A stochastic fluid-structure interaction problem describing Stokes flow interacting with a membrane

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Fluid-structure interaction

- The coupled dynamical interaction between a **viscous incompressible fluid** and a deformable **structure**
- **Fully coupled problems** are difficult: mixed parabolic-hyperbolic system
- Two types of couplings: **linear coupling** (assume fixed fluid domain) and **nonlinear coupling**

Literature review

Deterministic models of **fluid-structure interaction**

- with **linear coupling** (Du-Gunzburger-Hou-Lee '03, Barbu-Grujić-Lasiecka-Tuffaha '07 and '08, Kukavica-Tufaha-Ziane '10, Kuan-Čanić ‘21)

- with **nonlinear coupling** (Muha-Čanić '13, da Veiga '04, Lequeurre '11, Chambolle-Desjardins-Esteban-Grandmont ’05)

Solutions of **stochastic partial differential equations**:

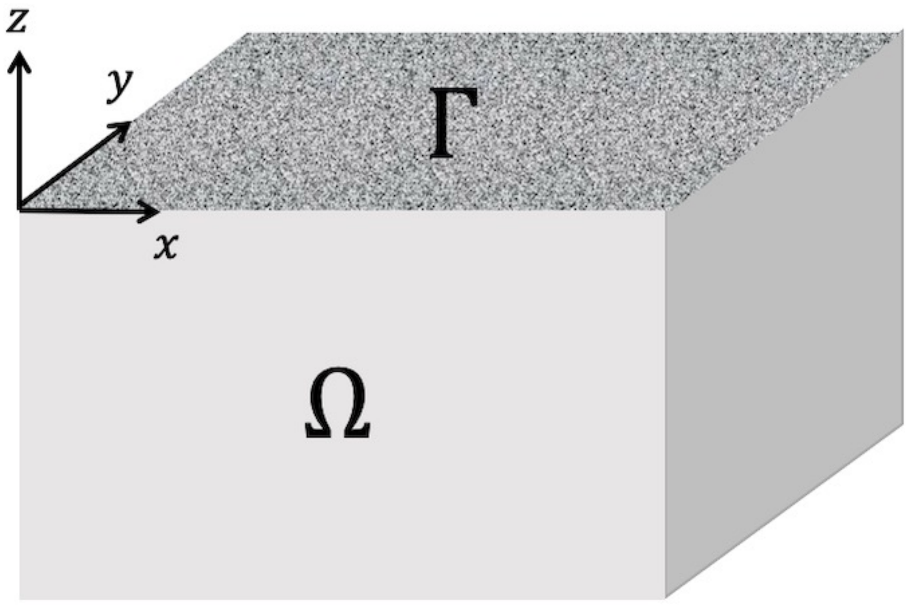
- Stochastic wave equations (Dalang-Frangos ‘98, Dalang ‘99, Karczewska-Zabcyk ‘99, Conus-Dalang ‘08, Dalang-Sanz-Solé ‘09)
- Stochastic Navier-Stokes equation (Bensoussan-Temam ‘73, Capinski-Gatarek ‘94, Flandoli-Gatarek ‘95)
- Stochastic one layer shallow water equations (Link-Nguyen-Temam ’17)

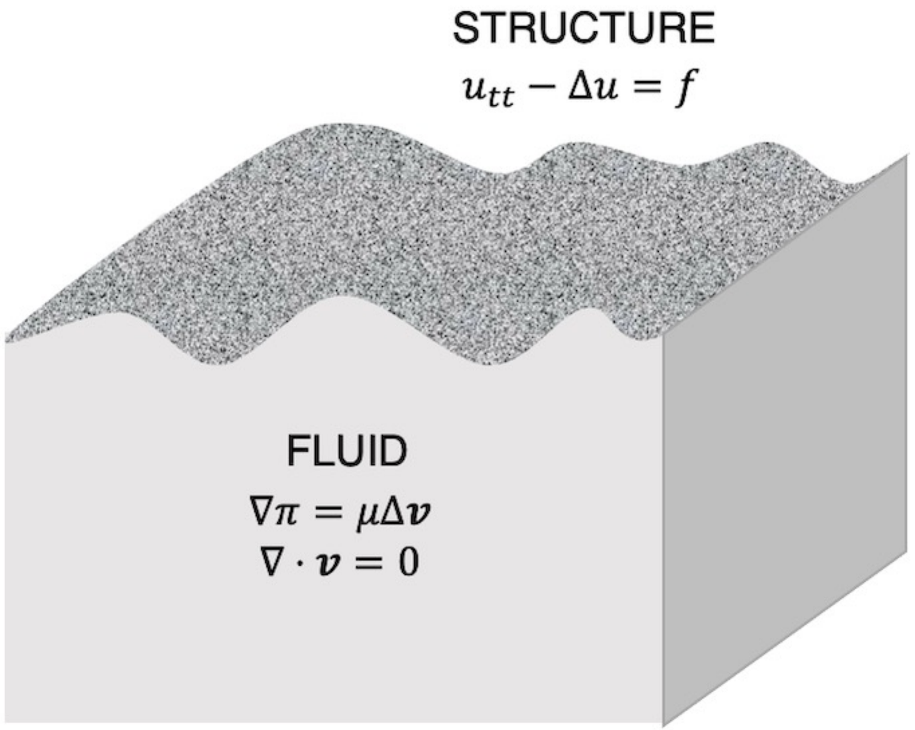
Not much past work on stochastic fluid-structure interaction

Stochastic fluid-structure interaction

Stochastic viscous wave equation (Kuan-Čanić ‘21)

$$u_{tt} + \sqrt{-\Delta} u_t - \Delta u = W(dx, dt) \qquad \text{in } \mathbb{R}^n$$





Stochastic fluid-structure interaction

Viscous nonlinear wave equation (Kuan-Čanić ’21 in Trans. AMS)

$$u_{tt} + \sqrt{-\Delta} u_t - \Delta u + u^p = 0, \qquad \text{in } \mathbb{R}^n$$

- *Probabilistic* **local and global well-posedness** of the *deterministic* viscous wave equation with a power nonlinearity (Kuan-Čanic ’21 in Trans. AMS, Kuan-Oh ‘21)
- Existence of **mild solution in dimensions one and two** to *stochastic* viscous wave equation and Hölder regularity properties (Kuan-Čanic ‘21)
- Local and global well-posedness **for singular stochastic viscous nonlinear wave equations** (Liu-Oh ‘21)

GOAL: Extend existence results to fully coupled FSI.

Stochastic fluid-structure interaction

Stochastic viscous wave equation (Kuan-Čanić ‘21)

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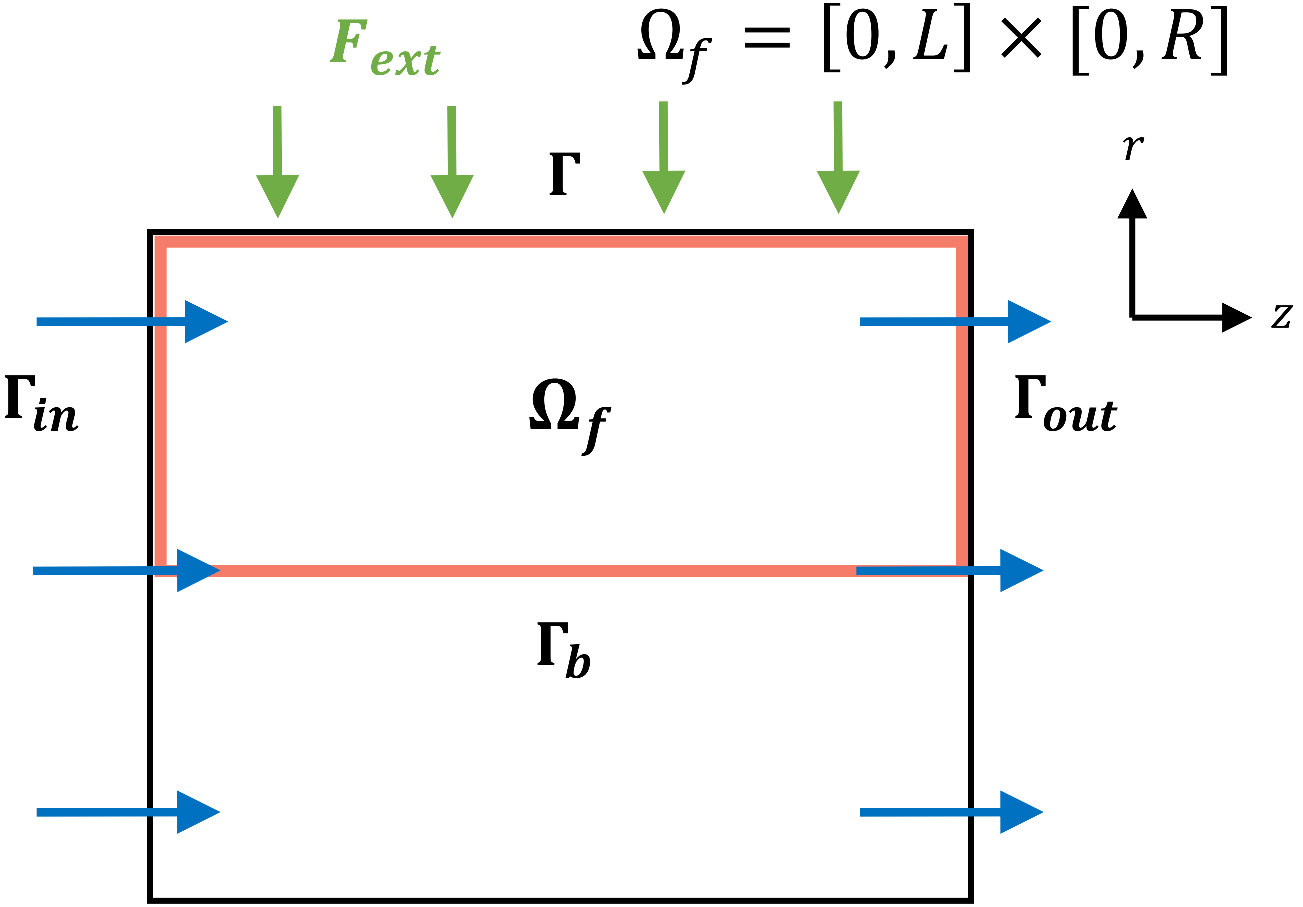
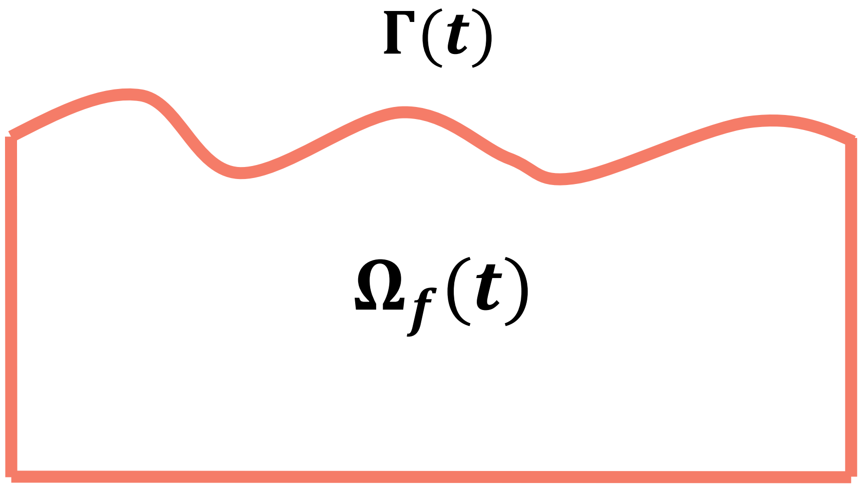
GOAL: Extend existence results to fully coupled FSI.

Description of the model

$$\Omega_f = [0, L] \times [0, R]$$



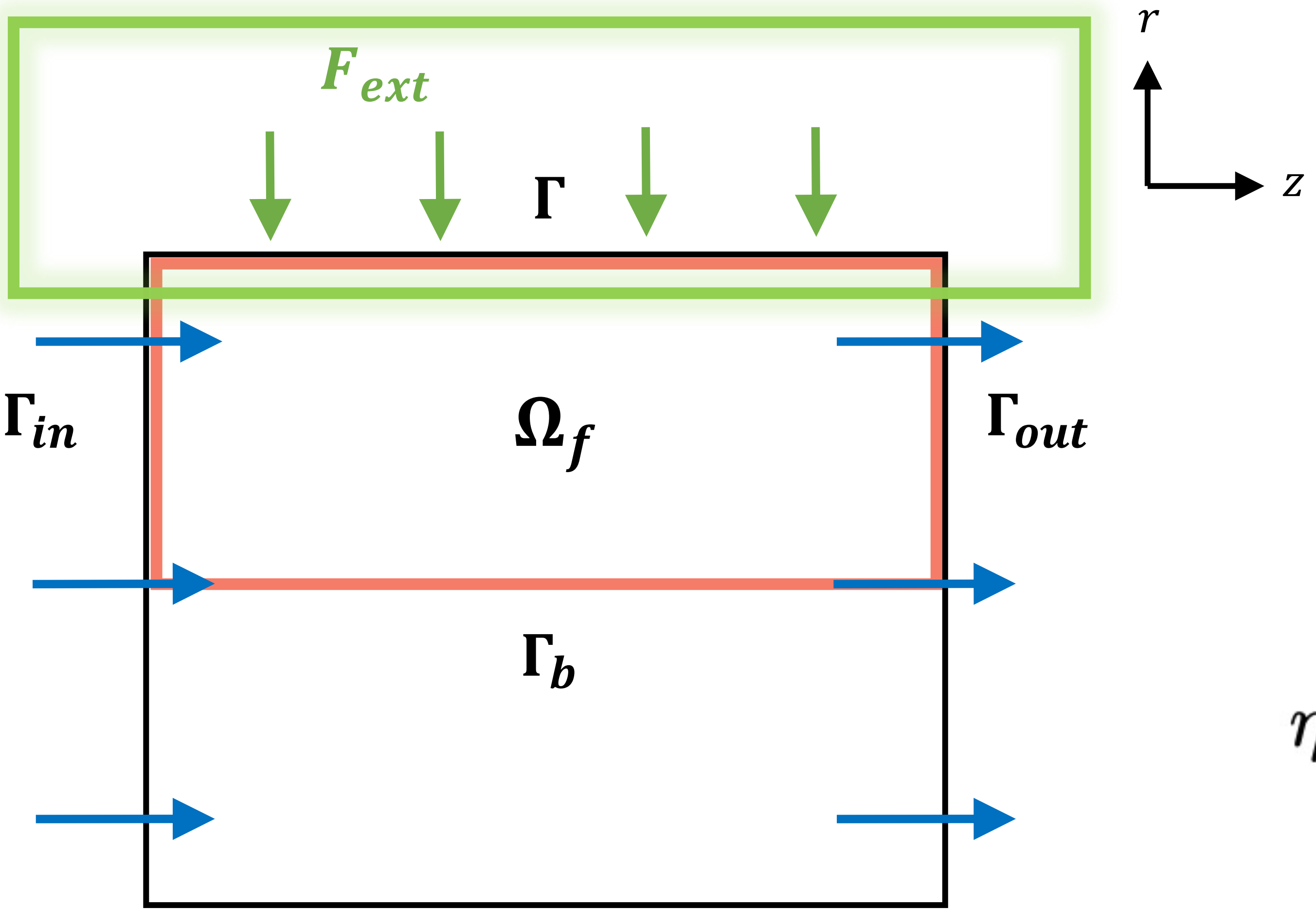


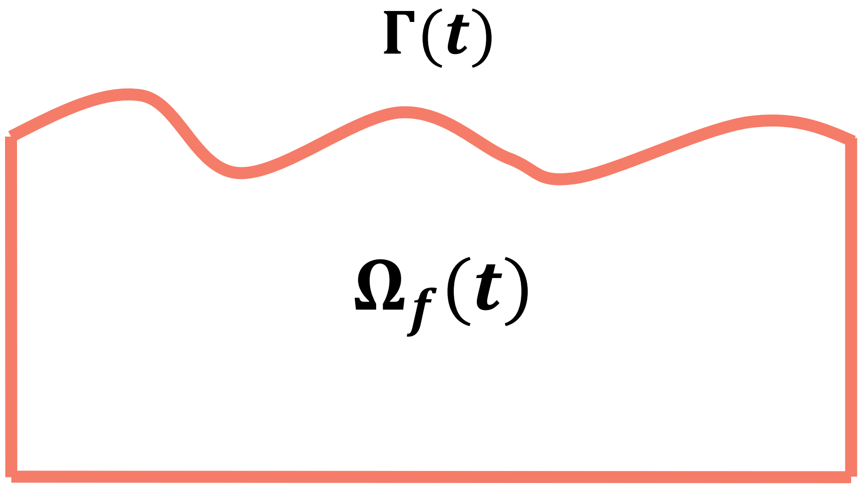
Structure subproblem

$$\Omega_f = [0, L] \times [0, R]$$

Structure displacement: $\eta(t, z)$

(Assume displacement in only radial direction)





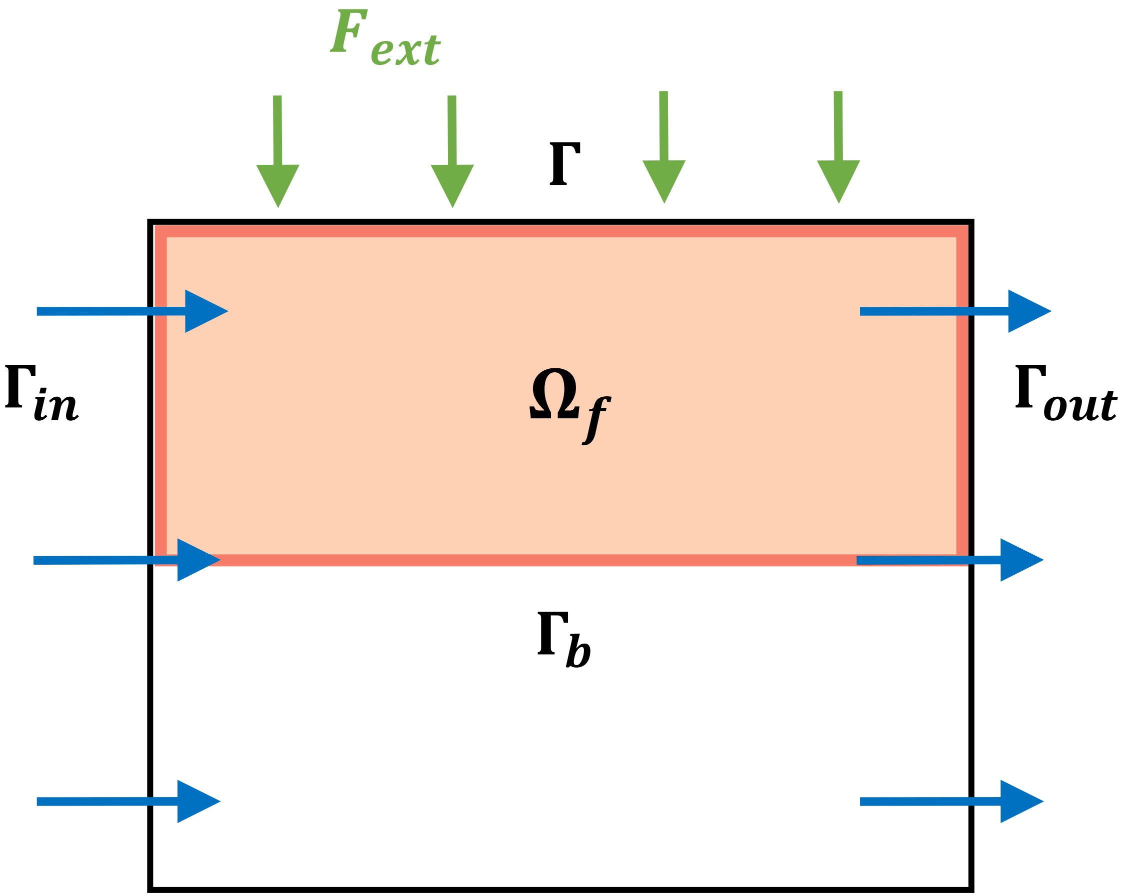
$$\eta_{tt} - \Delta\eta = f, \qquad \text{on } \Gamma,$$

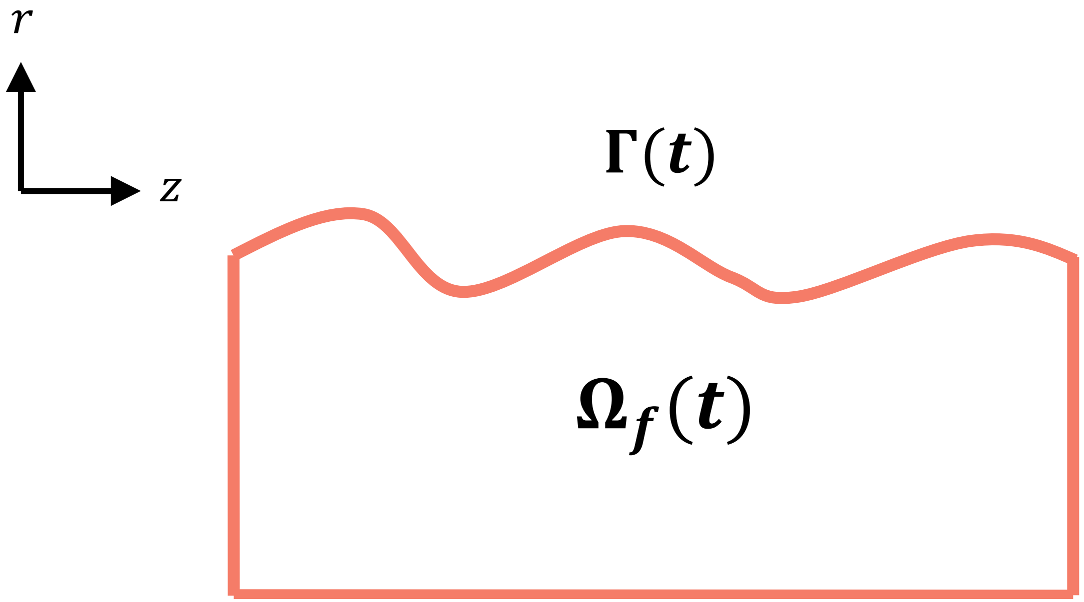
$$\eta(0) = \eta(L) = 0.$$

Initial condition: $\eta(t=0) = \eta_0 \in H_0^1(\Gamma), \quad \partial_t \eta(t=0) = v_0 \in L^2(\Gamma)$

Fluid subproblem

Fluid velocity: $\boldsymbol{u}(t,z,r) = \left(u_z(t,z,r), u_r(t,z,r)\right)$



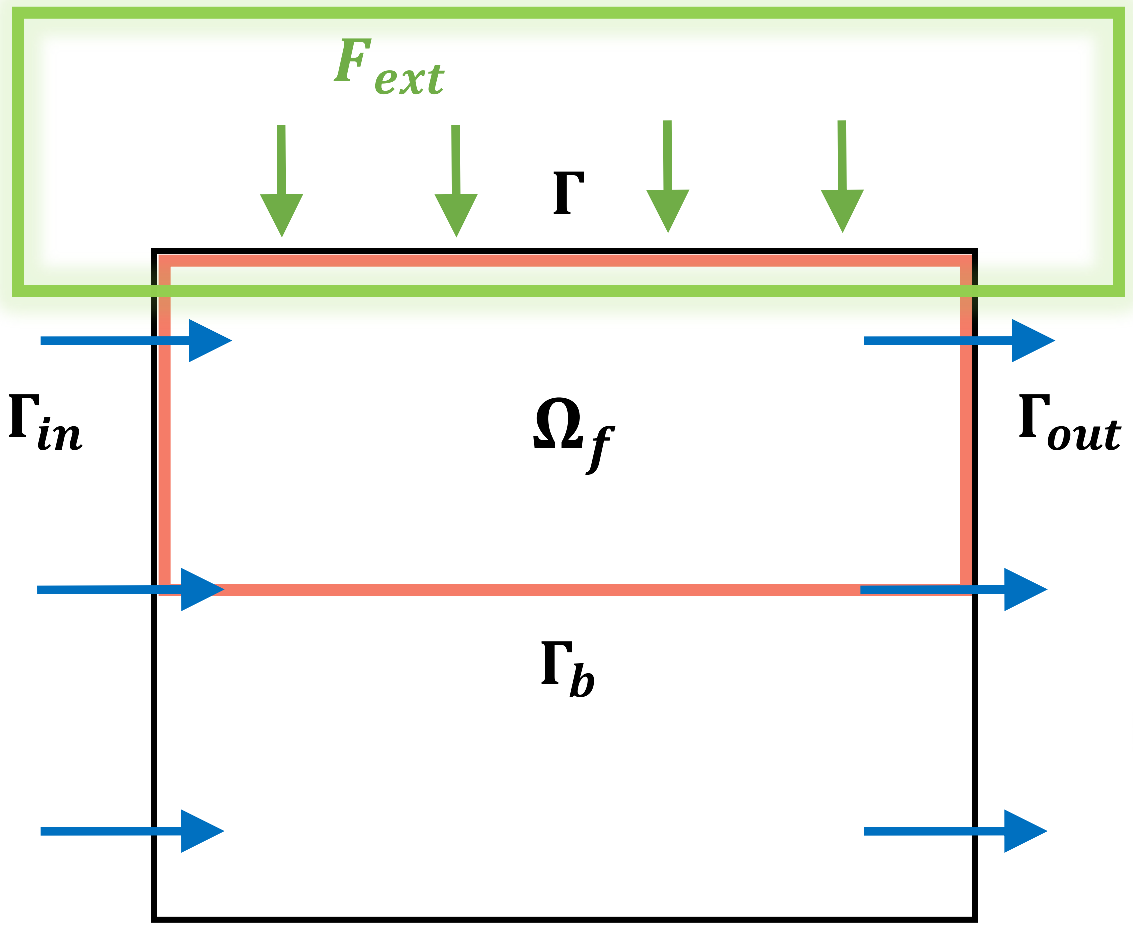


$$\left.\begin{array}{rcl} \partial_t \boldsymbol{u} - \mu \Delta \boldsymbol{u} + \nabla p & = & 0, \\ \nabla \cdot \boldsymbol{u} & = & 0, \end{array}\right\} \quad \text{in } \Omega_f.$$

$$\begin{array}{rrl} & u_r = 0, & \text{on } \Gamma_{in} \cup \Gamma_{out} \\ & p = P_{in/out}(t), & \text{on } \Gamma_{in/out} \\ u_r = 0, & \partial_r u_z = 0, & \text{on } \Gamma_b \end{array}$$

Initial condition: $\boldsymbol{u}(t=0) = \boldsymbol{u_0} \in \boldsymbol{L}^2(\Omega_f)$

Coupling conditions



Kinematic coupling condition

(continuity of velocities)

$$\boldsymbol{u} = \eta_t \boldsymbol{e_r}, \qquad \text{on } \Gamma$$

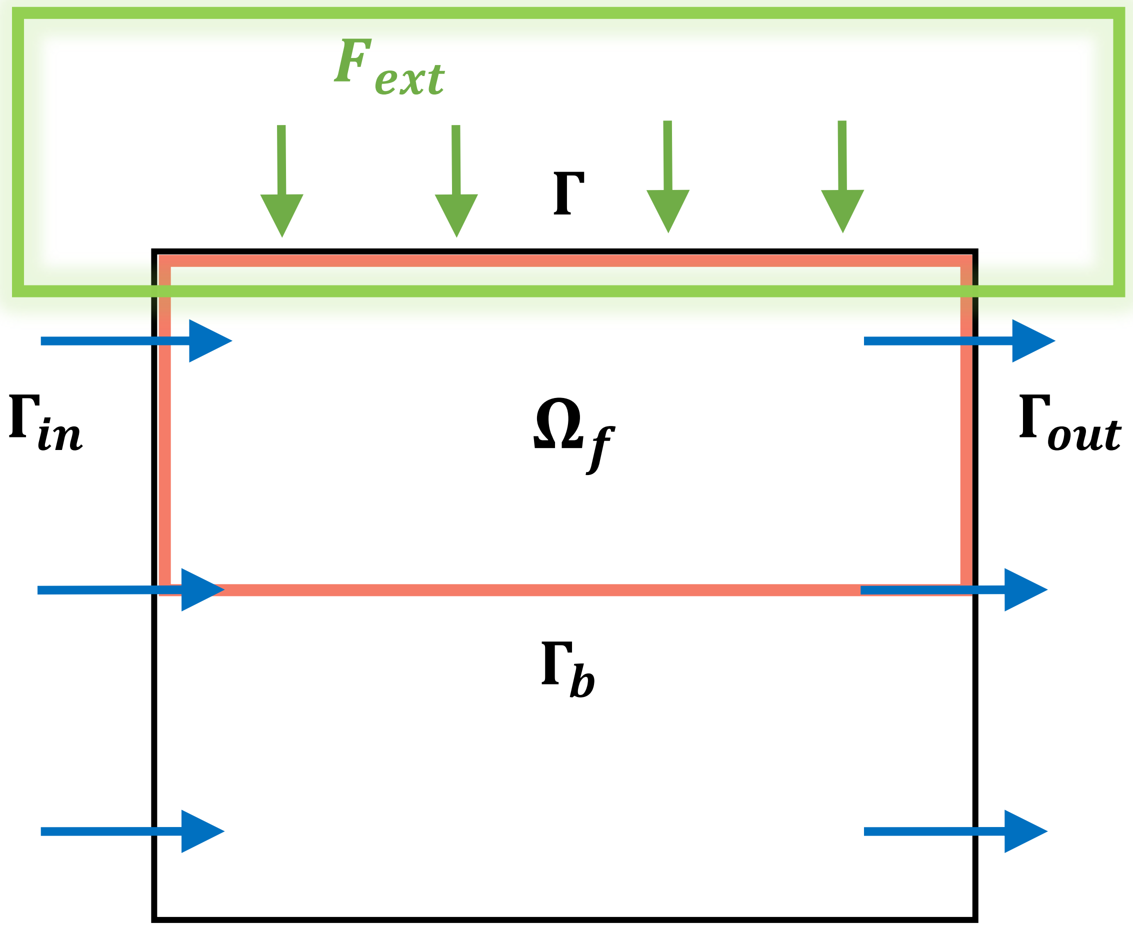
Dynamic coupling condition

(jump in internal and external loading on the structure)

$$\eta_{tt} - \Delta\eta = -\boldsymbol{\sigma} \boldsymbol{e_r} \cdot \boldsymbol{e_r} + F_{ext}, \qquad \text{on } \Gamma$$

where $\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\mu \boldsymbol{D}(\boldsymbol{u})$

Stochastic effects



Dynamic coupling condition

(jump in internal and external loading on the structure)

$$\eta_{tt} - \Delta\eta = -\boldsymbol{\sigma}\boldsymbol{e_r}\cdot\boldsymbol{e_r} + \boxed{F_{ext},} \qquad \text{on } \Gamma$$

Let *W*(*t*) be a one-dimensional Brownian motion with respect to a probability space with *complete** filtration $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$

$$F_{ext} = \dot{W}(t)$$

* $\mathcal{F}_t$ contains all null sets so the almost sure limit of $\mathcal{F}_t$ measurable random variables is also $\mathcal{F}_t$ measurable.

A priori energy estimate

ENERGY

$$\boxed{E(T) := \frac{1}{2}\int_{\Gamma} |\nabla \eta|^2 dz + \frac{1}{2}\int_{\Gamma} |v|^2 dz + \frac{1}{2}\int_{\Omega_f} |\boldsymbol{u}|^2 d\boldsymbol{x}}$$

$$d\left(\frac{1}{2}\int_{\Gamma} |\nabla \eta|^2 dz + \frac{1}{2}\int_{\Gamma} |v|^2 dz + \frac{1}{2}\int_{\Omega_f} |\boldsymbol{u}|^2 d\boldsymbol{x}\right)$$

$$= \left(\frac{L}{2} - 2\mu \int_{\Omega_f} |\boldsymbol{D}(\boldsymbol{u})|^2 d\boldsymbol{x} + \int_{\Gamma_{in}} p u_z dr - \int_{\Gamma_{out}} p u_z dr\right) dt + \left(\int_{\Gamma} v dz\right) dW.$$

Apply Itô’s formula.

STOCHASTICITY **FLUID DISSIPATION**

FINAL ESTIMATE
(for C independent of T)

$$\boxed{\mathbb{E}\left(\max_{0\le t\le T} E(t) + \mu\int_0^t \int_{\Omega_f} |\boldsymbol{D}(\boldsymbol{u})|^2 d\boldsymbol{x}\right) \le C\left(T + E(0) + ||P_{in}(t)||^2_{L^2(0,T)} + ||P_{out}(t)||^2_{L^2(0,T)}\right)}$$

Solution space and test space

$$\mathbb{E}\left(\max_{0\le t\le T} E(t) + \mu\int_0^t\int_{\Omega_f} |\boldsymbol{D}(\boldsymbol{u})|^2 d\boldsymbol{x}\right) \le C\left(T + E(0) + ||P_{in}(t)||^2_{L^2(0,T)} + ||P_{out}(t)||^2_{L^2(0,T)}\right)$$

$$\boxed{E(t) := \frac{1}{2}\int_\Gamma |\nabla\eta|^2 dz + \frac{1}{2}\int_\Gamma |v|^2 dz + \frac{1}{2}\int_{\Omega_f} |\boldsymbol{u}|^2 d\boldsymbol{x}}$$

$$\boxed{\mu\int_0^T\int_{\Omega_f} |\boldsymbol{D}(\boldsymbol{u})|^2 d\boldsymbol{x}}$$

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$$\boxed{\mu\int_0^T\int_{\Omega_f} |\boldsymbol{D}(\boldsymbol{u})|^2 d\boldsymbol{x}}$$

FLUID

$$\mathcal{V}_F = \{\boldsymbol{u} = (u_z, u_r) \in H^1(\Omega_f)^2 : \nabla\cdot\boldsymbol{u} = 0,\ u_z = 0 \text{ on } \Gamma,\ u_r = 0 \text{ on } \partial\Omega_f\backslash\Gamma\}.$$

$$\mathcal{W}_F(0,T) = L^2(\Omega; L^\infty(0,T;L^2(\Omega_f))) \cap L^2(\Omega; L^2(0,T;\mathcal{V}_F)).$$

STRUCTURE

$$\mathcal{V}_S = H_0^1(\Gamma).$$

$$\mathcal{W}_S(0,T) = L^2(\Omega; W^{1,\infty}(0,T;L^2(\Gamma))) \cap L^2(\Omega; L^\infty(0,T;\mathcal{V}_S)).$$

Solution space and test space

$$E(t) := \frac{1}{2}\int_\Gamma |\nabla\eta|^2 dz + \frac{1}{2}\int_\Gamma |v|^2 dz + \frac{1}{2}\int_{\Omega_f} |\boldsymbol{u}|^2 d\boldsymbol{x}$$

$$\mathbb{E}\left(\max_{0\le t\le T} E(t) + \mu\int_0^t\int_{\Omega_f} |\boldsymbol{D}(\boldsymbol{u})|^2 d\boldsymbol{x}\right) \le C\left(T + E(0) + ||P_{in}(t)||^2_{L^2(0,T)} + ||P_{out}(t)||^2_{L^2(0,T)}\right)$$

$$\mu\int_0^T\int_{\Omega_f} |\boldsymbol{D}(\boldsymbol{u})|^2 d\boldsymbol{x}$$

FLUID

$$\mathcal{V}_F = \{\boldsymbol{u} = (u_z, u_r) \in H^1(\Omega_f)^2 : \nabla\cdot\boldsymbol{u} = 0,\ u_z = 0 \text{ on } \Gamma,\ u_r = 0 \text{ on } \partial\Omega_f\backslash\Gamma\}.$$

$$\mathcal{W}_F(0,T) = L^2(\Omega; L^\infty(0,T;L^2(\Omega_f))) \cap L^2(\Omega; L^2(0,T;\mathcal{V}_F)).$$

STRUCTURE

$$\mathcal{V}_S = H^1_0(\Gamma).$$

$$\mathcal{W}_S(0,T) = L^2(\Omega; W^{1,\infty}(0,T;L^2(\Gamma))) \cap L^2(\Omega; L^\infty(0,T;\mathcal{V}_S)).$$

$$\mathcal{W}(0,T) = \{(\boldsymbol{u},\eta) \in \mathcal{W}_F(0,T)\times\mathcal{W}_S(0,T) : \boldsymbol{u}|_\Gamma = \eta_t\boldsymbol{e_r} \text{ for almost every } t\in[0,T], \text{ a.s.}\}.$$

$$\mathcal{Q}(0,T) = \{(\boldsymbol{q},\psi) \in C^1_c([0,T);\mathcal{V}_F\times\mathcal{V}_S) : \boldsymbol{q}(t,z,R) = \psi(t,z)\boldsymbol{e_r}.\}$$

Definition of a solution

$$\eta_{tt} - \Delta\eta = -\boldsymbol{\sigma}\boldsymbol{e_r} \cdot \boldsymbol{e_r} + F_{ext}(t) \qquad \text{on } \Gamma$$

$$\left.\begin{array}{rcl} \partial_t \boldsymbol{u} & = & \nabla \cdot \boldsymbol{\sigma}, \\ \nabla \cdot \boldsymbol{u} & = & 0, \end{array}\right\} \quad \text{in } \Omega_f$$

Definition of a solution

$$\eta_{tt} - \Delta\eta = -\boldsymbol{\sigma}\boldsymbol{e_r}\cdot\boldsymbol{e_r} + F_{ext}(t) \qquad \text{on } \Gamma$$

$$\left.\begin{array}{rcl} \partial_t \boldsymbol{u} &=& \nabla\cdot\boldsymbol{\sigma}, \\ \nabla\cdot\boldsymbol{u} &=& 0, \end{array}\right\} \quad \text{in } \Omega_f$$

$$\begin{aligned} -\int_0^T\int_{\Omega_f} \boldsymbol{u}\cdot\partial_t\boldsymbol{q}\,d\boldsymbol{x}dt + 2\mu\int_0^T\int_{\Omega_f}\boldsymbol{D}(\boldsymbol{u}):\boldsymbol{D}(\boldsymbol{q})d\boldsymbol{x}dt - \int_0^T\int_\Gamma \partial_t\eta\partial_t\psi dzdt + \int_0^T\int_\Gamma \nabla\eta\cdot\nabla\psi dzdt \\ = \int_0^T P_{in}(t)\left(\int_{\Gamma_{in}} q_z dr\right)dt - \int_0^T P_{out}(t)\left(\int_{\Gamma_{out}} q_z dr\right)dt \\ + \int_{\Omega_f}\boldsymbol{u_0}\cdot\boldsymbol{q}(0)d\boldsymbol{x} + \int_\Gamma v_0\psi(0)dz + \int_0^T\left(\int_\Gamma \psi dz\right)F_{ext}(t)dt, \end{aligned}$$

for all (deterministic) test functions in $\mathcal{Q}(0,T) = \{(\boldsymbol{q},\psi)\in C_c^1([0,T);\mathcal{V}_F\times\mathcal{V}_S) : \boldsymbol{q}(t,z,R) = \psi(t,z)\boldsymbol{e_r}.\}$

Definition of a solution

$$-\int_0^T\int_{\Omega_f}\boldsymbol{u}\cdot\partial_t\boldsymbol{q}\,d\boldsymbol{x}dt+2\mu\int_0^T\int_{\Omega_f}\boldsymbol{D}(\boldsymbol{u}):\boldsymbol{D}(\boldsymbol{q})d\boldsymbol{x}dt-\int_0^T\int_\Gamma\partial_t\eta\partial_t\psi dzdt+\int_0^T\int_\Gamma\nabla\eta\cdot\nabla\psi dzdt$$
$$=\int_0^T P_{in}(t)\left(\int_{\Gamma_{in}}q_z dr\right)dt-\int_0^T P_{out}(t)\left(\int_{\Gamma_{out}}q_z dr\right)dt$$
$$+\int_{\Omega_f}\boldsymbol{u_0}\cdot\boldsymbol{q}(0)d\boldsymbol{x}+\int_\Gamma v_0\psi(0)dz+\boxed{\int_0^T\left(\int_\Gamma\psi dz\right)F_{ext}(t)dt,}$$

In the case where $\quad F_{ext}(t)=\dot{W}(t),$

we interpret $\boxed{\int_0^T\left(\int_\Gamma\psi dz\right)F_{ext}(t)dt,}$ as $\boxed{\int_0^T\left(\int_\Gamma\psi dz\right)dW(t).}$

Stochastic basis

A **stochastic basis** is an ordered collection, consisting of

$$\mathcal{S} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, W),$$

where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, $\{\mathcal{F}_t\}_{t\geq 0}$ is a complete filtration with respect to this probability space, and W is a one-dimensional Brownian motion on the probability space with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$, meaning that

- W has continuous paths, almost surely,
- W is adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$,
- $W(t) - W(s)$ is independent of $\mathcal{F}_s$ for all $t \geq s$ and $W(t) - W(s) \sim N(0, t-s)$ for all $0 \leq s \leq t$.

Weak solution

measurable with respect to new probability space

Definition 2.1. An ordered triple $(\tilde{\mathcal{S}}, \tilde{\boldsymbol{u}}, \tilde{\eta})$ is a *weak solution in a probabilistically* ***weak*** *sense* if

$$\tilde{\mathcal{S}} = (\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}_{t\geq 0}, \tilde{\mathbb{P}}, \tilde{W}),$$

is a stochastic basis and $(\tilde{\boldsymbol{u}}, \tilde{\eta}) \in \mathcal{W}(0,T)$ with paths almost surely in $C(0,T;\mathcal{Q}')$ satisfies:

- $(\tilde{\boldsymbol{u}}, \tilde{\eta})$ is adapted to the filtration $\{\tilde{\mathcal{F}}_t\}_{t\geq 0}$,
- $\tilde{\eta}(0) = \eta_0$ almost surely, and
- for all $(\boldsymbol{q}, \psi) \in \mathcal{Q}(0,T)$,

$$\begin{aligned}
-\int_0^T \int_{\Omega_f} \tilde{\boldsymbol{u}} \cdot \partial_t \boldsymbol{q} \, d\boldsymbol{x} dt + 2\mu \int_0^T \int_{\Omega_f} \boldsymbol{D}(\tilde{\boldsymbol{u}}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x} dt - \int_0^T \int_\Gamma \partial_t \tilde{\eta} \partial_t \psi \, dz dt + \int_0^T \int_\Gamma \nabla \tilde{\eta} \cdot \nabla \psi \, dz dt \\
= \int_0^T P_{in}(t) \left(\int_{\Gamma_{in}} q_z dr \right) dt - \int_0^T P_{out}(t) \left(\int_{\Gamma_{out}} q_z dr \right) dt \\
+ \int_{\Omega_f} \boldsymbol{u_0} \cdot \boldsymbol{q}(0) d\boldsymbol{x} + \int_\Gamma v_0 \psi(0) dz + \int_0^T \left(\int_\Gamma \psi dz \right) d\tilde{W},
\end{aligned}$$

almost surely.

Strong solution

measurable with respect to *original* probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$

Definition 2.2. An ordered pair $(\boldsymbol{u}, \eta)$ is a *weak solution in a probabilistically* ***strong*** *sense* if $(\boldsymbol{u}, \eta) \in \mathcal{W}(0,T)$ with paths almost surely in $C(0,T;\mathcal{Q}')$ satisfies:

- $(\boldsymbol{u}, \eta)$ is adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$
- $\eta(0) = \eta_0$ almost surely, and
- for all $(\boldsymbol{q}, \psi) \in \mathcal{Q}(0,T)$,

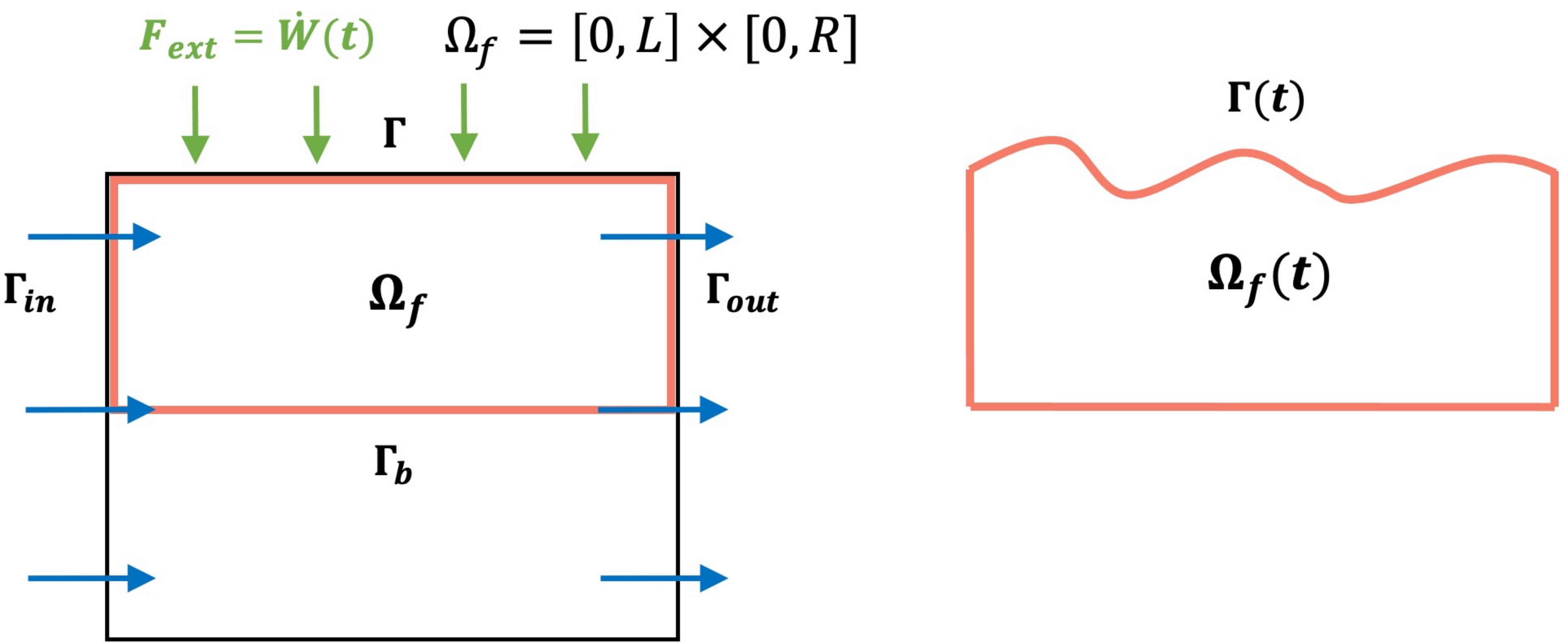
$$
\begin{aligned}
-\int_0^T \int_{\Omega_f} \boldsymbol{u}\cdot\partial_t \boldsymbol{q} \, d\boldsymbol{x} dt + 2\mu \int_0^T \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x} dt - \int_0^T \int_\Gamma \partial_t \eta \partial_t \psi dz dt + \int_0^T \int_\Gamma \nabla\eta\cdot\nabla\psi dz dt \\
= \int_0^T P_{in}(t) \left(\int_{\Gamma_{in}} q_z dr \right) dt - \int_0^T P_{out}(t) \left(\int_{\Gamma_{out}} q_z dr \right) dt \\
+ \int_{\Omega_f} \boldsymbol{u_0} \cdot \boldsymbol{q}(0) d\boldsymbol{x} + \int_\Gamma v_0 \psi(0) dz + \int_0^T \left(\int_\Gamma \psi dz \right) dW.
\end{aligned}
$$

almost surely.

STRATEGY: Construct weak solution, use standard *Göngy-Krylov theorem argument* to return to original probability space

Main theorem

Theorem 2.1. Let $\boldsymbol{u}_0 \in L^2(\Omega_f)$, $v_0 \in L^2(\Gamma)$, and $\eta_0 \in H_0^1(\Gamma)$. Let $P_{in/out} \in L^2_{loc}(0,\infty)$ and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a Brownian motion W with respect to a complete filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Then, for any $T > 0$, there exists a unique weak solution in a probabilistically strong sense in the sense of Definition 2.2 to the given stochastic fluid-structure interaction problem.



The splitting scheme

- Discretize in time, splitting the fluid, stochastic, and structure elements in the problem at each time step
- Fluid and structure splitting motivated by Muha and Čanić (2013)
- Stochastic splitting up method motivated by Bensoussan, Glowinski, Răşcanu (1992)

$$\boxed{dy + A(t, y)dt = B(t, y)dW}$$

deterministic

$$\boxed{dy_1 + A(t, y_1)dt = 0}$$

stochastic

$$\boxed{dy_2 = B(t, y_2)dW}$$

Solve on each time interval $[k\Delta t, (k+1)\Delta t]$

General scheme

Let $\Delta t = \frac{T}{N}$, $t_N^n = n\Delta t$. For each time step, iterate three subproblems.

At each step, $n = 0, 1, \dots, N-1$ and $i = 1, 2, 3$, keep track of $\boldsymbol{X}_N^{n+\frac{i}{3}} = \begin{pmatrix} \boldsymbol{u}_N^{n+\frac{i}{3}} \\ v_N^{n+\frac{i}{3}} \\ \eta_N^{n+\frac{i}{3}} \end{pmatrix}$,

and start the scheme with the initial data: $\boldsymbol{X}_N^0 = \begin{pmatrix} \boldsymbol{u}_0 \\ v_0 \\ \eta_0 \end{pmatrix}$, for all N.

GOAL: Take the limit of approximate solutions as $N \to \infty$.
This requires uniform bounds independent of N.

1. Structure subproblem

Update $\eta^{n+\frac{1}{3}}$ and $v^{n+\frac{1}{3}}$. Keep $\boldsymbol{u}^{n+\frac{1}{3}} = \boldsymbol{u}^n$.

$$\int_\Gamma \frac{\eta^{n+\frac{1}{3}} - \eta^n}{\Delta t} \phi dz = \int_\Gamma v^{n+\frac{1}{3}} \phi dz, \qquad \text{for all } \phi \in L^2(\Gamma),$$

$$\int_\Gamma \frac{v^{n+\frac{1}{3}} - v^n}{\Delta t} \psi dz + \int_\Gamma \nabla \eta^{n+\frac{1}{3}} \cdot \nabla \psi dz = 0, \qquad \text{for all } \psi \in H_0^1(\Gamma),$$

where we solve this system separately for each $\omega \in \Omega$.

Proposition 4.1. Suppose that η^n and v^n are $\mathcal{F}_{t^n}$ measurable random variables taking values in $H_0^1(\Gamma)$ and $L^2(\Gamma)$ respectively. Then, the structure problem (18) has a unique solution $(\eta^{n+\frac{1}{3}}, v^{n+\frac{1}{3}})$ that is a random variable taking values in $H_0^1(\Gamma) \times H_0^1(\Gamma)$ that is measurable with respect to $\mathcal{F}_{t^n}$.

2. Stochastic subproblem

$$\eta_{tt} - \Delta\eta = dW \longrightarrow \begin{cases} \eta_t = v, \\ v_t = \Delta\eta + dW. \end{cases}$$

Update $v^{n+\frac{2}{3}}$. Keep $\eta^{n+\frac{2}{3}} = \eta^{n+\frac{1}{3}}$ and $\boldsymbol{u}^{n+\frac{2}{3}} = \boldsymbol{u}^{n+\frac{1}{3}}$.

$$v^{n+\frac{2}{3}} = v^{n+\frac{1}{3}} + [W((n+1)\Delta t) - W(n\Delta t)].$$

Proposition 4.3. Suppose that $v^{n+\frac{1}{3}}$ is an $\mathcal{F}_{t^n}$ measurable random variable taking values in $H_0^1(\Gamma)$. Then, $v^{n+\frac{2}{3}}$ is an $\mathcal{F}_{t^{n+1}}$ measurable random variable taking values in $H^1(\Gamma)$.

3. Fluid subproblem

Update $\boldsymbol{u}^{n+1}$ and v^{n+1}. Keep $\eta^{n+1} = \eta^{n+\frac{2}{3}}$.

$$\mathcal{V} = \{(\boldsymbol{u}, v) \in \mathcal{V}_F \times L^2(\Gamma) : \boldsymbol{u}|_\Gamma = v\boldsymbol{e_r}\},$$

$$\mathcal{V}_F = \{\boldsymbol{u} = (u_z, u_r) \in H^1(\Omega_f)^2 : \nabla \cdot \boldsymbol{u} = 0,\ u_z = 0 \text{ on } \Gamma,\ u_r = 0 \text{ on } \Omega_f \backslash \Gamma\}.$$

$$\mathcal{Q} = \{(\boldsymbol{q}, \psi) \in \mathcal{V}_F \times H_0^1(\Gamma) : \boldsymbol{q}|_\Gamma = \psi\boldsymbol{e}_r\},$$

3. Fluid subproblem

Update $\boldsymbol{u}^{n+1}$ and v^{n+1}. Keep $\eta^{n+1} = \eta^{n+\frac{2}{3}}$.

For all test functions $(\boldsymbol{q}, \psi) \in \mathcal{Q}$,

$$\int_{\Omega_f} \frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^{n+\frac{2}{3}}}{\Delta t} \cdot \boldsymbol{q} d\boldsymbol{x} + 2\mu \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}^{n+1}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x} + \int_{\Gamma} \frac{v^{n+1} - v^{n+\frac{2}{3}}}{\Delta t} \psi dz$$
$$= P_{in}^{n} \int_0^R (q_z)|_{z=0} dr - P_{out}^{n} \int_0^R (q_z)|_{z=L} dr,$$

pathwise for each outcome $\omega \in \Omega$, where

$$P_{in/out}^{n} = \frac{1}{\Delta t} \int_{n\Delta t}^{(n+1)\Delta t} P_{in/out}(t) dt.$$

Proposition 4.5. Suppose that $\boldsymbol{u}^{n+\frac{2}{3}}$ and $v^{n+\frac{2}{3}}$ are $\mathcal{F}_{t^{n+1}}$ measurable random variables taking values in $\mathcal{V}_F$ and $H^1(\Gamma)$ respectively. Then, the fluid subproblem (21) has a unique solution $(\boldsymbol{u}^{n+1}, v^{n+1})$ that is an $\mathcal{F}_{t^{n+1}}$ measurable random variable taking values in $\mathcal{V}$.

Semidiscrete problem

The approximate solution satisfies the following **semidiscrete problem**:

$$\int_{\Omega_f} \frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^n}{\Delta t} \cdot \boldsymbol{q} d\boldsymbol{x} + 2\mu \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}^{n+1}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x} + \int_\Gamma \frac{v^{n+1} - v^n}{\Delta t} \psi dz + \int_\Gamma \nabla \eta^{n+1} \cdot \nabla \psi dz$$
$$= \int_\Gamma \frac{W((n+1)\Delta t) - W(n\Delta t)}{\Delta t} \psi dz + P_{in}^n \int_0^R (q_z)|_{z=0} dr - P_{out}^n \int_0^R (q_z)|_{z=L} dr,$$
$$\text{for all } (\boldsymbol{q}, \psi) \in \mathcal{Q},$$

$$\int_\Gamma \frac{\eta^{n+1} - \eta^n}{\Delta t} \phi dz = \int_\Gamma v^{n+\frac{1}{3}} \phi dz, \qquad \text{for all } \phi \in L^2(\Gamma),$$

where

$$P_{in/out}^n = \frac{1}{\Delta t} \int_{n\Delta t}^{(n+1)\Delta t} P_{in/out}(t) dt.$$

Semidiscrete problem

The approximate solution satisfies the following **semidiscrete problem**:

$$\int_{\Omega_f} \frac{\boldsymbol{u}^{n+1}-\boldsymbol{u}^n}{\Delta t}\cdot \boldsymbol{q}d\boldsymbol{x} + 2\mu\int_{\Omega_f}\boldsymbol{D}(\boldsymbol{u}^{n+1}):\boldsymbol{D}(\boldsymbol{q})d\boldsymbol{x} + \boxed{\int_\Gamma \frac{v^{n+1}-v^n}{\Delta t}\psi dz + \int_\Gamma \nabla\eta^{n+1}\cdot\nabla\psi dz}$$
$$= \int_\Gamma \frac{W((n+1)\Delta t)-W(n\Delta t)}{\Delta t}\psi dz + P_{in}^n\int_0^R (q_z)|_{z=0}dr - P_{out}^n\int_0^R (q_z)|_{z=L}dr,$$
$$\text{for all } (\boldsymbol{q},\psi)\in\mathcal{Q},$$

$$\boxed{\int_\Gamma \frac{\eta^{n+1}-\eta^n}{\Delta t}\phi dz = \int_\Gamma v^{n+\frac{1}{3}}\phi dz,} \qquad \text{for all } \phi\in L^2(\Gamma),$$

where

$$P_{in/out}^n = \frac{1}{\Delta t}\int_{n\Delta t}^{(n+1)\Delta t} P_{in/out}(t)dt.$$

structure subproblem

Semidiscrete problem

The approximate solution satisfies the following **semidiscrete problem**:

$$\int_{\Omega_f} \frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^n}{\Delta t} \cdot \boldsymbol{q} d\boldsymbol{x} + 2\mu \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}^{n+1}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x} + \int_\Gamma \frac{v^{n+1} - v^n}{\Delta t} \psi dz + \int_\Gamma \nabla \eta^{n+1} \cdot \nabla \psi dz$$

$$= \int_\Gamma \frac{W((n+1)\Delta t) - W(n\Delta t)}{\Delta t} \psi dz + P_{in}^n \int_0^R (q_z)|_{z=0} dr - P_{out}^n \int_0^R (q_z)|_{z=L} dr,$$

for all $(\boldsymbol{q}, \psi) \in \mathcal{Q}$,

$$\int_\Gamma \frac{\eta^{n+1} - \eta^n}{\Delta t} \phi dz = \int_\Gamma v^{n+\frac{1}{3}} \phi dz, \qquad \text{for all } \phi \in L^2(\Gamma),$$

where

$$P_{in/out}^n = \frac{1}{\Delta t} \int_{n\Delta t}^{(n+1)\Delta t} P_{in/out}(t) dt.$$

structure subproblem **stochastic subproblem**

Semidiscrete problem

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$$\int_{\Omega_f} \frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^n}{\Delta t} \cdot \boldsymbol{q} d\boldsymbol{x} + 2\mu \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}^{n+1}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x} + \int_\Gamma \frac{v^{n+1} - v^n}{\Delta t} \psi dz + \int_\Gamma \nabla \eta^{n+1} \cdot \nabla \psi dz$$

$$= \int_\Gamma \frac{W((n+1)\Delta t) - W(n\Delta t)}{\Delta t} \psi dz + P_{in}^n \int_0^R (q_z)|_{z=0} dr - P_{out}^n \int_0^R (q_z)|_{z=L} dr,$$

for all $(\boldsymbol{q}, \psi) \in \mathcal{Q}$,

$$\int_\Gamma \frac{\eta^{n+1} - \eta^n}{\Delta t} \phi dz = \int_\Gamma v^{n+\frac{1}{3}} \phi dz, \qquad \text{for all } \phi \in L^2(\Gamma),$$

where

$$P_{in/out}^n = \frac{1}{\Delta t} \int_{n\Delta t}^{(n+1)\Delta t} P_{in/out}(t) dt.$$

structure subproblem

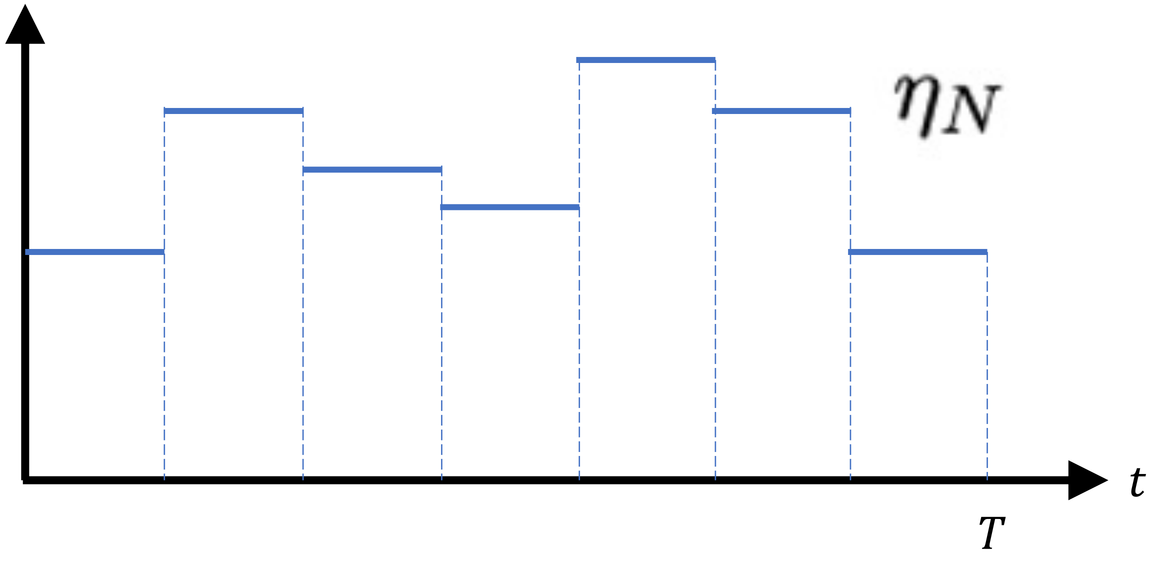
stochastic subproblem

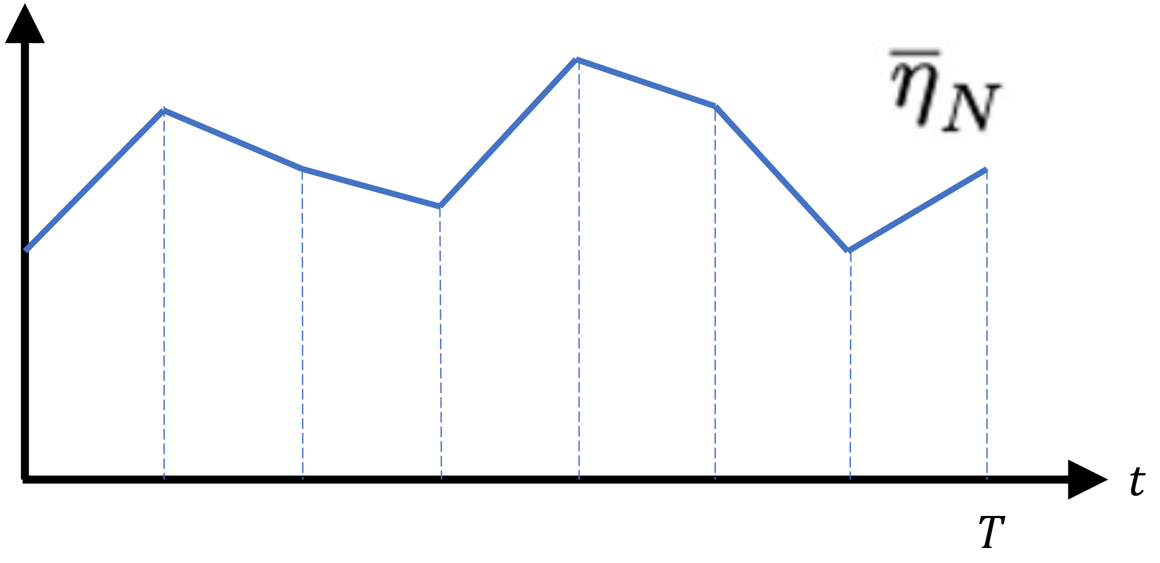
fluid subproblem

Approximate solutions

$$\boldsymbol{u}_N(t,\cdot) = \boldsymbol{u}_N^{n-1},$$

$$\eta_N(t,\cdot) = \eta_N^{n-1}, \qquad v_N(t,\cdot) = v_N^{n-1}, \qquad v_N^*(t,\cdot) = v_N^{n-\frac{2}{3}}, \qquad \text{for } t \in ((n-1)\Delta t, n\Delta t]$$





$$\overline{\eta}_N(n\Delta t) = \eta_N^n, \qquad \overline{\boldsymbol{u}}_N(n\Delta t) = \boldsymbol{u}_N^n, \qquad \overline{v}_N(n\Delta t) = v_N^n, \qquad \text{for } n = 0, 1, \ldots, N.$$

Note that: $\partial_t \overline{\eta}_N = v_N^*.$

Discrete energy identities

$$E_N^{n+\frac{i}{3}} = \frac{1}{2}\left(\int_{\Omega_f} |\boldsymbol{u}_N^{n+\frac{i}{3}}|^2 d\boldsymbol{x} + ||v_N^{n+\frac{i}{3}}||^2_{L^2(\Gamma)} + ||\nabla \eta_N^{n+\frac{i}{3}}||^2_{L^2(\Gamma)}\right)$$

kinetic and potential energy

$$D_N^{n+1} = (\Delta t)\mu \int_{\Omega_f} |\boldsymbol{D}(\boldsymbol{u}_N^{n+1})|^2 d\boldsymbol{x}.$$

fluid dissipation

Discrete energy identities

$$E_N^{n+\frac{i}{3}} = \frac{1}{2}\left(\int_{\Omega_f} |\boldsymbol{u}_N^{n+\frac{i}{3}}|^2 d\boldsymbol{x} + ||v_N^{n+\frac{i}{3}}||^2_{L^2(\Gamma)} + ||\nabla \eta_N^{n+\frac{i}{3}}||^2_{L^2(\Gamma)}\right)$$

kinetic and potential energy

$$D_N^{n+1} = (\Delta t)\mu \int_{\Omega_f} |\boldsymbol{D}(\boldsymbol{u}_N^{n+1})|^2 d\boldsymbol{x}.$$

fluid dissipation

numerical dissipation

stochastic contribution

$$E_N^{n+\frac{1}{3}} + \frac{1}{2}\left(||v_N^{n+\frac{1}{3}} - v_N^n||^2_{L^2(\Gamma)}\right) + \frac{1}{2}\left(||\nabla \eta_N^{n+\frac{1}{3}} - \nabla \eta_N^n||^2_{L^2(\Gamma)}\right) = E_N^n,$$

$$E_N^{n+\frac{2}{3}} = E_N^{n+\frac{1}{3}} + [W((n+1)\Delta t) - W(n\Delta t)] \int_\Gamma v_N^{n+\frac{1}{3}} dz + \frac{L}{2}[W((n+1)\Delta t) - W(n\Delta t)]^2,$$

$$E_N^{n+1} + 2\mu(\Delta t) \int_{\Omega_f} |\boldsymbol{D}(\boldsymbol{u}_N^{n+1})|^2 d\boldsymbol{x} + \frac{1}{2}\left(||\boldsymbol{u}_N^{n+1} - \boldsymbol{u}_N^{n+\frac{2}{3}}||^2_{L^2(\Omega_f)}\right) + \frac{1}{2}\left(||v_N^{n+1} - v_N^{n+\frac{2}{3}}||^2_{L^2(\Gamma)}\right)$$

$$= E_N^{n+\frac{2}{3}} + (\Delta t)\left(P_{in}^n \int_0^R (\boldsymbol{u}_N^{n+1})_z|_{z=0} dr - P_{out}^n \int_0^R (\boldsymbol{u}_N^{n+1})_z|_{z=L} dr\right).$$

Uniform energy estimate

1. ***Uniform semidiscrete kinetic energy and elastic energy estimate:***

$$\mathbb{E}\left(\max_{n=0,1,\dots,N-1} E_N^{n+\frac{1}{3}}\right) \le C, \quad \mathbb{E}\left(\max_{n=0,1,\dots,N-1} E_N^{n+\frac{2}{3}}\right) \le C, \text{ and } \quad \mathbb{E}\left(\max_{n=0,1,\dots,N-1} E_N^{n+1}\right) \le C.$$

2. ***Uniform semidiscrete viscous fluid dissipation estimate:***

$$\sum_{j=1}^{N} \mathbb{E}(D_N^j) \le C.$$

3. ***Uniform numerical dissipation estimate:***

$$\sum_{n=0}^{N-1}\left(\mathbb{E}\left(||v_N^{n+\frac{1}{3}} - v_N^n||^2_{L^2(\Gamma)}\right) + \mathbb{E}\left(||\nabla\eta_N^{n+\frac{1}{3}} - \nabla\eta_N^n||^2_{L^2(\Gamma)}\right)\right) \le C.$$

$$\sum_{n=0}^{N-1} \mathbb{E}\left(||v^{n+\frac{2}{3}} - v^{n+\frac{1}{3}}||^2_{L^2(\Gamma)}\right) \le C.$$

$$\sum_{n=0}^{N-1}\left(\mathbb{E}\left(||\boldsymbol{u}_N^{n+1} - \boldsymbol{u}_N^{n+\frac{2}{3}}||^2_{L^2(\Omega_f)}\right) + \mathbb{E}\left(||v_N^{n+1} - v_N^{n+\frac{2}{3}}||^2_{L^2(\Gamma)}\right)\right) \le C.$$

Uniform boundedness

Proposition 5.2. We have the following uniform boundedness results.

- $(\eta_N)_{N\in\mathbb{N}}$ is uniformly bounded in $L^2(\Omega; L^\infty(0,T; H_0^1(\Gamma)))$.
- $(v_N)_{N\in\mathbb{N}}$ is uniformly bounded in $L^2(\Omega; L^\infty(0,T; L^2(\Gamma)))$ and $L^2(\Omega; L^2(0,T; H^{1/2}(\Gamma)))$.
- $(v_N^*)_{N\in\mathbb{N}}$ is uniformly bounded in $L^2(\Omega; L^\infty(0,T; L^2(\Gamma)))$.
- $(\boldsymbol{u}_N)_{N\in\mathbb{N}}$ is uniformly bounded in $L^2(\Omega; L^\infty(0,T; L^2(\Omega_f)))$ and $L^2(\Omega; L^2(0,T; H^1(\Omega_f)))$.

Proposition 5.3. The sequence of linear interpolations of the structure displacements, $(\overline{\eta}_N)_{N\in\mathbb{N}}$, is uniformly bounded in $L^2(\Omega; L^\infty(0,T; H_0^1(\Gamma))) \cap L^2(\Omega; W^{1,\infty}(0,T; L^2(\Gamma)))$.

Passing to the limit

For all $(\boldsymbol{q}, \psi) \in \mathcal{Q}(0,T)$,

$$\int_0^T \int_{\Omega_f} \partial_t \overline{\boldsymbol{u}}_N \cdot \boldsymbol{q} d\boldsymbol{x} + 2\mu \int_0^T \int_{\Omega_f} \boldsymbol{D}(\tau_{\Delta t} \boldsymbol{u}_N) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x} dt + \int_0^T \int_\Gamma \partial_t \overline{v}_N \psi dz dt$$

$$+ \int_0^T \int_\Gamma \nabla(\tau_{\Delta t} \eta_N) \cdot \nabla \psi dz dt = \sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \int_\Gamma \frac{W((n+1)\Delta t) - W(n\Delta t)}{\Delta t} \psi dz dt$$

$$+ \sum_{n=0}^{N-1} \left(\int_{n\Delta t}^{(n+1)\Delta t} P_{in}^n \int_0^R (q_z)|_{z=0} dr - \int_{n\Delta t}^{(n+1)\Delta t} P_{out}^n \int_0^R (q_z)|_{z=L} dr dt \right)$$

Passing to the limit

For all $(\boldsymbol{q},\psi) \in \mathcal{Q}(0,T)$,

$$\int_0^T \int_{\Omega_f} \partial_t \overline{\boldsymbol{u}}_N \cdot \boldsymbol{q} d\boldsymbol{x} + \boxed{2\mu \int_0^T \int_{\Omega_f} \boldsymbol{D}(\tau_{\Delta t}\boldsymbol{u}_N) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x} dt} + \int_0^T \int_\Gamma \partial_t \overline{v}_N \psi dz dt$$
$$+ \int_0^T \int_\Gamma \nabla(\tau_{\Delta t}\eta_N) \cdot \nabla\psi dz dt = \sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \int_\Gamma \frac{W((n+1)\Delta t) - W(n\Delta t)}{\Delta t} \psi dz dt$$
$$+ \sum_{n=0}^{N-1} \left(\int_{n\Delta t}^{(n+1)\Delta t} P_{in}^n \int_0^R (q_z)|_{z=0} dr - \int_{n\Delta t}^{(n+1)\Delta t} P_{out}^n \int_0^R (q_z)|_{z=L} dr dt \right)$$

$\Big\downarrow$???

Only know that $\tau_{\Delta t}\boldsymbol{u}_N$ converges to $\boldsymbol{u}$ weakly in $L^2\left(\Omega; L^2\left(0,T; H^1(\Omega_f)\right)\right)$

****Need stronger form of convergence**

$$-\int_0^T \int_{\Omega_f} \boldsymbol{u} \cdot \partial_t \boldsymbol{q} d\boldsymbol{x} dt + \boxed{2\mu \int_0^T \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x} dt} - \int_0^T \int_\Gamma \partial_t \eta \partial_t \psi dz dt + \int_0^T \int_\Gamma \nabla\eta \cdot \nabla\psi dz dt$$
$$= \int_0^T P_{in}(t) \left(\int_{\Gamma_{in}} q_z dr \right) dt - \int_0^T P_{out}(t) \left(\int_{\Gamma_{out}} q_z dr \right) dt$$
$$+ \int_{\Omega_f} \boldsymbol{u}_0 \cdot \boldsymbol{q}(0) d\boldsymbol{x} + \int_\Gamma v_0 \psi(0) dz + \int_0^T \left(\int_\Gamma \psi dz \right) dW.$$

General outline

- Use **compactness arguments** to obtain tightness of measures corresponding to approximate solutions.
- Use tightness to obtain **weak convergence of probability measures**.
- Use **Skorokhod representation theorem** to get almost sure convergence on a different probability space.
- Use **Gyöngy-Krylov theorem argument** to obtain almost sure convergence on original probability space.

Probability measures on phase space

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ One dimensional Brownian motion $\{W_t\}_{t\geq 0}$ with respect to $(\mathcal{F}_t)_{t\geq 0}$

$$\boxed{\mathcal{X} = [L^2(0,T;L^2(\Gamma))]^2 \times [L^2(0,T;L^2(\Omega_f)) \times L^2(0,T;L^2(\Gamma))]^3 \times C(0,T;\mathbb{R}).}$$

$$\boxed{\mu_N = \mu_{\eta_N} \times \mu_{\bar{\eta}_N} \times \mu_{\boldsymbol{u}_N} \times \mu_{v_N} \times \mu_{\boldsymbol{u}_N} \times \mu_{v_N^*} \times \mu_{\bar{\boldsymbol{u}}_N} \times \mu_{\bar{v}_N} \times \mu_W .}$$

$B \subset L^2(0,T;L^2(\Gamma))$, Borel measurable $\qquad \mu_{\eta_N}(B) = \mathbb{P}(\eta_N \in B)$

Probability measures on phase space

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ One dimensional Brownian motion $\{W_t\}_{t\geq 0}$ with respect to $(\mathcal{F}_t)_{t\geq 0}$

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$$\boxed{\mu_N = \mu_{\eta_N} \times \mu_{\bar{\eta}_N} \times \mu_{\boldsymbol{u}_N} \times \mu_{v_N} \times \mu_{\boldsymbol{u}_N} \times \mu_{v_N^*} \times \mu_{\bar{\boldsymbol{u}}_N} \times \mu_{\bar{v}_N} \times \mu_W .}$$

$B \subset L^2(0,T;L^2(\Gamma))$, Borel measurable $\qquad \mu_{\eta_N}(B) = \mathbb{P}(\eta_N \in B)$

GOAL: Show the measures μ_N are **tight** as probability measures on $\mathcal{X}$

Compactness arguments

Probability measures $\{\mu_n\}_{n\geq 0}$ on a Banach space B are **tight** if for every $\epsilon > 0$, there exists a compact set $K \subset B$ such that $\mu_n(K) \geq 1 - \epsilon$, for all n.

Compactness arguments

Probability measures $\{\mu_n\}_{n\geq 0}$ on a Banach space B are **tight** if for every $\epsilon > 0$, there exists a compact set $K \subset B$ such that $\mu_n(K) \geq 1 - \epsilon$, for all n.

Why do we need compactness arguments?

For **real-valued** random variables $\{X_n\}_{n\geq 0}$, their laws are tight if $\mathbb{E}(|X_n|^2) \leq C$ uniformly in n by Chebychev's inequality and the fact that **any closed ball in $\mathbb{R}$ is compact**.

Compactness arguments

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Why do we need compactness arguments?

For **real-valued** random variables $\{X_n\}_{n\geq 0}$, their laws are tight if $\mathbb{E}(|X_n|^2) \leq C$ uniformly in n by Chebychev's inequality and the fact that **any closed ball in $\mathbb{R}$ is compact**.

This is no longer true for general Banach spaces!

For example, $\mathbb{E}(\|X_n\|_B^2) \leq C$ does NOT guarantee tightness because $\{\boldsymbol{f} \in \boldsymbol{B}; \|\boldsymbol{f}\|_{\boldsymbol{B}} \leq \boldsymbol{R}\}$ is not compact in $\boldsymbol{B}$.

Need to embed Banach space into weaker space via compact embedding.

Compactness: structure displacement

$$\mathcal{X} = [L^2(0,T;L^2(\Gamma))]^2 \times [L^2(0,T;L^2(\Omega_f)) \times L^2(0,T;L^2(\Gamma))]^3 \times C(0,T;\mathbb{R}).$$

$$\mu_N = \mu_{\eta_N} \times \mu_{\overline{\eta}_N} \times \mu_{\boldsymbol{u}_N} \times \mu_{v_N} \times \mu_{\boldsymbol{u}_N} \times \mu_{v_N^*} \times \mu_{\overline{\boldsymbol{u}}_N} \times \mu_{\overline{v}_N} \times \mu_W.$$

Proposition 5.3. The sequence of linear interpolations of the structure displacements, $(\overline{\eta}_N)_{N\in\mathbb{N}}$, is uniformly bounded in $L^2(\Omega; L^\infty(0,T;H_0^1(\Gamma))) \cap L^2(\Omega; W^{1,\infty}(0,T;L^2(\Gamma)))$.

By **Aubin-Lions compactness lemma**:

Lemma 6.1. We have the following compact embedding.

$$[W^{1,\infty}(0,T;L^2(\Gamma)) \cap L^\infty(0,T;H_0^1(\Gamma))] \subset\subset L^\infty(0,T;L^2(\Gamma)),$$

Compactness: fluid and structure velocity

$$\mathcal{K} = \{(\boldsymbol{u}, v) \in L^2(0,T;L^2(\Omega_f)) \times L^2(0,T;L^2(\Gamma)) :$$
$$\boldsymbol{u} = \boldsymbol{u}_N(\omega) \text{ and } v = v_N(\omega) \text{ for some } \omega \in \Omega \text{ and } N \in \mathbb{N}\}.$$

For $R > 0$, let $\mathcal{K}_R$ be the paths $\left(\boldsymbol{u}_N(\omega), v_N(\omega)\right)$ for which ω and N satisfy:

$$||(\boldsymbol{u}_N, v_N)||_{L^2(0,T;H^1(\Omega_f))\times L^2(0,T;H^{1/2}(\Gamma))} \leq R,$$

$$||\eta_N||_{L^\infty(0,T;H_0^1(\Gamma))} \leq R.$$

$$(\Delta t)\sum_{n=1}^{N}\int_{\Omega_f} |\boldsymbol{D}(\boldsymbol{u}_N^n)|^2 d\boldsymbol{x} \leq R.$$

$$\sum_{n=0}^{N-1} ||\boldsymbol{u}_N^{n+1} - \boldsymbol{u}_N^{n+\frac{2}{3}}||^2_{L^2(\Omega_f)} \leq R, \qquad \sum_{n=0}^{N-1} ||v_N^{n+\frac{1}{3}} - v_N^{n}||^2_{L^2(\Gamma)} \leq R,$$

$$\sum_{n=0}^{N-1} ||v_N^{n+\frac{2}{3}} - v_N^{n+\frac{1}{3}}||^2_{L^2(\Gamma)} \leq R, \qquad \sum_{n=0}^{N-1} ||v_N^{n+1} - v_N^{n+\frac{2}{3}}||^2_{L^2(\Gamma)} \leq R.$$

$$\sup_{s,t\in[0,T], s\neq t} \frac{|W(t) - W(s)|}{|t-s|^{1/4}} \leq R.$$

Lemma 6.2. For any arbitrary positive constant R, the set $\mathcal{K}_R$ is precompact in $L^2(0,T;L^2(\Omega_f)) \times L^2(0,T;L^2(\Gamma))$.

Tightness result

$$\mathcal{X} = [L^2(0,T;L^2(\Gamma))]^2 \times [L^2(0,T;L^2(\Omega_f)) \times L^2(0,T;L^2(\Gamma))]^3 \times C(0,T;\mathbb{R}).$$

$$\mu_N = \mu_{\eta_N} \times \mu_{\overline{\eta}_N} \times \mu_{\boldsymbol{u}_N} \times \mu_{v_N} \times \mu_{\boldsymbol{u}_N} \times \mu_{v_N^*} \times \mu_{\overline{\boldsymbol{u}}_N} \times \mu_{\overline{v}_N} \times \mu_W.$$

Conclusion: The probability measures μ_N defined on $\mathcal{X}$ are tight.

Proposition 6.1. Along a subsequence (which we will continue to denote by N), μ_N converges weakly as probability measures to a probability measure μ on $\mathcal{X}$.

Skorokhod representation theorem

weak convergence of probability measures → almost sure convergence of random variables on another probability space with equivalence of laws

Skorokhod representation theorem

weak convergence of probability measures $\Rightarrow$ almost sure convergence of random variables on another probability space with equivalence of laws

Suppose that the probability measures $\{\mu_n\}_{n\geq 0}$ on a separable Banach space B converge weakly to a probability measure μ. Then, there exists $\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}\right)$ and B-valued random variables X_n and X on this probability space such that $X_n \to X$ almost surely, with laws given by μ_n and μ, respectively.

Skorokhod representation theorem

Suppose that the probability measures $\{\mu_n\}_{n\geq 0}$ on a separable Banach space B converge weakly to a probability measure μ. Then, there exists $(\widetilde{\Omega}, \tilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ and B-valued random variables X_n and X on this probability space such that $X_n \to X$ almost surely, with laws given by μ_n and μ, respectively.

Lemma 6.7. There exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and $\mathcal{X}$-valued random variables

$$(\tilde{\eta}_N, \tilde{\bar{\eta}}_N, \tilde{\boldsymbol{u}}_N, \tilde{v}_N, \tilde{\boldsymbol{u}}^*_N, \tilde{v}^*_N, \tilde{\bar{\boldsymbol{u}}}_N, \tilde{\bar{v}}_N, \tilde{W}_N), \text{ for each } N,$$

$$(\tilde{\eta}, \tilde{\bar{\eta}}, \tilde{\boldsymbol{u}}, \tilde{v}, \tilde{\boldsymbol{u}^*}, \tilde{v^*}, \tilde{\bar{u}}, \tilde{\bar{v}}, \tilde{W}),$$

such that

$$(\tilde{\eta}_N, \tilde{\bar{\eta}}_N, \tilde{\boldsymbol{u}}_N, \tilde{v}_N, \tilde{\boldsymbol{u}}^*_N, \tilde{v}^*_N, \tilde{\bar{\boldsymbol{u}}}_N, \tilde{\bar{v}}_N, \tilde{W}_N) =_d (\eta_N, \bar{\eta}_N, \boldsymbol{u}_N, v_N, \boldsymbol{u}_N, v^*_N, \bar{\boldsymbol{u}}_N, \bar{v}_N, W),$$

for all N, and

$$(\tilde{\eta}_N, \tilde{\bar{\eta}}_N, \tilde{\boldsymbol{u}}_N, \tilde{v}_N, \tilde{\boldsymbol{u}}^*_N, \tilde{v}^*_N, \tilde{\bar{\boldsymbol{u}}}_N, \tilde{\bar{v}}_N, \tilde{W}_N) \to (\tilde{\eta}, \tilde{\bar{\eta}}, \tilde{\boldsymbol{u}}, \tilde{v}, \tilde{\boldsymbol{u}^*}, \tilde{v^*}, \tilde{\bar{u}}, \tilde{\bar{v}}, \tilde{W}), \qquad \text{a.s. in } \mathcal{X}, \text{ as } N \to \infty.$$

Weak solution

For all $(\boldsymbol{q}, \psi) \in \mathcal{Q}(0,T)$,

$$\int_0^T \int_{\Omega_f} \partial_t \tilde{\tilde{\boldsymbol{u}}}_N \cdot \boldsymbol{q} d\boldsymbol{x} + 2\mu \int_0^T \int_{\Omega_f} \boldsymbol{D}(\tau_{\Delta t} \tilde{\boldsymbol{u}}_N) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x} dt + \int_0^T \int_\Gamma \partial_t \tilde{\tilde{v}}_N \psi dz dt$$
$$+ \int_0^T \int_\Gamma \nabla(\tau_{\Delta t} \tilde{\eta}_N) \cdot \nabla \psi dz dt = \sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \int_\Gamma \frac{\tilde{W}_N((n+1)\Delta t) - \tilde{W}_N(n\Delta t)}{\Delta t} \psi dz dt$$
$$+ \sum_{n=0}^{N-1} \left(\int_{n\Delta t}^{(n+1)\Delta t} P_{in}^n \int_0^R (q_z)|_{z=0} dr - \int_{n\Delta t}^{(n+1)\Delta t} P_{out}^n \int_0^R (q_z)|_{z=L} dr dt \right)$$

For all $(\boldsymbol{q}, \psi) \in \mathcal{Q}(0,T)$,

$$-\int_0^T \int_{\Omega_f} \tilde{\boldsymbol{u}} \cdot \partial_t \boldsymbol{q} d\boldsymbol{x} dt + 2\mu \int_0^T \int_{\Omega_f} \boldsymbol{D}(\tilde{\boldsymbol{u}}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x} dt - \int_0^T \int_\Gamma \partial_t \tilde{\eta} \partial_t \psi dz dt + \int_0^T \int_\Gamma \nabla \tilde{\eta} \cdot \nabla \psi dz dt$$
$$= \int_0^T P_{in}(t) \left(\int_{\Gamma_{in}} q_z dr \right) dt - \int_0^T P_{out}(t) \left(\int_{\Gamma_{out}} q_z dr \right) dt$$
$$+ \int_{\Omega_f} \boldsymbol{u_0} \cdot \boldsymbol{q}(0) d\boldsymbol{x} + \int_\Gamma v_0 \psi(0) dz + \int_0^T \left(\int_\Gamma \psi dz \right) d\tilde{W},$$

Towards a strong solution

- We want to bring the solution back to the original probability space.
- Use a **Gyöngy-Krylov argument** along with a uniqueness result.

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Lemma 7.2 (Gyöngy-Krylov lemma). Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in a Banach space B. For positive integers m and n, define the joint probability measures $\nu_{m,n}$ on $B \times B$ by

$$\nu_{m,n}(A_1 \times A_2) = \mathbb{P}(X_m \in A_1, X_n \in A_2).$$

Suppose that the following *diagonal condition* holds: for any arbitrary subsequences $\{m_k\}_{k=1}^{\infty}$ and $\{n_k\}_{k=1}^{\infty}$, there exists a further subsequence such that the joint probability laws $\nu_{m_{k_l}, n_{k_l}}$ along this subsequence as $l \to \infty$ converge weakly to a limiting probability measure ν such that

$$\nu(\Delta) = 1,$$

where $\Delta = \{(x, x) : x \in B\}$ denotes the diagonal of $B \times B$. Then, X_n converges in probability to some B-valued random variable X as $n \to \infty$.

converges almost surely ***in the original topology*** along a subsequence

Gyöngy-Krylov argument

$$\mathcal{X} = [L^2(0,T;L^2(\Gamma))]^2 \times [L^2(0,T;L^2(\Omega_f)) \times L^2(0,T;L^2(\Gamma))]^3 \times C(0,T;\mathbb{R}).$$

$$\mu_N = \mu_{\eta_N} \times \mu_{\overline{\eta}_N} \times \mu_{\boldsymbol{u}_N} \times \mu_{v_N} \times \mu_{\boldsymbol{u}_N} \times \mu_{v_N^*} \times \mu_{\overline{\boldsymbol{u}}_N} \times \mu_{\overline{v}_N} \times \mu_W.$$

Define probability measures on $\mathcal{X}\times\mathcal{X}$: $\nu_{M,N} = \mu_M \times \mu_N$

tight and hence converge weakly along any subsequence to ν

We get *almost sure convergence* along a subsequence *in the original topology* on original probability space once we verify the **diagonal condition**:

$$\nu(\{(x,x) : x \in \mathcal{X}\}) = 1.$$

Gyöngy-Krylov argument

$$\mathcal{X} = [L^2(0,T;L^2(\Gamma))]^2 \times [L^2(0,T;L^2(\Omega_f)) \times L^2(0,T;L^2(\Gamma))]^3 \times C(0,T;\mathbb{R}).$$

$$\mu_N = \mu_{\eta_N} \times \mu_{\bar{\eta}_N} \times \mu_{\boldsymbol{u}_N} \times \mu_{v_N} \times \mu_{\boldsymbol{u}_N} \times \mu_{v_N^*} \times \mu_{\bar{\boldsymbol{u}}_N} \times \mu_{\bar{v}_N} \times \mu_W.$$

Given $\nu_{M,N} = \mu_M \times \mu_N$, verify $\nu(\{(x,x) : x \in \mathcal{X}\}) = 1.$

Gyöngy-Krylov argument

$$\mathcal{X} = [L^2(0,T;L^2(\Gamma))]^2 \times [L^2(0,T;L^2(\Omega_f)) \times L^2(0,T;L^2(\Gamma))]^3 \times C(0,T;\mathbb{R}).$$

$$\mu_N = \mu_{\eta_N} \times \mu_{\overline{\eta}_N} \times \mu_{\boldsymbol{u}_N} \times \mu_{v_N} \times \mu_{\boldsymbol{u}_N} \times \mu_{v_N^*} \times \mu_{\overline{\boldsymbol{u}}_N} \times \mu_{\overline{v}_N} \times \mu_W.$$

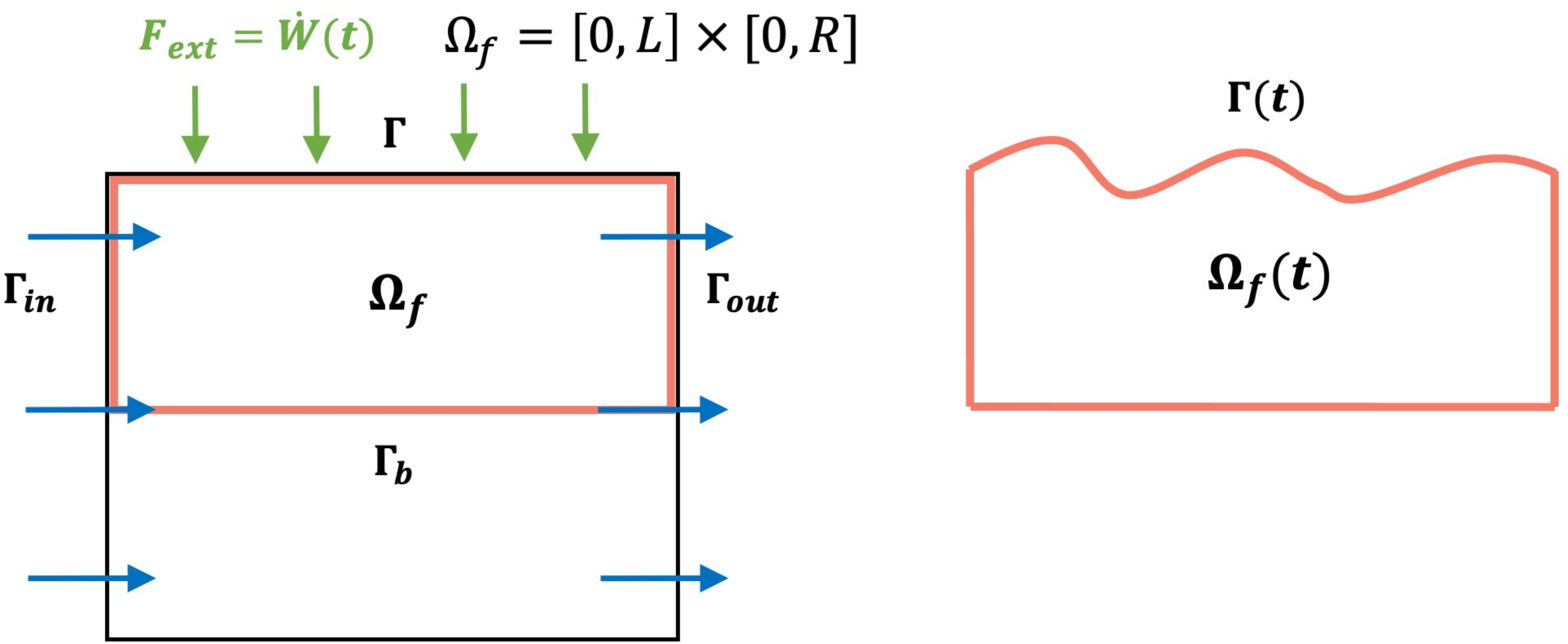
Given $\nu_{M,N} = \mu_M \times \mu_N$, verify $\nu(\{(x,x) : x \in \mathcal{X}\}) = 1.$

Use **Skorokhod representation theorem** to get $(X_{m_k}, X_{n_k}) \to (X_1, X_2)$ almost surely on a different probability space $(\widetilde{\Omega}, \tilde{\mathcal{F}}, \widetilde{\mathbb{P}})$, where limit has law given by weak limit ν.

Both X_1 and X_2 satisfy the linear stochastic problem. **Diagonal condition** follows from uniqueness in law of weak solution to linear stochastic problem since this implies that the laws of X_1 and X_2 are the same.

Main theorem

Theorem 2.1. Let $\boldsymbol{u}_0 \in L^2(\Omega_f)$, $v_0 \in L^2(\Gamma)$, and $\eta_0 \in H^1_0(\Gamma)$. Let $P_{in/out} \in L^2_{loc}(0,\infty)$ and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a Brownian motion W with respect to a given complete filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Then, for any $T > 0$, there exists a unique weak solution in a probabilistically strong sense in the sense of Definition 2.2 to the given stochastic fluid-structure interaction problem.



Significance of results

- Despite the **very rough Brownian forcing**, the stochastic fluid-structure interaction system still supports a solution
- The FSI model of time-dependent Stokes with the wave equation is **robust** under stochastic perturbations
- Provides a method for **construction of solutions** in stochastic PDEs that works well with fully coupled stochastic problems
- Can be used as a basis for a **numerical scheme** for stochastic FSI
- Extension to moving boundary FSI problems with **nonlinear coupling**