Section 3.1 12, 13, 19, 20, 21, 37, 40

12) $\begin{vmatrix} 3 & 0 & 0 & 0 \\ 7 & -2 & 0 & 0 \\ 2 & 6 & 3 & 0 \\ 3 & -8 & 4 & -3 \end{vmatrix} = 3(-2)(3)(-3) = \boxed{54}$ since lower triangular

13) $\begin{vmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{vmatrix} = -2 \begin{vmatrix} 4 & 0 & 3 & -5 \\ 7 & 3 & 4 & -8 \\ 5 & 0 & 2 & -3 \\ 0 & 0 & -1 & 2 \end{vmatrix}$ $= 2$

$= -2(3) \begin{vmatrix} 4 & 3 & -5 \\ 5 & 2 & -3 \\ 0 & -1 & 2 \end{vmatrix}$

$= -2(3)\left(4 \begin{vmatrix} 2 & -3 \\ -1 & 2 \end{vmatrix} - 5 \begin{vmatrix} 3 & -5 \\ -1 & 2 \end{vmatrix}\right)$

$= -6(4(1) - 5(1)) = \boxed{6}$

19) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \quad \begin{vmatrix} c & d \\ a & b \end{vmatrix} = bc - ad$

multiply by -1

20) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \quad \begin{vmatrix} a+kc & b+kd \\ c & d \end{vmatrix} = ad - bc$

no effect

21) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \quad \begin{vmatrix} a & b \\ kc & kd \end{vmatrix} = k(ad - bc)$

multiply by k

37) $5A = \begin{bmatrix} 15 & 5 \\ 20 & 10 \end{bmatrix}$, no $\det(5A) = 25 \det(A)$.

40) a. True b. False (it is the product of the diagonal entrie

Section 3.2 9, 20, 28, 39, 40

9) $\begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ -1 & 0 & 5 & 3 \\ 3 & -3 & -2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & -1 & 2 & 3 \\ 0 & 0 & 7 & 3 \end{vmatrix} \begin{matrix} \\ \\ R_1+R_3 \\ R_4-3R_1 \end{matrix}$

$= \begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & 7 & 7 \\ 0 & 0 & 7 & 3 \end{vmatrix} R_2+R_3 = \begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & 7 & 7 \\ 0 & 0 & 0 & -4 \end{vmatrix} \begin{matrix} \\ \\ \\ R_4-R_3 \end{matrix}$

$= 1 \cdot 1 \cdot 7 \cdot (-4) = \boxed{-28}$

20) $\begin{vmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{vmatrix} = 2 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$

$\leftarrow$ R_2+R_1

$= 2 \cdot 7 = \boxed{14}$

28) (a) False. $(-1)^3 = -1$ factor.

(b) False.

(c) False (can be linearly dependent)

(d) False $\det A^{-1} = \frac{1}{\det A}$.

39) (a) $\det AB = (\det A)(\det B) = -3 \cdot 4 = -12$

(b) $\det 5A = 5^3 \det A = -375$

(c) $\det B^T = \det B = 4$

(d) $\det A^{-1} = \frac{1}{\det A} = \frac{-1}{3}$ (e) $\det A^3 = (\det A)^3 = -27$

40) (a) $\det AB = (\det A)(\det B) = 3$

(b) $\det B^5 = (\det B)^5 = -1$ (c) $\det 2A = 2^4 \det A = -48$

(d) $\det A^T BA = \det(A^T) \det B \det A = (-3) \cdot (-1) \cdot (-3)$

$= \boxed{-9}$

(e) $\det B^{-1}AB = \det(B^{-1}) \det(A) \det B = \frac{1}{-1} \cdot (-3) \cdot (-1)$

$= \boxed{-3}$

Section 1.4

22) Yes since $A = \begin{bmatrix} 0 & 0 & 4 \\ 0 & -3 & -1 \\ -2 & 8 & -5 \end{bmatrix}$ is invertible

(it clearly has a pivot in each row).

Section 1.5

37) $\begin{cases} x - 4y = 0 \\ 2x - 8y = 0 \end{cases}$

Let $\vec{b} = (0,1)$. Then $\begin{matrix} x - 4y = 0 \\ 2x - 8y = 1 \end{matrix}$

is inconsistent. This does not contradict Theorem 6 since Theorem 6 requires a particular solution to exist to work.

Section 1.7

12) $\begin{bmatrix} 2 & -6 & 8 \\ -4 & 7 & h \\ 1 & -3 & 4 \end{bmatrix}$ When does $Ax = 0$ have nontrivial solution?

$\downarrow$

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -5 & h+16 \\ 1 & -3 & 4 \end{bmatrix} \begin{matrix} R_1 - 2R_3 \\ R_2 + 4R_3 \\ \end{matrix} \rightarrow \begin{bmatrix} 1 & -3 & 4 \\ 0 & 1 & -\frac{1}{5}h - \frac{16}{5} \\ 0 & 0 & 0 \end{bmatrix}$

$\boxed{\text{all } h}$ x_3 always free

Section 2.1

17) $\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} = \begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix}$

$\left.\begin{array}{l} x_1 - 2x_2 = -1 \\ -2x_1 + 5x_2 = 6 \end{array}\right\} \rightarrow x_2 = 4, x_1 = 7$

$\left.\begin{array}{l} y_1 - 2y_2 = 2 \\ -2y_1 + 5y_2 = -9 \end{array}\right\} \rightarrow y_2 = -5 \quad y_1 = -8$

$\begin{bmatrix} 7 \\ 4 \end{bmatrix}, \begin{bmatrix} -8 \\ -5 \end{bmatrix}$

22) If columns of B are linearly dependent, $Bx = 0$ has a nontrivial solution x_0, so $Bx_0 = 0$. Then $(AB)x_0 = 0$ too. So $ABx = 0$ has a nontrivial solution $x = x_0$. So the columns of AB are linearly dependent.

Section 2.2

18) $A = PBP^{-1}$

$P^{-1}A = P^{-1}PBP^{-1} = BP^{-1}$

$P^{-1}AP = BP^{-1}P \Rightarrow \boxed{B = P^{-1}AP}$

Additional Problems

1) (a) True, always has trivial solution.

(b) False, $\det(I_2 + I_2) = \det\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right) = 4$

$\neq \det(I_2) + \det(I_2) = 1 + 1 = 2$

(c) True

(d) True, $\det(P^2AP^{10}) = (\det P)^2 \det A (\det P)^{10}$
$= \det A.$

(e) False, $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

(f) True. $(\det A)^3 = 1$, so $\det A = 1$ (since $\det A$ must be real).

2) The last row is $-R_1 - R_2 - \ldots - R_{n-1}$.
So if we do the row operations
$R_n + R_1, R_n + R_2, \ldots, R_n + R_{n-1}$
to the last row, we get a row of zeros.
So the determinant of the coefficient matrix is 0.
Thus, the system has a unique solution.

3) (a) Taking det along a row for A is the same as taking det along a column for A^t.
For the transpose, $\det A^t = \det A$ since the cofactor expansion is just reflected across the diagonal.

(b) If the columns of A are linearly independent, then $\det A \neq 0$. So $\det A^t = \det A \neq 0$.
So the columns of A^t are linearly independent and thus the rows of A are linearly independent.

4) (a) $(\det A)^n = 0 \Rightarrow \det A = 0$.

(b) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

5) $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$

$A + B = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$ $A - B = \begin{bmatrix} 2 & 0 \\ -1 & 0 \end{bmatrix}$

$(A+B)(A-B) = \begin{bmatrix} -2 & 0 \\ 2 & 0 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$B^2 = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

$A^2 - B^2 = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \neq (A+B)(A-B)$

$(A+B)(A-B) = A^2 + BA - AB + B^2$
so for $(A+B)(A-B) = A^2 - B^2$,
need A and B to commute.

6) The reduced row echelon form of A is I_m, and can be gotten from some row operations. Applying those row operations to the top m rows of the block matrix, we get $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \to \begin{bmatrix} I_m & * \\ 0 & B \end{bmatrix}$.

B can be sent to I_n by some row operations. Applying those row operations to the bottom n rows, $\begin{bmatrix} I_m & * \\ 0 & B \end{bmatrix} \to \begin{bmatrix} I_m & * \\ 0 & I_n \end{bmatrix}$. We can use row operations and I_n to clear out $*$: $\begin{bmatrix} I_m & * \\ 0 & I_n \end{bmatrix} \to \begin{bmatrix} I_m & 0 \\ 0 & I_n \end{bmatrix} = I_{m+n}$.
So M has I_{m+n} identity as reduced row echelon form so M is invertible.