\ &= \frac{4}{3}\pi u^{3/2}\Big|_0^{R^2} \\ &= \frac{4}{3}\pi R^3.\end{aligned}$$

We could also have made the trigonometric substitution $y = \sqrt{R^2 - x^2}\sin\theta$, $dy = \sqrt{R^2 - x^2}\cos\theta d\theta$ to evaluate this integral in Cartesian coordinates directly, along the lines we use in part (e) of this solution.

(c) Consider a four-dimensional space with coordinates (x, y, z, w). How would we define the quadruple integral over a box $\tilde{B} = \{(x, y, z, w)|a \leq x \leq b, c \leq y \leq d, r \leq z \leq s, u \leq w \leq v\}$? How would we define the quadruple integral of a continuous function over a general bounded region H in 4D space?

We would define the quadruple integral over some box as a limit of Riemann sums, where we break the box up into $ijk\ell$ different equally-sized sub-boxes, i in the x direction, j in the y direction, k in the z direction, and ℓ in the w direction, sample one point in each box and multiply f at that point times the volume of the box, and add up over all boxes. The limit as all of i, j, k, and ℓ go to infinity is how we define the quadruple integral over the box, just as we defined double and triple integrals as limits of 2D and 3D Riemann sums.

To define the quadruple integral of a continuous function f over a general bounded region H in 4D space, we proceed as we did in 2D and 3D by defining a new function $\tilde{f}(x, y, z, w)$ which is equal to f inside H and 0 outside H and defining $\iiiint_H f(x, y, z, w)d\tilde{V}$ to be the value $\iiiint_B \tilde{f}(x, y, z, w)d\tilde{V}$ for some box B containing H.

(d) Explain why the hypersurface (i.e. the 3D region in 4D space) defined by $x^2 + y^2 + z^2 + w^2 = R^2$ is called a hypersphere of radius R.

The sphere of radius R in $\mathbb{R}^3$ is given by the equation $x^2 + y^2 + z^2 = R^2$, and it consists of all points in 3D space that are at a distance of R from the origin. The equation $x^2 + y^2 + z^2 + w^2 = R^2$ is the exact analogue of this equation in four variables instead of three, and it gives all the points at a distance of R from the origin in 4D space, so it makes sense to call this a “hypersphere” since it is very much like a sphere in one more dimension.

(e) Set up and evaluate a quadruple integral of the function $f(x, y, z, w) = 1$ to find the four-dimensional “hypervolume” of inside four-dimensional hypersphere of radius R.

Setting up analogously to how we set up in part (a), we have

$$\int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \int_{-\sqrt{R^2-x^2-y^2}}^{\sqrt{R^2-x^2-y^2}} \int_{-\sqrt{R^2-x^2-y^2-z^2}}^{\sqrt{R^2-x^2-y^2-z^2}} dwdzdydx.$$

We evaluate this iterated integral one integral at a time:

$$\int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \int_{-\sqrt{R^2-x^2-y^2}}^{\sqrt{R^2-x^2-y^2}} \int_{-\sqrt{R^2-x^2-y^2-z^2}}^{\sqrt{R^2-x^2-y^2-z^2}} dwdzdydx = \int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \int_{-\sqrt{R^2-x^2-y^2}}^{\sqrt{R^2-x^2-y^2}} 2\sqrt{R^2 - x^2$$

We now let $z = \sqrt{R^2 - x^2 - y^2} \sin\theta,\ dz = \sqrt{R^2 - x^2 - y^2} \cos\theta d\theta$, so that the new bounds on the innermost interal are $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. We have

$$\int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \int_{-\sqrt{R^2-x^2-y^2}}^{\sqrt{R^2-x^2-y^2}} 2\sqrt{R^2 - x^2 - y^2 - z^2} dzdydx = \int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \int_{-\pi/2}^{\pi/2} 2\sqrt{R^2 - x^2 - y^2}$$

$$= \int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} 2(R^2 - x^2 - y^2) \int_{-\pi/2}^{\pi/2} \mathrm{c}$$

We know $\int_{-\pi/2}^{\pi/2} \cos^2\theta = \frac{\pi}{2}$ (we could use the cosine double-angle formula and the identity $\sin^2\theta + \cos^2\theta = 1$ to see this, for example), so we have gotten down to a double integral which we can convert to polar coordinates in the xy-plane and solve:

$$\begin{aligned}
\int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} 2(R^2 - x^2 - y^2) \int_{-\pi/2}^{\pi/2} \cos^2\theta d\theta dydx &= \pi \int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} (R^2 - x^2 - y^2) dydx \\
&= \pi \int_0^{2\pi} \int_0^R (R^2 - r^2) r dr d\theta \\
&= 2\pi^2 \int_0^R R^2 r - r^3 dr \\
&= 2\pi^2 \frac{R^2 r^2}{2} - \frac{r^4}{4} \Big|_0^R \\
&= \frac{\pi^2}{2} R^4.
\end{aligned}$$