Discussion 10/5/18 Solutions

Problem Set 1

1) $f(x) = \dfrac{\sin(2x)}{1+\cos(2x)}$

$$f'(x) = \frac{(1+\cos(2x))(2\cos(2x)) - \sin(2x)(-2\sin(2x))}{(1+\cos(2x))^2}$$

$$= \frac{2\cos(2x) + 2\cos^2(2x) + 2\sin^2(2x)}{(1+\cos(2x))^2}$$

$$= \frac{2\cos(2x)+2}{(1+\cos(2x))^2} = \frac{2}{1+\cos(2x)}$$

$f(x) = e^{-3x} + \tan(x)$

$f'(x) = -3e^{-3x} + \sec^2 x$

$f(x) = \sqrt{e^x + \sin(x)} + xe^{2x}$

$$f'(x) = \frac{1}{2\sqrt{e^x+\sin x}}(e^x + \cos x) + 2xe^{2x} + e^{2x}$$

2) (a) Chain rule: $f(x) = \sin^2 x = (\sin x)^2$

$f'(x) = 2\sin x \cos x$

Product rule: $f(x) = \sin x \cdot \sin x$

$f'(x) = \sin x \cos x + \cos x \sin x$

$= 2\sin x \cos x$

(b) Chain rule: $f(x) = \sin^2 x + \cos^2 x = (\sin x)^2 + (\cos x)^2$

$f'(x) = 2\sin x \cos x + 2\cos x(-\sin x) = 0$

Trig identity: $f(x) = \sin^2 x + \cos^2 x = 1$

$f'(x) = 0.$

(c) Quotient rule:

$$f(x) = \frac{1+x}{x^2}$$

$$f'(x) = \frac{x^2(1) - (1+x)(2x)}{(x^2)^2}$$

$$= \frac{x^2 - 2x - 2x^2}{x^4}$$

$$= \frac{-2x - x^2}{x^4} = -\frac{2}{x^3} - \frac{1}{x^2}$$

Power rule:

$$f(x) = \frac{1+x}{x^2} = \frac{1}{x^2} + \frac{1}{x}$$

$$f'(x) = -\frac{2}{x^3} - \frac{1}{x^2}$$

(d) Chain rule 1:

$$f(x) = e^{(2x)} \qquad e^u \text{ with } u = 2x$$

$$= 2e^{2x}$$

Chain rule 2:

$$f(x) = (e^x)^2 \qquad u^2 \text{ with } u = e^x$$

$$= 2(e^x) \cdot e^x = 2e^{2x}$$

<u>Problem Set 3</u>

3) (a) $g(x) = e^x \sec(x^2+x)$

$g(0) = e^0 \sec(0) = 1$

$g'(x) = e^x \sec(x^2+x)\tan(x^2+x) \cdot (2x+1)$
$\quad + e^x \sec(x^2+x)$

$g'(0) = \cancel{e^0 \sec(0)\tan(0) \cdot 1} + e^0 \sec(0) = 1$

$\boxed{y = x+1}$

(b) $h(x)=e^{(e^x)}\sin(\sin(x))$

$$h'(x)=e^{(e^x)}\cos(\sin x)\cdot\cos x$$
$$+e^xe^{(e^x)}\sin(\sin(x))$$

$$h(0)=e^{(e^0)}\sin(\sin(0))$$
$$=e^1\sin(0)=0.$$

$$h'(0)=e^{(e^0)}\cos(\sin 0)\cdot\cos 0$$
$$\cancel{+e^0e^{(e^0)}\sin(\sin 0)}$$
$$=e^1\cos 0\cdot\cos 0=e.$$

$$\boxed{y=ex}$$

Question 4

$$f(x)=\frac{1}{1-x} \qquad u=1-x \quad \frac{1}{u}.$$

$$f'(x)=-\frac{1}{(1-x)^2}\cdot(-1)=\frac{1}{(1-x)^2} \quad \text{(Chain rule)}$$

$$f''(x)=-\frac{2}{(1-x)^3}(-1) \qquad u=1-x \quad \frac{1}{u^2}$$
$$=\frac{2}{(1-x)^3}$$

$$f'''(x)=-\frac{3\cdot 2}{(1-x)^4}\cdot(-1)=\frac{6}{(1-x)^4}$$

So $\boxed{f^{(20)}(x)=\dfrac{20!}{(1-x)^{21}}}$

$f(x) = \frac{1}{1-5x}$ $\quad u = 1-5x \quad \frac{1}{u}$

$f'(x) = -\frac{1}{(1-5x)^2}(-5)$

$= \frac{5}{(1-5x)^2}$ $\quad u = 1-5x \quad \frac{5}{u^2}$

$f''(x) = \frac{5\cdot(-2)}{(1-5x)^3}\cdot(-5)$

$= \frac{5^2\cdot 2}{(1-5x)^3}$

$f'''(x) = \frac{5^2\cdot 2\cdot(-3)}{(1-5x)^4}(-5)$

$= \frac{5^3\cdot 3\cdot 2}{(1-5x)^4}$

So $\boxed{f^{(20)}(x) = \frac{5^{20}\cdot 20!}{(1-5x)^{21}}}$

Problem Set 3

5) $f' = 3f$. Use Chain Rule

$\Rightarrow \boxed{f = e^{3x}}$

$f''' = -8f$. Use Chain Rule

$\Rightarrow \boxed{f = e^{-2x}}$

$f'' = -\frac{1}{4}f$. Use Chain Rule

$\Rightarrow \boxed{f = \cos\left(\frac{1}{2}x\right)}$

$\frac{f'}{f} = \frac{2}{x}$ Use Power Rule. $\boxed{f(x)=x^2}$

$\frac{f'}{f} = 2x$. Use Chain Rule. $\boxed{f(x)=e^{x^2}}$

6) (a) $f(f^{-1}(x)) = x$

$$\frac{d}{dx}[f(f^{-1}(x))] = \frac{d}{dx}x$$

$$f'(f^{-1}(x)) \cdot \frac{d}{dx}(f^{-1}(x)) = 1.$$

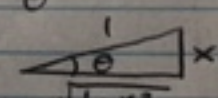
So $$\frac{d}{dx}(f^{-1}(x)) = \frac{1}{f'(f^{-1}(x))}.$$

(b) $f(x) = \sin x$, $f^{-1}(x) = \arcsin x$. $f'(x) = \cos x$

$$\frac{d}{dx}(\arcsin(x)) = \frac{1}{\cos(\underbrace{\arcsin x}_{\theta})}$$

$\sin\theta = \frac{x}{1}$

$$= \frac{1}{\sqrt{1-x^2}}.$$

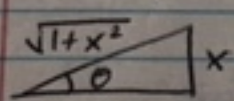


$\cos\theta = \sqrt{1-x^2}$

$f'(x) = \sec^2 x$ $\quad f(x) = \tan x$, $f^{-1}(x) = \arctan x$

$$\frac{d}{dx}(\arctan(x)) = \frac{1}{\sec^2(\arctan(x))}$$

$\tan\theta = \frac{x}{1}$



$$= \cos^2(\underbrace{\arctan x}_{\theta})$$ since $\sec x = \frac{1}{\cos x}$

$\cos\theta = \frac{1}{\sqrt{1+x^2}}$

so $\cos^2\theta = \frac{1}{1+x^2}$

$$= \frac{1}{1+x^2}.$$