1. Chebyshev’s inequality

(1) True/false practice:
   (a) Chebyshev’s inequality holds only for discrete random variables.
   False. One of the great strengths of Chebyshev’s inequality is that it applies for all random variables; discrete or continuous, normal or not.

   (b) Z-scores are strictly stronger than Chebyshev’s inequality, so there’s no reason to learn Chebyshev’s inequality.
   False. While it is true that we can do better than Chebyshev’s inequality if we know the random variable $X$ is normal (for example, the probability that of $X$ being more than 2 standard deviations from the mean of a normal distribution is around 0.05, whereas Chebyshev’s inequality only guarantees that the probability of $X$ being more than 2 standard deviations from the mean is at most 0.25), we can only use our z-score table for random variables we know to be normally distributed. Chebyshev’s inequality works for all random variables, so it makes sense that it’s not as precise as the statements we could make for specific kinds of random variables like normal random variables.

(2) (adapted from Rosen 7.4.36) Suppose a coin is biased so that it comes up heads $\frac{1}{4}$ of the time and tails $\frac{3}{4}$ of the time. We flip this coin 1600 times. Give a lower bound for the probability that we get between 366 and 434 heads. 

Hint: $\sqrt{3} \approx 1.732$.

We have a binomial random variable with $p = \frac{1}{4}$ and $n = 1600$. We know the mean of this random variable will be $np = 1600 \cdot \frac{1}{4} = 400$ and the variance of this random variable will be $np(1-p) = 1600 \cdot \frac{1}{4} \cdot \frac{3}{4} = 300$. This means the standard deviation of this random variable will be $\sqrt{300} = 10\sqrt{3} \approx 17.3$. Since we know we have an integer number of heads, the probability of getting more than 400 $- 2 \cdot 10\sqrt{3}$ heads and less than 400 $+ 2 \cdot 20\sqrt{3}$ heads will be equal to the probability of getting between 400 $- 34 = 366$ heads and 400 $+ 34 = 434$ heads. By Chebyshev’s inequality, we know that $P(|X - \mu| < 2\sigma) \geq 1 - \frac{1}{2} = 0.75$.

We could also use Chebyshev’s inequality in the form $P(|X - \mu| < r) \geq 1 - \frac{\text{Var}[X]}{r^2}$ to get a very similar result.

Note: with this many coin flips, the fraction of flips that come up heads will be approximately normally distributed by the central limit theorem, so we could do much better than this crude estimate from Chebyshev’s inequality.

(3) (original) A number is chosen uniformly at random between $-\sqrt{3}$ and $\sqrt{3}$. Give an upper bound for the probability that this number is between 1.5 and $\sqrt{3}$.

We have a continuous uniform random variable $X$ on the interval $[-\sqrt{3}, \sqrt{3}]$. The mean of $X$ will be $\mu = 0$, as the PDF is symmetric around $x = 0$. The variance of $X$ will be $\frac{(\sqrt{3} - (-\sqrt{3}))^2}{12} = 1$, so the standard deviation of $X$ will be $\sigma = 1$.

Chebyshev’s inequality tells us that $P(|X| \geq 1.5) = P(|X - \mu| \geq 1.5\sigma) \leq \frac{1}{1.5^2} = \frac{4}{9}$. Since the maximum value in the range of $X$ is $\sqrt{3}$, $P(1.5 \leq X \leq \sqrt{3}) = P(X \geq 1.5)$. Moreover, since the PDF is symmetric around $x = 0$, we have that $P(X \geq 1.5) = P(X \leq -1.5) = \frac{1}{2}P(|X| \geq 1.5)$. This means we can divide the value we get from Chebyshev’s inequality by 2 to get a bound $P(1.5 \leq X \leq \sqrt{3}) \leq \frac{1}{2} \cdot \frac{4}{9} = \frac{2}{9}$.
(4) (adapted from Rosen 7.4.38b) Suppose that on an exam, the mean is 54 and the standard deviation is 21. Give an upper bound for the fraction of students who either got a 12 or lower or got a 96 or higher. Explain why the real fraction of students who either got a 12 or lower or got a 96 or higher is probably much smaller.

Let \( X \) be the random variable giving a student’s score on the exam. With a mean of \( \mu = 54 \) and a standard deviation of \( \sigma = 21 \), we have that 12 = 54 − 42 = \( \mu - 2\sigma \) and 96 = 54 + 42 = \( \mu + 2\sigma \). The fraction of students who either got a 12 or lower or got a 96 or higher is equal to the probability \( P(X \leq 12 \text{ or } X \geq 96) \), which we can rewrite in the form \( P(|X - \mu| \geq 2\sigma) \). Chebyshev’s inequality tells us that \( \frac{1}{2\sigma} = \frac{1}{2 \cdot 21} = \frac{1}{42} \) is an upper bound for this probability.

We know that scores on exams tend to be approximately normally distributed, and we know that the total probability \( P(|X - \mu| \geq 2\sigma) \) for a normal distribution is around 0.05, so we would expect to have a much smaller fraction of students who either got a 12 or lower or got a 96 or higher.

2. NORMAL DISTRIBUTION

(5) True/False practice:

(a) The function \( f(x) = \frac{1}{\sqrt{2\pi}3}e^{- (x-1)^2 / 18} \) is a PDF for a random variable with mean 2 and standard deviation 3.

[False] The function \( f(x) = \frac{1}{\sqrt{2\pi}3}e^{- (x-1)^2 / 18} \) is a PDF for a random variable with mean 1 and standard deviation 3. Recall that a normal random variable with mean \( \mu \) and standard deviation \( \sigma \) has PDF \( f(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-(x-\mu)^2/(2\sigma^2)} \).

(b) If police are instructed to ticket motorists driving 125 kilometers per hour or more, what percentage of motorists are targeted?

We find this probability by transforming to a standard normal random variable (i.e. one with mean 0 and standard deviation 1) and using a z-score table. We want to find \( P(X \leq 100) \) for \( X \) a normal random variable with mean 112 and standard deviation 8.

We have \( P(X \leq 100) = P(X - 112 \leq -12) = P(\frac{X-112}{8} \leq -1.5) \), and we know that the random variable \( (X - 112)/8 \) is a standard normal random variable (which we denote by \( Z \)). We know that looking at the probability corresponding to \( z = 1.5 \) will give us \( P(0 \leq Z \leq 1.5) = P(-1.5 \leq Z \leq 0) \), and we know \( P(Z \leq 0) = \frac{1}{2} \) since the standard normal distribution is symmetric around \( X = 0 \). From our table, we see \( P(0 \leq Z \leq 1.5) = 0.4332 \), so we conclude that \( P(\frac{X-112}{8} \leq -1.5) = \frac{1}{2} - 0.4332 = 0.0668 \), and thus that the probability that a randomly-chosen vehicle is traveling at a legal speed is 0.0668.

(6) (Stewart/Day 12.5.69) The speed of vehicles on a highway with speed limit 100 kilometers per hour are normally distributed with mean 112 kilometers per hour and standard deviation 8 kilometers per hour.

(a) What is the probability that a randomly-chosen vehicle is traveling at a legal speed?

We now want to find \( P(X \geq 125) \) for \( X \) a normal random variable with mean 112 and standard deviation 8. As above, we transform \( X \) into a standard normal random variable and use our z-score table.

We have \( P(X \geq 125) = P(X - 112 \geq 13) = P(\frac{X-112}{8} \geq \frac{13}{8}) \). We know that looking at the probability corresponding to \( z = \frac{13}{8} \) will give us \( P(0 \leq Z \leq \frac{13}{8}) \), and we know that \( P(Z \geq \frac{13}{8}) = \frac{1}{2} - P(0 \leq Z \leq \frac{13}{8}) \). From our table, we see that \( P(0 \leq Z \leq \frac{13}{8}) = 0.4479 \), so we have that \( P(\frac{X-112}{8} \geq \frac{13}{8}) = \frac{1}{2} - 0.4479 = 0.0521 \), and thus that about 5.21% of motorists are targeted?

3. ACKNOWLEDGMENTS
