

1) Find all solutions of

$$e^{-x} \frac{dy}{dx} = (y+2)(x+1)^2$$

then find the particular solution that passes through the point  $(-1, 2)$ .

① Separate variables to  $\int \frac{1}{y+2} dy = \int e^x (x+1)^2 dx$   
find non-constant solutions.

$$\begin{aligned} \text{ASIDE: } \int e^x (x+1)^2 dx & \quad u = x+1 \\ & \quad x = u-1 \quad dx = du \\ & = \int e^{u-1} u^2 du \\ & = \frac{1}{e} \int u^2 e^u du \end{aligned}$$

$$= \frac{1}{e} \left( u^2 e^u - 2 \int u e^u du \right)$$

$\begin{matrix} v = u^2 & dw = e^u du \\ dv = 2u du & w = e^u \end{matrix}$

↑ can integrate by parts again

$$= \frac{1}{e} (u^2 e^u - 2(u e^u - e^u)) + C$$

$$= \frac{1}{e} u^2 e^u - \frac{2}{e} u e^u + \frac{2}{e} e^u + C$$

$$= u^2 e^{u-1} - 2u e^{u-1} + 2e^{u-1} + C$$

$$= (x+1)^2 e^x - 2(x+1)e^x + 2e^x + C$$

So non-constant solution:

$$\ln|y+2| = (x+1)^2 e^x - 2(x+1)e^x + 2e^x + C$$

$$|y+2| = e^C e^{((x+1)^2 e^x - 2(x+1)e^x + 2e^x)}$$

$$y+2 = \pm e^C e^{((x+1)^2 e^x - 2(x+1)e^x + 2e^x)}$$

$$y = -2 \pm e^C e^{((x+1)^2 e^x - 2(x+1)e^x + 2e^x)}$$

② Find constant solutions.

$$y = k \frac{dy}{dx} = 0.$$

$$e^{-x} \frac{dy}{dx} \frac{1}{(x+1)^2} = y+2$$

$$0 = k+2 \quad k = -2$$

constant solution:  $y = -2$

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Orthogonal trajectories of  
 $\arctan x + \ln(1+y^3) = C.$

Implicitly differentiate.

$$\frac{d}{dx} (\arctan x + \ln(1+y^3)) = \frac{d}{dx} (C)$$

$$\frac{1}{1+x^2} + \frac{3y^2}{1+y^3} \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{1}{1+x^2} \left( \frac{1+y^3}{3y^2} \right)$$

↑ for family of curves

So for orthogonal trajectory,

$$\frac{dy}{dx} = (1+x^2) \left( \frac{3y^2}{1+y^3} \right)$$

① Non-constant solution

$$\left(\frac{1+y^3}{3y^2}\right) dy = (1+x^2) dx$$

$$\int \left(\frac{1}{3y^2} + \frac{y}{3}\right) dy = \int (1+x^2) dx$$

$$\boxed{-\frac{1}{3y} + \frac{1}{6}y^2 = x + \frac{1}{3}x^3 + C}$$

② Constant solutions

$$\frac{1}{1+x^2} \frac{dy}{dx} = \frac{3y^2}{1+y^3}$$

$$y = k, \frac{dy}{dx} = 0.$$

$$0 = \frac{3k^2}{1+k^3} \quad 3k^2 = 0 \quad k = 0$$

constant solution:  $\boxed{y = 0}$

$$\frac{dP}{dt} = kP \left(1 - \frac{P}{M}\right) \Rightarrow P = \frac{M}{1 + Ae^{-kt}}$$

$$\frac{dP}{dt} = 0.1P \left(1 - \frac{P}{5000}\right)$$

$$k = 0.1 \\ M = 5000$$

$$P(t) = \frac{5000}{1 + Ae^{-0.1t}}$$

$$P(10) = 2P(0)$$

$$\frac{5000}{1 + Ae^{-1}} = 2 \left(\frac{5000}{1 + A}\right) \quad \text{Solve for } A.$$

$$\frac{1}{1+Ae^{-t}} = \frac{2}{1+A}$$

$$1+A = 2+2Ae^{-t}$$

$$A(1-2e^{-t}) = 1$$

$$A = \frac{1}{1-2e^{-t}}$$

$$0.9(5000) = \frac{5000}{1+Ae^{-0.1t}}$$

$$\frac{9}{10} = \frac{1}{1+Ae^{-0.1t}}$$

$$\frac{10}{9} = 1+Ae^{-0.1t}$$

$$Ae^{-0.1t} = \frac{1}{9}$$

$$e^{-0.1t} = \frac{1}{9A}$$

$$-0.1t = \ln\left(\frac{1}{9A}\right)$$

$$t = -10 \ln\left(\frac{1}{9A}\right)$$

$$= 10 \ln(9A)$$

$$= \boxed{10 \ln\left(\frac{9}{1-2e^{-t}}\right)}$$

$$2) \frac{dy}{dx} + xy = -x^3$$

$$e^{\int x dx} = e^{\frac{1}{2}x^2}$$

$$e^{\frac{x^2}{2}} \frac{dy}{dx} + x e^{\frac{x^2}{2}} y = -x^3 e^{\frac{x^2}{2}}$$

$$\frac{d}{dx} (e^{\frac{x^2}{2}} y) = -x^3 e^{\frac{x^2}{2}}$$

$$e^{\frac{x^2}{2}} y = \int -x^3 e^{\frac{x^2}{2}} dx$$

ASIDE:  $u = \frac{x^2}{2} \quad du = x dx$

$$x^2 = 2u$$

$$\begin{aligned} \text{So } \int -x^3 e^{\frac{x^2}{2}} dx &= \int -x^2 e^{\frac{x^2}{2}} (x dx) \\ &= \int -2u e^u du \end{aligned}$$

$$= -2(u e^u - e^u) + C$$

$$\begin{aligned} &= -2u e^u + 2e^u + C \\ &= -x^2 e^{\frac{x^2}{2}} + 2e^{\frac{x^2}{2}} + C \end{aligned}$$

$$\text{So } e^{\frac{x^2}{2}} y = -x^2 e^{\frac{x^2}{2}} + 2e^{\frac{x^2}{2}} + C$$

$$y = -x^2 + 2 + C e^{-\frac{x^2}{2}}$$

$$\frac{dy}{dx} + \frac{y}{x} = \cos x$$

$$e^{\int \frac{1}{x} dx} = e^{\ln x} = x$$

$$x \frac{dy}{dx} + y = x \cos x$$

$$\frac{d}{dx} (xy) = x \cos x$$

$$xy = \int x \cos x dx$$

$$xy = x \sin x + \cos x + C$$

^ Integrate by parts.

$$y = \sin x + \frac{\cos x}{x} + \frac{C}{x}$$

$$\frac{dy}{dx} + \frac{y}{2x} = x + \frac{1}{x\sqrt{x}}$$

$$e^{\int \frac{1}{2x} dx} = e^{\frac{1}{2} \ln x} = e^{\ln \sqrt{x}} = \sqrt{x}$$

$$\sqrt{x} \frac{dy}{dx} + \frac{1}{2\sqrt{x}} y = x^{3/2} + \frac{1}{x}$$

$$\frac{d}{dx} (\sqrt{x} y) = x^{3/2} + \frac{1}{x}$$

$$\sqrt{x} y = \frac{2}{5} x^{5/2} + \ln|x| + C$$

$$y = \frac{2}{5} x^2 + \frac{\ln|x|}{\sqrt{x}} + \frac{C}{\sqrt{x}}$$

$$\frac{dy}{dx} + \frac{2xy}{1+x^2} = e^x$$

$$e^{\int \frac{2x}{1+x^2} dx} = e^{\ln(1+x^2)} = (1+x^2)$$

$$(1+x^2) \frac{dy}{dx} + 2xy = (1+x^2)e^x$$

$$\frac{d}{dx} ((1+x^2)y) = e^x + x^2 e^x$$

$$(1+x^2)y = \int e^x + x^2 e^x dx$$

$$= e^x + \int x^2 e^x dx$$

$$= e^x + x^2 e^x - 2x e^x + 2e^x + C$$

(integrate by parts twice)

$$y = \frac{(x^2 - 2x + 3)e^x + C}{1+x^2}$$

$$\frac{dy}{dx} + \frac{y}{(\arctan x)(1+x^2)} = x$$

$$e^{\int \frac{1}{(\arctan x)(1+x^2)} dx} = e^{\ln(\arctan x)} = \arctan x$$

$$\arctan x \frac{dy}{dx} + \frac{y}{1+x^2} = x \arctan x$$

$$\frac{d}{dx} (\arctan x y) = x \arctan x$$

$$(\arctan x) y = \int x \arctan x dx$$

$$u = \arctan x \quad dv = x$$

$$du = \frac{1}{1+x^2} dx \quad v = \frac{1}{2} x^2$$

ASIDE  $\int x \arctan x dx = \frac{1}{2} x^2 \arctan x - \frac{1}{2} \int \frac{x^2}{1+x^2} dx$

$$\begin{array}{r} \frac{1 - \frac{1}{x^2+1}}{x^2+0x+1} \\ \underline{-(x^2+0x+1)} \\ -1 \end{array} = \frac{1}{2} x^2 \arctan x - \frac{1}{2} \left( \int 1 - \frac{1}{1+x^2} dx \right)$$
$$= \frac{1}{2} x^2 \arctan x - \frac{1}{2} x + \frac{1}{2} \arctan x + C$$

So  $(\arctan x) y = \frac{1}{2} x^2 \arctan x - \frac{1}{2} x + \frac{1}{2} \arctan x + C$

$$y = \frac{1}{2} x^2 - \frac{x}{2 \arctan x} + \frac{1}{2} + \frac{C}{\arctan x}$$