A stochastic fluid-structure interaction model given by a stochastic viscous wave equation in $\mathbb{R}^2$

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Stochastic viscous wave equation

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Stochastic viscous wave equation

$$u_{tt} + \sqrt{-\Delta} u_t - \Delta u = f(u) W(dx, dt) \qquad \text{in } \mathbb{R}^n,$$

$$u(0, \cdot) = g(\cdot), \qquad u_t(0, \cdot) = h(\cdot)$$

$W(dx, dt)$ denotes spacetime white noise, and $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function.

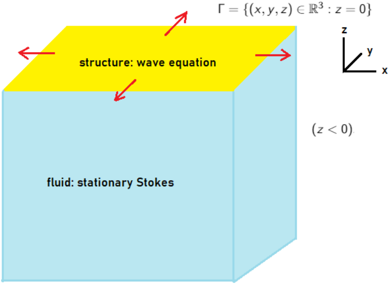
Derivation of equation: structure and fluid

- **Structure subproblem:** Infinite, prestressed, elastic membrane with reference configuration $\mathbb{R}^2$.

$$u_{tt} - \Delta u = f \qquad \text{on } \mathbb{R}^2.$$

- **Fluid subproblem:** Viscous, incompressible, Newtonian fluid.

$$\left.\begin{array}{rcl} \nabla\pi &=& \mu\Delta \boldsymbol{v}, \\ \nabla\cdot\boldsymbol{v} &=& 0, \end{array}\right\} \quad \text{in } (z<0).$$



Derivation of equation: coupling conditions

- **Kinematic coupling condition:** Continuity of velocities at the interface $\mathbb{R}^2$,

$$\boldsymbol{v} = \frac{\partial u}{\partial t}\boldsymbol{e_z} \qquad \text{on } \mathbb{R}^2.$$

We assume displacement of the structure in only the z direction and linear coupling.

- **Dynamic coupling condition:** Balance of forces, fluid load on structure

$$u_{tt} - \Delta u = -\boldsymbol{\sigma}\boldsymbol{e_z} \cdot \boldsymbol{e_z} + F_{ext} \qquad \text{on } \mathbb{R}^2,$$

where $\boldsymbol{\sigma} = -\pi \boldsymbol{I} + 2\mu \boldsymbol{D}(\boldsymbol{v})$.

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- Since $v_x = v_y = 0$ on $\Gamma = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$, by the divergence free condition,

$$-\boldsymbol{\sigma} \boldsymbol{e_z} \cdot \boldsymbol{e_z}|_\Gamma = \pi - 2\mu \frac{\partial v_z}{\partial z}\bigg|_\Gamma = \pi|_\Gamma.$$

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- **Goal:** Calculate $\pi|_\Gamma$ in terms of u_t, from

$$\left.\begin{array}{rcl} \nabla \pi & = & \mu \Delta \boldsymbol{v}, \\ \nabla \cdot \boldsymbol{v} & = & 0, \end{array}\right\} \quad \text{in } (z < 0),$$

with $\boldsymbol{v} = u_t \boldsymbol{e_z}$ on Γ.

Derivation of equation: Solution to Stokes problem

$$\left.\begin{array}{rcl} \nabla \pi & = & \mu \Delta \boldsymbol{v}, \\ \nabla \cdot \boldsymbol{v} & = & 0, \end{array}\right\} \quad \text{in } (z < 0), \qquad \boldsymbol{v} = u_t \boldsymbol{e_z} \;\; \text{on } \Gamma.$$

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- π is harmonic in $(z < 0)$ with Neumann boundary condition

$$\left.\frac{\partial \pi}{\partial z}\right|_{\Gamma} = \mu \Delta_{x,y} v_z + \mu \left.\frac{\partial^2 v_z}{\partial z^2}\right|_{\Gamma}.$$

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- To find v_z, note $\Delta^2 v_z = 0$, $v_z|_\Gamma = u_t$, $\frac{\partial v_z}{\partial z}|_\Gamma = 0$.
- Take Fourier transform in x, y.

$$|\boldsymbol{\xi}|^4 \widehat{v_z}(\boldsymbol{\xi}, z) - 2|\boldsymbol{\xi}|^2 \frac{\partial^2}{\partial z^2}\widehat{v_z}(\boldsymbol{\xi}, z) + \frac{\partial^4}{\partial z^4}\widehat{v_z}(\boldsymbol{\xi}, z) = 0.$$

$$\widehat{v_z}(\boldsymbol{\xi}, z) = C_1(\boldsymbol{\xi})e^{|\boldsymbol{\xi}|z} + C_2(\boldsymbol{\xi})ze^{|\boldsymbol{\xi}|z} + C_3(\boldsymbol{\xi})e^{-|\boldsymbol{\xi}|z} + C_4(\boldsymbol{\xi})ze^{-|\boldsymbol{\xi}|z}.$$

Derivation of equation: Solution to Stokes problem

$$\left.\begin{array}{rcl} \nabla\pi & = & \mu\Delta\boldsymbol{v}, \\ \nabla\cdot\boldsymbol{v} & = & 0, \end{array}\right\} \quad \text{in } (z<0), \quad \boldsymbol{v} = u_t\boldsymbol{e_z} \ \text{ on } \Gamma.$$

- Use boundary conditions.

$$\widehat{v_z}(\boldsymbol{\xi}, z) = \widehat{u_t}(\boldsymbol{\xi})e^{|\boldsymbol{\xi}|z} - |\xi|\widehat{u_t}(\boldsymbol{\xi})ze^{|\boldsymbol{\xi}|z}.$$

- Thus, π is harmonic with Neumann boundary condition

$$\left.\frac{\partial\widehat{\pi}}{\partial z}\right|_\Gamma = -\mu|\boldsymbol{\xi}|^2\widehat{v_z} + \mu\left.\frac{\partial^2\widehat{v_z}}{\partial z^2}\right|_\Gamma = -2\mu|\boldsymbol{\xi}|^2\widehat{u_t}(\boldsymbol{\xi}).$$

- Dirichlet to Neumann operator on $(z<0)$ is $\sqrt{-\Delta}$. So Neumann to Dirichlet operator is $\frac{1}{|\boldsymbol{\xi}|}$ Fourier multiplier.

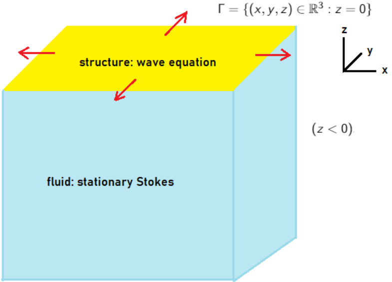
$$\widehat{\pi}|_\Gamma = -2\mu|\boldsymbol{\xi}|\widehat{u_t}(\boldsymbol{\xi}) \Longrightarrow -\boldsymbol{\sigma}\boldsymbol{e_z}\cdot\boldsymbol{e_z} = \pi|_\Gamma = -2\mu\sqrt{-\Delta}u_t.$$

Derivation of equation: Stochastic variation in external pressure

$$u_{tt} - \Delta u = -\boldsymbol{\sigma} \mathbf{e_z} \cdot \mathbf{e_z} + F_{ext} \implies u_{tt} + 2\mu\sqrt{-\Delta}u_t - \Delta u = F_{ext}.$$

In real-life FSI, small random deviations in pressure are observed in data.

$$F_{ext} = f(u)W(dx, dt) \implies u_{tt} + 2\mu\sqrt{-\Delta}u_t - \Delta u = f(u)W(dx, dt).$$



Summary of main results

Let $n = 1$ or $n = 2$. Let g, h be continuous functions in $H^2(\mathbb{R}^n)$, and let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function.

$$u_{tt} + \sqrt{-\Delta} u_t - \Delta u = f(u) W(dx, dt) \qquad \text{in } \mathbb{R}^n.$$

Theorem 1 (J.K., S. Čanić, '21)

There exists a unique function valued mild solution to the stochastic viscous wave equation with initial data $u(0, x) = g(x)$, $\partial_t u(0, x) = h(x)$.

Theorem 2 (J.K., S. Čanić, '21)

Let $n = 2$. For each $\alpha \in [0, 1/2)$, the mild solution with initial data g, h has a modification that is (locally) α-Hölder continuous on $\mathbb{R}^+ \times \mathbb{R}^n$.

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Let $n = 2$. For each $\alpha \in [0, 1/2)$, the mild solution with initial data g, h has a modification that is (locally) α-Hölder continuous on $\mathbb{R}^+ \times \mathbb{R}^n$.

This is unlike stochastic heat and wave equations with spacetime white noise, which have function valued mild solution only in dimension one.

Viscous wave operator (Kuan-Čanić '20)

$$u_{tt} + \sqrt{-\Delta}u_t - \Delta u + u^p = 0$$

- Deterministically ill-posed for initial data in $H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n)$ and $0 < s < s_{cr} = \frac{n}{2} - \frac{2}{p-1}$, since solution map is unbounded from

$$H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n) \to C([0,T]; H^s(\mathbb{R}^n)) \times C([0,T]; H^{s-1}(\mathbb{R}^n)).$$

- Same critical exponent as NLWE (Christ-Colliander-Tao '03).
- After a randomization of arbitrary given initial data, the problem is "well-posed" with high probability.

J. Kuan and S. Canic. Deterministic ill-posedness and probabilistic well-posedness of the viscous nonlinear wave equation describing fluid-structure interaction. **Transactions of the American Mathematical Society** Accepted 2020.

Literature review

- **FSI**: incompressible, viscous Newtonian fluid and an elastic structure
  - **Linear coupling** (Du-Gunzburger-Hou-Lee '03, Barbu-Grujić-Lasiecka-Tuffaha '07 and '08, Kukavica-Tufaha-Ziane '10, Kuan-Čanić '21, and many more)
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- **Spatially homogeneous Gaussian noise in higher dimensions** for the heat and wave equations (Dalang-Frangos '98, Dalang '99, Karczewska-Zabcyzk '99)

- **Hölder continuity** properties (Conus-Dalang '08, Dalang-Sanz-Solé '09, Sanz-Solé-Sarrà '02)
  - For $n = 1$, with spacetime white noise: heat equation (up to $1/4$-Hölder in time, $1/2$-Hölder in space) and wave equation (up to $1/2$-Hölder in both time and space).

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Deterministic linear viscous wave equation

$$u_{tt} + \sqrt{-\Delta} u_t - \Delta u = F, \qquad \text{on } \mathbb{R}^n,$$
$$u(0,x) = 0, \qquad \partial_t u(0,x) = 0.$$

By the Fourier transform,

$$\widehat{u}(t,\xi) = \int_0^t \widehat{F}(\tau,\xi) e^{-\frac{|\xi|}{2}(t-\tau)} \frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|(t-\tau)\right)}{\frac{\sqrt{3}}{2}|\xi|} d\tau.$$

Define the convolution kernel,

$$K_t(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-\frac{|\xi|}{2}t} \frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|t\right)}{\frac{\sqrt{3}}{2}|\xi|} d\xi,$$

so therefore,

$$u(t,x) = \int_0^t \int_{\mathbb{R}^n} K_{t-s}(x-y) F(s,y) ds dy.$$

The concept of mild solution

Main question:

How would we define a solution to the equation

$$u_{tt} + \sqrt{-\Delta} u_t - \Delta u = f(u) W(dx, dt), \qquad \text{on } \mathbb{R}^n,$$

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Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we formally want to say that u satisfies

$$u(t, x, \omega) = \int_0^t \int_{\mathbb{R}^n} K_{t-s}(x - y) f(u(s, y, \omega)) W(dy, ds).$$

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An adapted, jointly measurable process $\{u(t, x, \omega)\}_{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n}$ satisfying this, where this stochastic integral is well-defined, is a **mild solution**.

Spacetime white noise

$\{W(A)\}_{A \in \mathcal{B}(\mathbb{R}^+ \times \mathbb{R}^n)}$ is a mean zero Gaussian process with covariance

$$\mathbb{E}(W(A)W(B)) = \lambda(A \cap B).$$

- For Borel subsets $A_1, A_2, ..., A_k$, $(W(A_1), W(A_2), ..., W(A_k))$ is multivariate Gaussian.
- $W(A) \sim N(0, \lambda(A))$.
- If $A \cap B = \varnothing$, then $W(A)$ and $W(B)$ are independent Gaussians.
- For a given $A, B \in \mathcal{B}(\mathbb{R}^+ \times \mathbb{R}^n)$,

$$W(A \cup B) = W(A) + W(B) - W(A \cap B), \qquad \text{a.s.}$$

Stochastic integration against white noise

Define $W_t(A) = W([0, t] \times A)$ and consider the filtration

$$\mathcal{F}_t = \sigma(W_s(A), s \in [0, t], A \text{ bounded, in } \mathcal{B}(\mathbb{R}^+ \times \mathbb{R}^n)).$$

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$$\mathcal{F}_t = \sigma(W_s(A), s \in [0,t], A \text{ bounded, in } \mathcal{B}(\mathbb{R}^+ \times \mathbb{R}^n)).$$

For $X \in \mathcal{F}_a$ and a bounded Borel set A, define for fixed but arbitrary $0 \leq a < b$,

$$\int_0^t \int_{\mathbb{R}^n} X(\omega) 1_{(a,b]}(s) 1_A(x) W(dx, ds) = X(\omega)[W_{t \wedge b}(A) - W_{t \wedge a}(A)].$$

Extend by linearity to class of elementary integrands $\mathcal{S}$.

Stochastic integration against white noise

For all $f \in \mathcal{S}$,

$$\mathbb{E}\left[\left(\int_0^\infty \int_{\mathbb{R}^n} f(t,x,\omega) W(dx,dt)\right)^2\right] = \mathbb{E}\left(\int_0^\infty \int_{\mathbb{R}^n} |f(t,x,\omega)|^2 dx dt\right).$$

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Extend (by density) to a class $\mathcal{P}_W$ of predictable integrands with norm

$$||f||_{\mathcal{P}_W}^2 := \mathbb{E}\left(\int_0^\infty \int_{\mathbb{R}^n} f(t,x,\omega)^2 dxdt\right) < \infty.$$

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Main idea:

For appropriate (predictable) integrands, we can stochastically integrate if

$$\mathbb{E}\left(\int_0^\infty \int_{\mathbb{R}^n} |f(t,x,\omega)|^2 dxdt\right) < \infty.$$

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An illustrative example

Instead of considering the most general equation

$$u_{tt} + \sqrt{-\Delta}u_t - \Delta u = f(u)W(dx, dt), \qquad \text{on } \mathbb{R}^n,$$

we will consider the simplified model with zero initial data

$$u_{tt} + \sqrt{-\Delta}u_t - \Delta u = W(dx, dt), \qquad \text{on } \mathbb{R}^n.$$

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Instead of considering the most general equation

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$$u_{tt} + \sqrt{-\Delta} u_t - \Delta u = W(dx, dt), \qquad \text{on } \mathbb{R}^n.$$

This has a function-valued mild solution in **both** dimensions one and two, *unlike the corresponding stochastic heat and wave equations.*

Stochastic heat equation

Stochastic heat equation with additive noise

Consider

$$u_t - \Delta u = W(dx, dt), \qquad \text{on } \mathbb{R}^n,$$

with zero initial data.

The mild solution (if it exists) is given explicitly as

$$u(t,x) = \int_0^t \int_{\mathbb{R}^n} K^H_{t-s}(x-y) W(ds, dy),$$

where

$$K^H_t(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-|\xi|^2 t} d\xi = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{x^2}{4t}}.$$

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For stochastic integral to be defined, we require

$$\int_0^t \int_{\mathbb{R}^n} |K^H_{t-s}(x-y)|^2 dyds < \infty.$$

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Stochastic heat equation

For mild solution to be defined, $\int_0^t \int_{\mathbb{R}^n} |K_{t-s}^H(x-y)|^2 dyds < \infty$.

$$K_t^H(x) = t^{-\frac{n}{2}} K^H\left(\frac{x}{t^{1/2}}\right), \text{ where } K^H(x) := K_1^H(x).$$

Using this scaling property, we can rewrite the integrability condition as

$$\begin{aligned}\int_0^t \int_{\mathbb{R}^n} |K_{t-s}^H(x-y)|^2 dyds &= \int_0^t \int_{\mathbb{R}^n} (t-s)^{-n} \left| K^H\left(\frac{x-y}{(t-s)^{1/2}}\right)\right|^2 dyds \\ &= \left(\int_0^t (t-s)^{-n/2} ds\right) ||K^H||_{L^2(\mathbb{R}^n)}^2 < \infty.\end{aligned}$$

Mild solution only in $n = 1$ due to spacetime scaling.

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$$\begin{aligned}\int_0^t \int_{\mathbb{R}^n} |K_{t-s}^W(x-y)|^2 dyds &= \int_0^t \int_{\mathbb{R}^n} (t-s)^{2-2n} \left| K^W\left(\frac{x-y}{t-s}\right)\right|^2 dyds \\ &= \left(\int_0^t (t-s)^{2-n} ds\right) ||K^W||^2_{L^2(\mathbb{R}^n)} < \infty.\end{aligned}$$

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$$K_t^W(x) = t^{1-n} K^W\left(\frac{x}{t}\right), \text{ where } K^W(x) := K_1^W(x).$$

Using this scaling property, we can rewrite the integrability condition as

$$\begin{aligned}\int_0^t \int_{\mathbb{R}^n} |K_{t-s}^W(x-y)|^2 dyds &= \int_0^t \int_{\mathbb{R}^n} (t-s)^{2-2n} \left| K^W\left(\frac{x-y}{t-s}\right)\right|^2 dyds \\ &= \left(\int_0^t (t-s)^{2-n} ds\right) ||K^W||_{L^2(\mathbb{R}^n)}^2 < \infty.\end{aligned}$$

Mild solution only in $n = 1$ due to lack of decay of fundamental solution.

$$K_t^W(x) = \frac{1}{2} 1_{|x|<t} \qquad \text{for } n = 1,$$

$$K_t^W(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{t^2 - |x|^2}} 1_{|x|<t} \qquad \text{for } n = 2.$$

Stochastic viscous wave equation

Stochastic viscous wave equation with additive noise

Consider

$$u_{tt} + \sqrt{-\Delta}u_t - \Delta u = W(dx, dt), \qquad \text{on } \mathbb{R}^n,$$

with zero initial data.

The mild solution (if it exists) is given explicitly as

$$u(t,x) = \int_0^t \int_{\mathbb{R}^n} K_{t-s}(x-y)W(ds, dy),$$

where

$$K_t(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-\frac{|\xi|}{2}t} \frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|t\right)}{\frac{\sqrt{3}}{2}|\xi|} d\xi$$

Stochastic viscous wave equation

Stochastic viscous wave equation with additive noise

Consider

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For stochastic integral to be defined, we require

$$\int_0^t \int_{\mathbb{R}^n} |K_{t-s}(x-y)|^2 dyds < \infty.$$

Stochastic viscous wave equation

Same spacetime scaling as wave equation:

$$K_t(x) = t^{1-n}K\left(\frac{x}{t}\right), \text{ where } K(x) := K_1(x).$$

Using this scaling property (same as wave equation),

$$\int_0^t \int_{\mathbb{R}^n} |K_{t-s}(x-y)|^2 dyds = \left(\int_0^t (t-s)^{2-n} ds\right) ||K||^2_{L^2(\mathbb{R}^n)} < \infty.$$

But unlike the wave equation, we have the following.

Lemma

For all $n \geq 1$ and for all $1 \leq p \leq \infty$, $K(x) \in L^p(\mathbb{R}^n)$.

$\Longrightarrow$ Function valued mild solution in dimensions one and two.

A comparison of the various equations

Stochastic heat equation.

$$\int_0^t \int_{\mathbb{R}^n} |K_{t-s}^H(x-y)|^2 dyds = \left(\int_0^t (t-s)^{-n/2} ds \right) ||K^H||_{L^2(\mathbb{R}^n)}^2.$$

Has $K^H \in L^2(\mathbb{R}^n)$ but undesirable spacetime scaling.

Stochastic wave equation.

$$\int_0^t \int_{\mathbb{R}^n} |K_{t-s}^W(x-y)|^2 dyds = \left(\int_0^t (t-s)^{2-n} ds \right) ||K^W||_{L^2(\mathbb{R}^n)}^2.$$

Has desirable spacetime scaling but $K^W \notin L^2(\mathbb{R}^n)$ for $n \geq 2$.

Stochastic viscous wave equation.

$$\int_0^t \int_{\mathbb{R}^n} |K_{t-s}(x-y)|^2 dyds = \left(\int_0^t (t-s)^{2-n} ds \right) ||K||_{L^2(\mathbb{R}^n)}^2.$$

Has both $K \in L^2(\mathbb{R}^n)$ and desirable spacetime scaling.

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1. Introduction and derivation of the equation
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4. Existence and uniqueness
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Main result

$$u_{tt} + \sqrt{-\Delta} u_t - \Delta u = f(u) W(dx, dt), \qquad \text{in } \mathbb{R}^n,$$

with **deterministic** initial data $u(0, x) = g(x)$ and $\partial_t u(0, x) = h(x)$.

In this case, a mild solution satisfies

$$u(t, x, \omega) = u_{free}(t, x) + \int_0^t \int_{\mathbb{R}^n} K_{t-s}(x - y) f(u(s, y, \omega)) W(dy, ds)$$

Theorem 1 (J.K., S. Čanić, '21)

Let $n = 1$ or $n = 2$, and let g and h be continuous functions in $H^2(\mathbb{R}^n)$. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function. Then, there exists a unique function valued mild solution to the equation

$$u_{tt} + \sqrt{-\Delta} u_t - \Delta u = f(u) W(dx, dt) \qquad \text{on } \mathbb{R}^n$$

with initial data $u(0, x) = g(x)$, $\partial_t u(0, x) = h(x)$.

Proof sketch

- Picard iteration due to nonlinearity. First iterate u_0 is $u_{free}(t,x)$. Subsequent iterates are

$$u_n(t,x,\omega) = u_0(t,x) + \int_0^t \int_{\mathbb{R}^n} K_{t-s}(x-y) f(u_{n-1}(s,y,\omega)) W(dy,ds).$$

Proof sketch

- Picard iteration due to nonlinearity. First iterate u_0 is $u_{free}(t,x)$. Subsequent iterates are

$$u_n(t,x,\omega) = u_0(t,x) + \int_0^t \int_{\mathbb{R}^n} K_{t-s}(x-y) f(u_{n-1}(s,y,\omega)) W(dy,ds).$$

- Estimate the increments to show convergence. Define

$$H_k^2(t) := \sup_{0 \le s \le t, x \in \mathbb{R}^n} \mathbb{E}[(u_k - u_{k-1})^2(s,x,\omega)].$$

Proof sketch

- Picard iteration due to nonlinearity. First iterate u_0 is $u_{free}(t,x)$. Subsequent iterates are

$$u_n(t,x,\omega) = u_0(t,x) + \int_0^t \int_{\mathbb{R}^n} K_{t-s}(x-y) f(u_{n-1}(s,y,\omega)) W(dy,ds).$$

- Estimate the increments to show convergence. Define

$$H_k^2(t) := \sup_{0 \le s \le t, x \in \mathbb{R}^n} \mathbb{E}[(u_k - u_{k-1})^2(s,x,\omega)].$$

- Use the inductive inequality

$$H_k^2(t) \le L^2 \int_0^t \int_{\mathbb{R}^n} K_{t-s}^2(x-y) H_{k-1}^2(s) dyds$$

and use Gronwall to get convergence at each point in $L^2(\Omega)$ uniformly on bounded time intervals.

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Main theorem

$$u_{tt} + \sqrt{-\Delta} u_t - \Delta u = f(u) W(dx, dt), \qquad \text{in } \mathbb{R}^n,$$

with **deterministic** initial data $u(0, x) = g(x)$ and $\partial_t u(0, x) = h(x)$.

$\{u(t, x, \omega)\}_{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n}$. Then, $\{\tilde{u}(t, x, \omega)\}_{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n}$ is a **modification** if

$$\mathbb{P}(\tilde{u}(t, x, \omega) = u(t, x, \omega)) = 1, \qquad \text{for all } (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n.$$

Observe that $(u(t_1, x_1, \omega), \ldots, u(t_k, x_k, \omega)) =_d (\tilde{u}(t_1, x_1, \omega), \ldots, \tilde{u}(t_k, x_k, \omega))$.

Theorem 2 (J.K., S. Čanić, '21)

Let $n = 2$. Let g, h be continuous functions in $H^2(\mathbb{R}^n)$, and let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function. For each $\alpha \in [0, 1/2)$, the mild solution with initial data g, h has a **modification** that is (locally) α-Hölder continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ in space and time.

Kolmogorov continuity criterion

Kolmogorov continuity theorem

Let $\{u(t,x,\omega)\}_{(t,x)\in\mathbb{R}^+\times\mathbb{R}^n,\omega\in\Omega}$ be a real-valued stochastic process. If there exist $\gamma,\epsilon>0$ such that for each compact set $K\subset\mathbb{R}^+\times\mathbb{R}^n$,

$$\mathbb{E}(|u(t,x,\omega)-u(s,y,\omega)|^\gamma)\le C_K|(t,x)-(s,y)|^{n+1+\epsilon}$$
$$\text{for all } (t,x),(s,y)\in K,$$

then for each α such that

$$0\le\alpha<\frac{\epsilon}{\gamma},$$

the stochastic process $\{u(t,x,\omega)\}_{(t,x)\in\mathbb{R}^+\times\mathbb{R}^n}$ has a modification that is locally α-Hölder continuous on $\mathbb{R}^+\times\mathbb{R}^n$.

Sketch of proof

Theorem 2 (J.K., S. Čanić, '21)

Let $n = 2$. Let g, h be continuous functions in $H^2(\mathbb{R}^n)$, and let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function. For each $\alpha \in [0, 1/2)$, the mild solution to the stochastic viscous wave equation with initial data g, h has a modification that is (locally) α-Hölder continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ in space and time.

By **Kolmogorov continuity theorem**, it suffices to estimate

$$\mathbb{E}(|u(t,x) - u(t',x)|^p) \leq C_{T,p,\delta}|t - t'|^{\frac{\delta p}{2}}, \text{ for all } t, t' \in [0, T], \text{ and } x \in \mathbb{R}^2,$$

$$\mathbb{E}(|u(t,x') - u(t,x)|^p) \leq C_{T,p,\delta}|x - x'|^{\frac{\delta p}{2}}, \text{ for all } t \in [0, T], \text{ and } x, x' \in \mathbb{R}^2,$$

where $T > 0$, $p \geq 2$, and $\delta \in (0, 1)$.

Thank you! Any questions?