1. Binomial Coefficients and Identities

(1) True/false practice:
(a) If we are given a complicated expression involving binomial coefficients, factorials, powers, and fractions that we can interpret as the solution to a counting problem, then we know that that expression is an integer.
   True. If an expression is the answer to a counting problem, it has to be an integer; there can’t be a fractional number of ways to make a committee/government/string/etc.

(b) We can use the binomial theorem to show that
   \[ \sum_{r=0}^{k} \binom{n}{r} \binom{m}{k-r} = \binom{n+m}{k}. \]
   True. We know that \((1 + x)^m = \sum_{s=0}^{m} \binom{m}{s} x^s\) and \((1 + 1)^n = \sum_{s=0}^{n} \binom{n}{s} x^s\). The coefficient on \(x^k\) in \((1 + x)^m(1 + x)^n\) is given by \(\sum_{r=0}^{k} \binom{n}{r} \binom{m}{k-r}\), since if \((1 + x)^m\) contributes \(x^r\) to the \(x^k\) then \((1 + x)^n\) contributes \(x^{k-r}\). However, since \((1 + x)^m(1 + x)^n = (1 + x)^{m+n}\), we can also compute the coefficient on \(x^k\) in \((1 + x)^{m+n}\) as \(\binom{n+m}{k}\).

(2) (textbook 6.4.7) What is the coefficient of \(x^9\) in \((2 - x)^{19}\)?

   The coefficient on \(x^9\) is, by the binomial theorem,
   \[ \binom{19}{9} \cdot 2^{19-9}(-1)^9 = -2^{10} \binom{19}{9} = -94595072. \]

(3) (textbook 6.4.17) What is the row of Pascal’s triangle containing the binomial coefficients \(\binom{n}{k}\), \(0 \leq k \leq 9\)?

   Either by writing out rows 0 through 8 of Pascal’s triangle or by directly computing the binomial coefficients, we see that the row is
   
   \[
   \begin{array}{cccccccccc}
   1 & 9 & 36 & 84 & 126 & 126 & 84 & 36 & 9 & 1
   \end{array}
   \]

(4) (textbook 6.4.25) Prove that if \(n\) and \(k\) are positive integers with \(1 \leq k \leq n\), then \(k\binom{n}{k} = n\binom{n-1}{k-1}\):

   (a) using a combinatorial argument

   Consider selecting a committee of \(k\) people with a designated chair out of a group of \(n\) people. There are \(\binom{n}{k}\) ways to first select the committee members, and then \(k\) ways to select the chair out of those. There are \(n\) ways to first select a leader for the committee, and then \(\binom{n-1}{k-1}\) ways to select the rest of the committee from the remaining people. Since these two ways of counting counting the same thing, we know that \(k\binom{n}{k} = n\binom{n-1}{k-1}\).

   (b) using the formula for binomial coefficients.

   We know
   \[
   k \binom{n}{k} = \frac{n!}{(n-k)!(k-1)!} = n \cdot \frac{(n-1)!}{(n-k)!(k-1)!} = n \frac{(n-1)!}{((n-1) - (k-1))!(k-1)!} = n \binom{n-1}{k-1}.
   \]

(5) (textbook 6.4.31 plus an extra challenge) Prove the hockeystick identity

   \[
   \sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{r}.
   \]
whenever $n$ and $r$ are positive integers:

(a) using a combinatorial argument

Suppose we want to pick a hockey team of $n + 1$ people out of $n + r + 1$ people; we know that this can be done in \( \binom{n+r+1}{n+1} = \binom{n+r+1}{r} \) ways.

We now count the number of ways to pick this team in another way. Line up the players in some arbitrary order from 1 to $n + r + 1$. If the first player on the team is player $r + 1$, then we need to select $n$ more players from the remaining $n$ players, which can be done in \( \binom{n}{n} = \binom{n}{0} \) ways. If the first player on the team is player $r$, then we need to select $n$ more players from the remaining $n + 1$ players, which can be done in \( \binom{n+1}{n} = \binom{n+1}{1} \) ways. In general, if the first player on the team is player $r + 1 - k$, then we need to select $n$ more players from the remaining $n + k$ players, which can be done in \( \binom{n+k}{n} = \binom{n+k}{k} \) ways. Since the first player must be player $r + 1 - k$ for some $k$ from 0 to $r$, and these cases are distinct there are \( \sum_{k=0}^{r} \binom{n+k}{k} \) ways to select the team.

Since both sides count the same thing (the number of ways to select an $n + 1$-person hockey team out of $n + r + 1$ people), we know

\[
\sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{r}
\]

as desired.

(b) using Pascal’s identity \( \binom{n}{k} + \binom{n+1}{k+1} = \binom{n+1}{k+1} \)

We know that \( \binom{n}{0} = \binom{n+1}{0} = 1 \). So we rewrite our sum on the left-hand side as

\[
\binom{n+1}{0} + \sum_{k=1}^{r} \binom{n+k}{k}.
\]

We know that \( \binom{n+1}{0} + \binom{n+1}{1} = \binom{n+2}{1} \) by Pascal’s identity. This means that we can rewrite out sum as

\[
\binom{n+2}{1} + \sum_{k=2}^{r} \binom{n+k}{k}.
\]

We can then pull out the \( \binom{n+2}{2} \) term from the sum and apply Pascal’s identity again, rewriting our sum as

\[
\binom{n+3}{2} + \sum_{k=3}^{r} \binom{n+k}{k}.
\]

Continuing this process, we will end up with a sum that reduces down to \( \binom{n+r}{r} + \binom{n+r}{r-1} \), which, by Pascal’s identity one final time, is \( \binom{n+r+1}{r+1} \).

(c) (challenge) by counting paths on Pascal’s triangle.

We know that \( \binom{n+r+1}{r} \) represents the number of paths from the top of Pascal’s triangle to the position in row $n + r + 1$ and “column” $r$. To get there, we must go through one of the elements in the diagonal $(n, 0), (n + 1, 1)$, and so on through $(n + r, r)$. Once we pass through this diagonal, we are in the diagonal containing \( \binom{n+r+1}{r} \), so we must go down and to the right for the remainder of our path. Since a path passes through the diagonal at each of the \( \binom{n+k}{k} \) at exactly one of the places in the left-hand sum, and there are \( \binom{n+k}{k} \) ways to make it to that point in the diagonal, after which the remaining moves are completely determined, we see that

\[
\sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{r}
\]

as desired.

If you sketch out a couple of examples of this process, you’ll see where the name “hockey stick” comes from.
2. Generalized permutations and combinations

(7) True/false practice:
(a) In a counting problem, the question of which objects are distinguishable and which objects are indistinguishable is very important.

True. For example, the number of ways to order \( n \) distinguishable people in a line is very different than the number of ways to order \( n \) identical 1s in a string.

(b) There are many different ways to conceptualize combinations with repetition: for example, the number of solutions with \( x_1, x_2, x_3 \) nonnegative integers to equations like \( x_1 + x_2 + x_3 = 10 \), the number of ways to give 10 indistinguishable treats to three distinguishable dogs, and the number of ways to put 2 bars in between 10 stars.

True. We can show that both of the last two problems above are equivalent to the first one, showing that all three are equivalent. If we let \( x_k \) represent the number of treats we are giving dog \( k \), we see that any assignment of treats to dogs can be represented as an ordered triple \( (x_1, x_2, x_3) \) with \( x_1 + x_2 + x_3 = 10 \), and vice versa, any ordered triple \( (x_1, x_2, x_3) \) with \( x_1 + x_2 + x_3 = 10 \) gives us a way to give the three dogs treats such that 10 treats in total are dispensed.

(8) (textbook 6.5.1) In how many ways can five elements be selected in order from a set with three elements when repetition is allowed?

We have 3 choices for the first element, 3 for the second, and so on through 3 for the fifth, for a total of \( 3^5 = 243 \) ways to select the elements. Alternatively, we can use our table; this is the case of distinguishable balls, distinguishable boxes, and no restriction on what kind of function.

(9) (textbook 6.5.7) How many ways are there to select three unordered elements from a set with five elements when repetition is allowed?

This is the case of indistinguishable balls and distinguishable bins without restrictions on what kind of function is used. There are thus \( \binom{3+5-1}{3} = 35 \) ways to select the elements.

(10) (textbook 6.5.13) A book publisher has 3000 copies of a discrete mathematics book. How many ways are there to store these books in their three warehouses if the copies of the book are indistinguishable?

This is the case of indistinguishable balls and distinguishable bins without restrictions on what kind of function is used. There are thus \( \binom{3000+3-1}{3} = 4504501 \) ways to store the books.

(11) (textbook 6.5.17) How many strings of 10 ternary digits (each digit is either 0, 1, or 2) are there with exactly two 0s, exactly three 1s, and exactly five 2s?

This is a time when we want to use the “MISSISSIPPI formula.” There are 10! ways to arrange the 10 symbols, but we need to divide by 2! since the 0s are indistinguishable, by 3! since the 1s are indistinguishable, and 5! since the 2s are indistinguishable. The answer is thus \( \frac{10!}{2!3!5!} = 2520 \).

3. Acknowledgments