Math 1B: Midterm 2 Review Guide

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March 19, 2019

Contents

1 Sequences

This whole section of Math 1B is about sequences and series. So the first thing to get out of the way - what are sequences, what are series, and what is the difference between them? At first, sequences and series seem very different, but they are very related.

A **sequence** is just an ordered list of numbers, written as

$$a_1, a_2, a_3, ...$$

We sometimes express a sequence by giving the nth term of the sequence. For example, the sequence defined by

$$a_n = 2n + 1$$

is given by plugging in $n = 1$, $n = 2$, $n = 3$, ... and putting the results in a list. So $a_n = 2n + 1$ defines the sequence

$$3, 5, 7, 9, ...$$

Or sequences can be defined recursively, so that each term is defined in terms of its previous terms. For example, the Fibonacci sequence is given by

$$a_1 = 1, \quad a_2 = 1, \quad a_n = a_{n-2} + a_{n-1}$$

So this sequence is $1, 1, 2, 3, 5, 8, ...$

Usually, when we have a sequence, we want to know its limit. To do this, we just use L'Hopital's rule. If the n variable bothers you, you can change it to an x to make it look more like L'Hopital's rule.

Example 1.1. Find the limit of the sequence

$$a_n = \frac{\ln(n)}{\sqrt{n}}$$

Solution. Just use L'Hopital's rule. We want to find $\lim_{n\to\infty} \frac{\ln(n)}{\sqrt{n}}$. Plugging in, we get the indeterminate form $\frac{\infty}{\infty}$. So use L'Hopital's rule (take the derivative of the numerator, then the denominator and calculate that new limit).

$$\lim_{n\to\infty} \frac{\ln(n)}{\sqrt{n}} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{2\sqrt{n}}} = \lim_{n\to\infty} \frac{2}{\sqrt{n}} = 0$$

So the limit is zero. $\square$

We also have the Squeeze Theorem as a tool for evaluating limits.

Example 1.2. Find the limit of the sequence

$$a_n = \frac{\sin(e^n)}{\ln(n+1)}$$

Solution. Whenever you see a sine or cosine, remember that sine and cosine take values between -1 and 1 inclusive. So

$$-1 \leq \sin(e^n) \leq 1$$

$$-\frac{1}{\ln(n+1)} \leq \frac{\sin(e^n)}{\ln(n+1)} \leq \frac{1}{\ln(n+1)}$$

Remember how the Squeeze theorem works. We have squeezed our original sequence terms between $-\frac{1}{\ln(n+1)}$ and $\frac{1}{\ln(n+1)}$. Since

$$\lim_{n\to\infty} -\frac{1}{\ln(n+1)} = 0$$

and

$$\lim_{n\to\infty} \frac{1}{\ln(n+1)} = 0$$

we conclude by the Squeeze Theorem that

$$\lim_{n\to\infty} \frac{\sin(e^n)}{\ln(n+1)} = 0$$

also. $\square$

The main fact that we will need about sequences later is the following very important fact:

$$\lim_{n\to\infty} a_n = 0 \text{ is equivalent to } \lim_{n\to\infty} |a_n| = 0$$

$$\lim_{n\to\infty} a_n \neq 0 \text{ is equivalent to } \lim_{n\to\infty} |a_n| \neq 0$$

This is because what is making a sequence have a limit of zero is not whether it is above zero or below zero at any given moment. It is its distance from zero, which is the absolute value of its terms.

Example 1.3. Show that the limit of

$$a_n = (-1)^{n^2} \frac{2n+1}{n-1}$$

is not zero.

Solution. Use the fact above. To show that $\lim_{n\to\infty} a_n \neq 0$, we can just show that $\lim_{n\to\infty} |a_n| \neq 0$ since these are equivalent conditions. Absolute values take off any powers of -1 since these correspond to just positive or negative signs. So

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \left| (-1)^{n^2} \frac{2n+1}{n-1} \right| = \lim_{n\to\infty} \frac{2n+1}{n-1} = 2$$

where the last limit is evaluated by noting that the degree on the top and bottom of n are the same (they are both degree 1), so the limit is the quotient of the coefficients in front (2 divided by 1). Since $\lim_{n\to\infty} |a_n| = 2 \neq 0$, we conclude by the above equivalence that $\lim_{n\to\infty} a_n \neq 0$ too. $\square$

2 Basic Series (Geometric, p-series, Telescoping)

Now that we know what a sequence is, we can now ask what a series is. A series is something very different, but related to sequences, as you already know. A series is basically an infinite sum, where we are adding an infinite bunch of numbers together. But what does it mean to add infinitely many things together? We have to define this carefully.

Given a series, we define the **kth partial sum** to be the sum of just the first k terms, and we denote it by S_k. For example, consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

Then, the first three partial sums are

$$S_1 = 1, \quad S_2 = 1 + \frac{1}{2}, \quad S_3 = 1 + \frac{1}{2} + \frac{1}{3}$$

Since it really does not make sense to add "infinitely many things" together, when we have a series, we really are taking finite partial sums and sending the number of things we are adding together to infinity. So in particular, we consider

$$\lim_{k\to\infty} S_k$$

to be the "value" of the series. If this limit exists (is finite), the series **converges** and is said to equal the value of that limit. If the limit does not exist, the series is said to **diverge**, and it makes no sense to say it has a value.

Example 2.1. Determine whether the following series converge or diverge by considering their partial sums. Find the value if the series converges.

$$\sum_{n=1}^{\infty} (-1)^n$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$$

Solution. It usually helps to just write out a few terms of the series to start with, to get an idea for what the series is doing.

$$\sum_{n=1}^{\infty} (-1)^n = -1 + 1 - 1 + 1 - 1 + 1 - \dots$$

Let us write out a few partial sums.

$$S_1 = -1, \quad S_2 = -1 + 1 = 0, \quad S_3 = -1 + 1 - 1 = -1, \quad S_4 = -1 + 1 - 1 + 1 = 0$$

So we have that the partial sums S_k are -1 if k is odd, and 0 if k is even. So the partial sums S_k are bouncing back and forth between -1 and 0. Hence, $\lim_{k\to\infty} S_k$ does not exist, so the series $\sum_{n=1}^{\infty} (-1)^n$ diverges by definition.

For the second series $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$, recall that there is a general formula for the partial sums of a geometric series. But we can just calculate some partial sums and see the pattern.

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+...$$

So the partial sums are

$$S_1 = \frac{1}{2},\quad S_2 = \frac{1}{2}+\frac{1}{4} = \frac{3}{4},\quad S_3 = \frac{1}{2}+\frac{1}{4}+\frac{1}{8} = \frac{7}{8}$$

So we can see the pattern is

$$S_k = \frac{2^k-1}{2^k} = 1-\frac{1}{2^k}$$

Therefore, we need to consider the limit as $k \to \infty$ of the partial sums.

$$\lim_{k\to\infty} S_k = \lim_{k\to\infty}\left(1-\frac{1}{2^k}\right) = 1-0 = 1$$

Since this limit is finite, the series $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$ converges and equals a value of 1. $\square$

The two series in the example before are examples of a specific type of series called a **geometric series**. Recall that a geometric series is a series where you multiply by a fixed factor to get from one number to the next. The first term can be anything. We call the first term a and the multiplying factor r. For example,

$$\sum_{n=1}^{\infty}(-1)^n = -1+1-1+1-1+1-...$$

is geometric with $a=-1$ (the first term), and $r=-1$ (you multiply by -1 to get to the next term), and

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+...$$

is geometric with $a=1/2$ and $r=1/2$. As another example, the series

$$\sum_{n=0}^{\infty}(-1)^n\frac{7}{4^n} = 7-\frac{7}{4}+\frac{7}{16}-\frac{7}{64}+...$$

is geometric with $a=7$ and $r=-1/4$. Here is an important fact about geometric series:

For a geometric series, if $|r|<1$, the series converges to a value of $\frac{a}{1-r}$.

For a geometric series, if $|r|\geq 1$, the series diverges.

For example, from this criterion, we can see that

$$\sum_{n=1}^{\infty}4\left(-\frac{3}{2}\right)^n$$

which is geometric with $a = 4(-3/2) = -6$ and $r = -3/2$ diverges since $|r| > 1$. However, the series

$$\sum_{n=0}^{\infty} (-1)^n \frac{7}{4^n} = 7 - \frac{7}{4} + \frac{7}{16} - \frac{7}{64} + ...$$

converges and it converges to a value of

$$\frac{a}{1-r} = \frac{7}{1-(-1/4)} = 7 \cdot \frac{4}{5} = \frac{28}{5}$$

since $a = 7$ and $r = -1/4$, so $|r| < 1$.

The next important type of series is called a **p-series**. This is any series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

Using the Integral Test, we can see the following important fact.

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \textbf{ converges if } p > 1$$

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \textbf{ diverges if } p \leq 1$$

The final important type of series is called a **telescoping series**. These are usually rational functions that, using partial fraction decomposition, can be reduced to series where terms cancel out. Let's do two examples.

Example 2.2. Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2 - n}$$

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$$

converge and find the value that they converge to.

Solution. Consider the first series. Do a partial fraction decomposition.

$$\frac{1}{n^2 - n} = \frac{A}{n} + \frac{B}{n-1}$$

$$1 = A(n-1) + Bn$$

$$0n + 1 = (A+B)n - A$$

Matching coefficients, we have that $-A = 1$, so $A = -1$. Also, $A + B = 0$, so $B = 1$. Thus,

$$\frac{1}{n^2 - n} = \frac{-1}{n} + \frac{1}{n-1}$$

So we have that

$$\sum_{n=1}^{\infty} \frac{1}{n^2 - n} = \sum_{n=1}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n} \right)$$

Writing some partial sums out and taking advantage of cancellations, we have that

$$S_1 = 1 - \frac{1}{2}$$

$$S_2 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) = 1 - \frac{1}{3}$$

$$S_3 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) = 1 - \frac{1}{4}$$

So we see that the general pattern is

$$S_k = 1 - \frac{1}{k+1}$$

Thus,

$$\text{lim}_{k\to\infty} S_k = \text{lim}_{k\to\infty} \left(1 - \frac{1}{k+1}\right) = 1 - 0 = 1$$

So the first series converges to a value of 1.

Next, consider the second series. The partial fraction decomposition for this series, as one can check, is

$$\frac{1}{n^2 - 1} = \frac{1}{2} \cdot \frac{1}{n-1} - \frac{1}{2} \cdot \frac{1}{n+1}$$

(Calculate this, and check this!). Thus,

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1} = \sum_{n=2}^{\infty} \frac{1}{2} \left(\frac{1}{n-1} - \frac{1}{n+1} \right)$$

So we have that the partial sums are (note the sum starts at $n = 2$, not $n = 1$)

$$S_1 = \frac{1}{2}\left(1 - \frac{1}{3}\right)$$

$$S_2 = \frac{1}{2}\left(1 - \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right)$$

$$S_3 = \frac{1}{2}\left(1 - \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right) + \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5}\right)$$

$$S_4 = \frac{1}{2}\left(1 - \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right) + \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{4} - \frac{1}{6}\right) = \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{5} - \frac{1}{6}\right)$$

where we took advantage of cancellations, wherever they happened. So we see that the partial sums for $k \geq 2$ are of the form

$$S_k = \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{k+1} - \frac{1}{k+2}\right)$$

Note that although this formula does not hold for $k = 1$, this does not matter when we are considering the limit of S_k as $k \to \infty$. We have that

$$\mathrm{lim}_{k\to\infty} S_k = \mathrm{lim}_{k\to\infty}\ \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{k+1} - \frac{1}{k+2}\right) = \frac{1}{2}\left(1 + \frac{1}{2} - 0 - 0\right) = \frac{3}{4}$$

So the series converges, and it converges to $3/4$. □

As a brief note, why are sequences and series related, even though they seem like different concepts? They are related because we cannot really add infinitely many things together. So we must interpret a series as the limit of a sequence of partial sums, $\mathrm{lim}_{k\to\infty} S_k$, where the series diverges if this limit does not exist, and converges to this limit if the limit is indeed finite.

Another connection between sequences and series arises because we usually consider the sequence a_n of the terms we are adding together in the series $\sum a_n$, as in the next section.

In these few cases, we could actually calculate the value that a series converged too because we had an expression for the kth partial sum in general. In most cases, this is not possible. Therefore, we will usually only be able to tell if a series converges or diverges, but we will not be able to calculate the value a series converges too. **If a question asks you to find the value a series converges to, most likely, that series is geometric or telescoping, or there is an easily identifiable formula for the kth partial sum S_k that you can find by writing out some partial sums and finding a pattern.**

3 The nth Term Test/Divergence Test

Let us now review the tests we have at our disposal to determine whether a series converges or diverges. The most basic test is the **nth term test**, also called the **divergence test**.

Series Test 3.1 (The nth term test/divergence test)**.** The nth term test says

If $\mathrm{lim}_{n\to\infty} a_n \neq 0$ (or equivalently, $\mathrm{lim}_{n\to\infty} |a_n| \neq 0$), then $\sum a_n$ diverges.

If $\mathrm{lim}_{n\to\infty} a_n = 0$, we know nothing about the convergence or divergence of $\sum a_n$.

I want to emphasize that if $\mathrm{lim}_{n\to\infty} a_n = 0$, you cannot say that $\sum a_n$ converges. For example, $\sum \frac{1}{n}$ diverges and $\sum \frac{1}{n^2}$ converges, but both $\mathrm{lim}_{n\to\infty} \frac{1}{n}$ and $\mathrm{lim}_{n\to\infty} \frac{1}{n^2}$ are zero. Also, note that in the first statement above, we used the fact that $\mathrm{lim}_{n\to\infty} a_n \neq 0$ is equivalent to $\mathrm{lim}_{n\to\infty} |a_n| \neq 0$.

This is the simplest test of divergence. It helps to try it first, even if you just verify it in your head and do not write it out explicitly. If you do not try it and the question could easily be solved by the nth term test, you will end up wasting quite a bit of time. Let's see some examples of the nth term test in action.

Example 3.1. Does the following series converge or diverge?

$$\sum_{n=1}^{\infty} \sec^2\left(\frac{\ln(n)}{n}\right)$$

Solution. Use the nth term test. Note that to evaluate

$$\lim_{n\to\infty} \sec^2\left(\frac{\ln(n)}{n}\right)$$

we can evaluate the limit of $\frac{\ln(n)}{n}$ and plug into $\sec^2$. Since plugging in gives $\frac{\infty}{\infty}$, we have by L'Hopital's rule that

$$\lim_{n\to\infty} \frac{\ln(n)}{n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$$

So by the continuity of $\sec^2$, we have that

$$\lim_{n\to\infty} \sec^2\left(\frac{\ln(n)}{n}\right) = \sec^2\left(\lim_{n\to\infty} \frac{\ln(n)}{n}\right) = \sec^2(0) = \frac{1}{\cos^2(0)} = \frac{1}{1} \neq 0$$

So since $\lim_{n\to\infty} a_n = 1 \neq 0$, we have that the original series diverges by the nth term test/divergence test. $\square$

Example 3.2. Does the following series converge or diverge?

$$\sum_{n=1}^{\infty} (-1)^n \frac{e^n}{2e^n - 3}$$

Solution. This example looks hard, because we want to apply the nth term test, but the $(-1)^n$ is creating sign changes. But remember that $\lim_{n\to\infty} a_n \neq 0$ is equivalent to $\lim_{n\to\infty} |a_n| \neq 0$. So let us instead show that $\lim_{n\to\infty} |a_n| \neq 0$. We have that

$$\lim_{n\to\infty} \left|(-1)^n \frac{e^n}{2e^n - 3}\right| = \lim_{n\to\infty} \frac{e^n}{2e^n - 3}$$

Plugging in, we get $\frac{\infty}{\infty}$. So we use L'Hopital's rule to get that this limit is

$$\lim_{n\to\infty} \frac{e^n}{2e^n} = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2} \neq 0$$

So we have shown that $\lim_{n\to\infty} |a_n| \neq 0$, since it is equal to $1/2$. So we conclude that $\lim_{n\to\infty} a_n \neq 0$ too. So the series diverges by the nth term test/divergence test. $\square$

As a note, some questions ask you to find the limit of a given sequence. If the sequence looks like something that might belong in a series problem, it helps to use the following form of the nth term test, which is equivalent to the previous form.

If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Let's see a quick example of this.

Example 3.3. Find the limit of the sequence

$$a_n = \frac{2^n}{n!}$$

Solution. Please forgive the forward jump in this review guide to the Ratio Test. But here, it seems hard to find the limit of the sequence $\frac{2^n}{n!}$. In fact, this seems like an expression better suited for a series problem, for example with the Ratio Test. But this is the key to doing the problem! Because we can consider the series

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}$$

To see whether this series converges or not, use the Ratio Test. We have that

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}}\right| &= \lim_{n\to\infty}\frac{2^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n} \\ &= \lim_{n\to\infty}\frac{2^{n+1}}{2^n}\cdot\frac{n!}{(n+1)!} = \lim_{n\to\infty} 2\cdot\frac{1}{n+1} = 2\cdot 0 = 0 < 1\end{aligned}$$

where we got rid of the absolute values because everything inside the absolute values was already positive. So since $0 < 1$, we have by the Ratio Test that the original series converges (absolutely). So since

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}$$

converges, we have by the nth term test in the form given right before this example that $\lim_{n\to\infty} \frac{2^n}{n!} = 0$. This gives us the solution. □

4 Tests for Nonnegative Series

We first begin for tests for the convergence or divergence of series that are non-negative. By this, we mean series of the form

$$\sum_{n=1}^{\infty} a_n$$

where $a_n \geq 0$. **When doing a series question, it is important to take moment to determine whether a series is positive or negative.** For example, a series might not "look" positive at first, but it might actually be, as in the following series.

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^3}\right)$$

At first glance, one might think that sine is a function that is both positive and negative, which is true. However, if we consider $\frac{1}{n^3}$, we see that for $n \geq 1$ (since $1/n^3$ is decreasing to zero as n increases), we have that

$$0 < \frac{1}{n^3} \leq 1 < \frac{\pi}{2}$$

so that in fact $\sin\left(\frac{1}{n^3}\right) \geq 0$ for all $n \geq 1$. Thus, this series

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^3}\right)$$

is actually a non-negative series, even though it does not seem like it on first glance. One can show that this series converges using the Limit Comparison Test (see subsection 4.3).

In the subsections that follow, we will consider three tests for convergence/divergence of nonnegative series.

4.1 Integral Test

The integral test basically says that we can convert the question of convergence or divergence of a nonnegative series into the question of convergence or divergence of an improper integral, which is something we already know how to do. Let us first state the integral test.

Series Test 4.1 (The integral test)**.** Suppose that $\sum_{n=k}^{\infty} a_n$ is a nonnegative series ($a_n \geq 0$), and

$$\sum_{n=k}^{\infty} a_n = \sum_{n=k}^{\infty} f(n)$$

for a function f that is nonnegative, continuous, and eventually decreasing (there exists N such that $f'(x) \geq 0$ for all $x \geq N$) for $x \geq k$. Then,

$$\sum_{n=k}^{\infty} a_n \text{ and } \int_k^{\infty} f(x)dx$$

either both converge or both diverge. So in particular, the convergence or divergence of the improper integral

$$\int_k^{\infty} f(x)dx$$

matches the convergence or divergence of the infinite series $\sum_{n=k}^{\infty} a_n$.

It is important to check all three conditions on f, that f is nonnegative, continuous, and eventually decreasing. To check that f is eventually decreasing, you can prove it by reasoning about what parts of f are increasing or decreasing, or if this fails, you can just calculate $f'(x)$ and show that $f'(x)$ is eventually less than or equal to 0.

Example 4.1. Does the following series converge or diverge?

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^{3/2}}$$

Solution. Consider the function

$$f(x) = \frac{1}{x(\ln(x))^{3/2}}$$

We observe that this is a function we can easily integrate using a u-substitution. But before we do that, we must check that the conditions of the integral test hold.

- f is nonnegative for $x \geq 2$ since x and $\ln(x)$ are nonnegative for $x \geq 2$.
- f is continuous for $x \geq 2$ since the zeros of the denominator are at $x = 0$ and $x = 1$, which are not in the range $x \geq 2$.
- f is decreasing because x and $\ln(x)$ (and hence $(\ln(x))^{3/2}$) are positive increasing functions, so $\frac{1}{x(\ln(x))^{3/2}}$ is decreasing.

So the conditions of the integral test hold, and we can apply the integral test.

Before we calculate the desired improper integral, let us first calculate the indefinite integral we need as a side calculation, using the u-substitution $u = \ln(x)$.

$$\int \frac{1}{x(\ln(x))^{3/2}}dx = \int \frac{1}{u^{3/2}}du = -2u^{-1/2} + C = -\frac{2}{(\ln(x))^{1/2}} + C$$

So we have that

$$\begin{aligned}\int_2^{\infty} \frac{1}{x(\ln(x))^{3/2}}dx = \lim_{N\to\infty} \int_2^{N} \frac{1}{x(\ln(x))^{3/2}}dx = \lim_{N\to\infty} \left(-\frac{2}{(\ln(N))^{1/2}} + \frac{2}{(\ln(2))^{1/2}}\right) \\ = 0 + \frac{2}{(\ln(2))^{1/2}} = \frac{2}{(\ln(2))^{1/2}}\end{aligned}$$

since $\ln(N) \to \infty$ and hence $(\ln(N))^{1/2} \to \infty$ as $N \to \infty$. So the improper integral converges. Thus, the original series converges too. $\square$

Example 4.2. Does the following series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{n^2}{e^n}$$

Solution. Consider the function

$$f(x) = \frac{x^2}{e^x} = x^2e^{-x}$$

We observe that this is a function we can easily integrate using two integration by parts. But before we do that, we must check that the conditions of the integral test hold.

- f is nonnegative because x^2 and e^{-x} are nonnegative functions.
- f is the product of two continuous functions x^2 and e^{-x}, so it is continuous too.
- We want to show f is decreasing. Since x^2 is increasing but e^{-x} is decreasing, we cannot show this using our usual qualitative methods. So we must find $f'(x)$.

  $$f'(x) = 2xe^{-x} - x^2e^{-x} = x(2-x)e^{-x}$$

  Note that $f'(x) < 0$ eventually, for $x > 2$. This is because for $x > 2$, indeed, $x > 0$ and $e^{-x} > 0$, and $2 - x < 0$. So $f'(x) < 0$ for $x > 2$. Thus, f is eventually decreasing.

So the conditions of the integral test hold, and we can apply the integral test.

First, let us calculate the indefinite integral, by integrating by parts twice.

$$\begin{aligned}\int x^2e^{-x}dx = -x^2e^{-x} + 2\int xe^{-x}dx = -x^2e^{-x} + 2\left(-xe^{-x} + \int e^{-x}dx\right) \\ = -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C\end{aligned}$$

where the two integration by parts we did are

$$u = x^2, \quad dv = e^{-x}dx$$

$$du = 2xdx, \quad v = -e^{-x}$$

and

$$u = x, \quad dv = e^{-x}dx$$

$$du = dx, \quad v = -e^{-x}$$

So we have that the improper integral is

$$\begin{aligned}\int_1^\infty x^2e^{-x}dx = \lim_{N\to\infty}\int_1^N x^2e^{-x}dx = \lim_{N\to\infty}\left(-N^2e^{-N} - 2Ne^{-N} - 2e^{-N} + e^{-1} + 2e^{-1} + 2e^{-1}\right) \\ = 5e^{-1} + \lim_{N\to\infty}\left(-N^2e^{-N} - 2Ne^{-N} - 2e^{-N}\right)\end{aligned}$$

We have some limits to compute. The following limit is an indeterminate form $\infty \cdot 0$ which we can turn into an indeterminate quotient that we can apply L'Hopital's rule to twice (since L'Hopital the first time still produces $\frac{\infty}{\infty}$).

$$\lim_{N\to\infty} N^2e^{-N} = \lim_{N\to\infty}\frac{N^2}{e^N} = \lim_{N\to\infty}\frac{2N}{e^N} = \lim_{N\to\infty}\frac{2}{e^N} = 0$$

The second limit is $\infty \cdot 0$, which we can turn into an indeterminate quotient $\frac{\infty}{\infty}$ which we can apply L'Hopital's rule to.

$$\lim_{N\to\infty} 2Ne^{-N} = \lim_{N\to\infty} \frac{2N}{e^N} = \lim_{N\to\infty} \frac{2}{e^N} = 0$$

The final limit is easy.

$$\lim_{N\to\infty} 2e^{-N} = 0$$

So we have that

$$\int_1^\infty x^2 e^{-x} dx = 5e^{-1} + (-0 - 0 - 0) = 5e^{-1}$$

So since the improper integral converges, the original series converges too. $\square$

4.2 Direct Comparison Test

Let's start with the intuitive idea of the Direct Comparison Test, before stating it. Suppose I gave you these two finite sums.

$$1 + 4 + 7 + 8 + 10$$

$$2 + 5 + 8 + 9 + 12$$

Can you tell which is larger, without calculating the sums? The bottom sum is clearly larger since every term being added in the bottom sum is greater than or equal to the corresponding term being added in the top sum. This is exactly the idea of the Direct Comparison Test.

In particular, suppose that $\sum a_n$ and $\sum b_n$ are both positive series, and we know that $a_n \leq b_n$. Then, intuitively,

$$0 \leq \sum a_n \leq \sum b_n$$

So if $\sum a_n$ diverges (is infinity in value), then $\sum b_n$ must also be infinity and must also diverge, since the only thing greater than or equal to infinity is infinity itself. And if $\sum b_n$ converges (is some finite nonnegative number), then because $\sum a_n$ is between 0 and $\sum b_n$, we must have that $\sum a_n$ is finite too, so that $\sum a_n$ converges.

Now, just like the comparison test for improper integrals, there are cases where this test is inconclusive. For example, if we know that $\sum a_n$ converges to some finite positive number, we do not know anything about $\sum b_n$, since $\sum b_n$ could simply be a larger finite positive number (and hence converge) or $\sum b_n$ could be infinity (and hence diverge). Similarly, if $\sum b_n$ diverges, this tells us nothing about $\sum a_n$, since $\sum a_n$ could be finite (and hence converge) or $\sum a_n$ could be infinity (and hence diverge) since $0 < \infty \leq \infty$ is valid.

We formally state the Direct Comparison Test as follows.

Series Test 4.2 (Direct Comparison Test)**.** Suppose $\sum a_n$ and $\sum b_n$ are nonnegative series (so $a_n \geq 0$, $b_n \geq 0$), with $a_n \leq b_n$.

- If $\sum b_n$ converges, then $\sum a_n$ converges too.
- If $\sum a_n$ diverges, then $\sum b_n$ diverges too.

- If $\sum b_n$ diverges, we know nothing about the convergence or divergence of $\sum a_n$.
- If $\sum a_n$ converges, we know nothing about the convergence or divergence of $\sum b_n$.

The trick to knowing what series to compare a given series to is finding the "essence" of a series. This consists of reducing the series to its barest parts, and thinking about the dominant terms only.

Example 4.3. Determine whether the series

$$\sum_{n=2}^{\infty} \frac{\ln(n)}{\sqrt{n}}$$

converges or diverges.

Guess. Start with what I call the "guess" phase. **This happens in your head and helps you know what you are aiming for (but don't write it down as an answer!)**. This is where you take a guess before you write up your solution, by getting an idea of how the different parts of the series are behaving. For example, the bottom is $n^{1/2}$. We know $\sum \frac{1}{n^{1/2}}$ is divergent, and since $\ln(n)$ is increasing to infinity as n increases, the $\ln(n)$ is making the series "even more divergent." So we guess that the series is divergent. $\square$

Solution. Now we need to give a rigorous solution. Use Direct Comparison. Since $\ln(x)$ is increasing, we have for $n \geq 2$ that

$$\ln(n) \geq \ln(2) > 0$$

for $n \geq 2$. (Note it is incorrect to say $\ln(n) \geq 1$ for $n \geq 2$ since $\ln(2)$ is actually less than 1). So then,

$$\frac{\ln(n)}{\sqrt{n}} \geq \frac{\ln(2)}{\sqrt{n}} > 0$$

Since $\sum \frac{\ln(2)}{\sqrt{n}} = \ln(2) \sum \frac{1}{\sqrt{n}}$ diverges by the p-test, the original series also diverges by Direct Comparison.

$\square$

Example 4.4. Determine whether the series

$$\sum_{n=4}^{\infty} \left(\frac{4}{3}\right)^n \sec\left(\frac{\pi}{n}\right)$$

converges or diverges.

Guess. Take a guess. We do not know what $\sec(\pi/n)$ is doing at first, but we immediately see the $(4/3)^n$, which suggests that this series should be compared to the divergent geometric series $\sum (4/3)^n$ somehow (divergent, since $r = 4/3$ has $|r| > 1$). So we guess that this series is divergent. $\square$

Solution. Let us understand what is going on with the $\sec(\pi/n)$ term. We note that π/n is decreasing to 0 as $n \to \infty$. So for $n \geq 4$,

$$0 < \frac{\pi}{n} \leq \frac{\pi}{4}$$

Note that $\sec(x)$ is an increasing function. Using this fact (or by looking at a graph of secant),

$$1 < \sec\left(\frac{\pi}{n}\right) \leq \sqrt{2} = \sec\left(\frac{\pi}{4}\right)$$

Since our guess was that this series diverges, use only the first part of this inequality. So for $n \geq 4$,

$$\sec\left(\frac{\pi}{n}\right) > 1$$

Thus,

$$\left(\frac{4}{3}\right)^n \sec\left(\frac{\pi}{n}\right) > \left(\frac{4}{3}\right)^n > 0$$

Since $\sum(4/3)^n$ diverges, by Direct Comparison, the original series diverges also. □

Example 4.5. Determine whether the series

$$\sum_{n=1}^{\infty} \frac{n-1}{n^3 + \ln(n)}$$

converges or diverges.

Guess. Take a guess. Remember that the behavior as $n \to \infty$ of the terms you are adding is what makes a series converge or diverge. So what is $\frac{n-1}{n^3+\ln(n)}$ behaving like as $n \to \infty$? On the top, the dominant term is n, and on the bottom the dominant term is n^3, since $\ln(n)$ grows much slower than n to any positive power. So the series is essentially $\sum \frac{n}{n^3} = \sum \frac{1}{n^2}$. So we expect the series to converge. □

Solution. Now, we show that the series converges rigorously, using Direct Comparison. Note that for $n \geq 1$,

$$\frac{n}{n^3} \geq \frac{n-1}{n^3} \geq \frac{n-1}{n^3 + \ln(n)} \geq 0$$

where in the first inequality, we used the fact that bigger numerator means bigger number, and in the second inequality, we used the fact that smaller denominator means bigger number. So since

$$0 \leq \frac{n-1}{n^3 + \ln(n)} \leq \frac{n}{n^3} = \frac{1}{n^2}$$

and since $\sum \frac{1}{n^2}$ converges, we have that the original series converges by Direct Comparison. □

4.3 Limit Comparison Test

The Limit Comparison Test, like the Direct Comparison Test, requires comparing to another series that we already know the convergence or divergence of (usually a geometric series or a p-series). However, it is different in that it works very well when the Direct Comparison Test fails (because the direct comparison inequality gives an inconclusive result, or because a Direct Comparison is not immediately obvious). The Limit Comparison Test is stated below.

Series Test 4.3 (Limit Comparison Test)**.** Let $\sum a_n$ and $\sum b_n$ be nonnegative series ($a_n \geq 0$, $b_n \geq 0$). If

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}$$

is a finite, positive number, then $\sum a_n$ and $\sum b_n$ converge or diverge the same. The statement also holds if we take $L = \lim_{n\to\infty} \frac{b_n}{a_n}$. **If $L = 0$ or $L = \infty$, the test is inconclusive.**

It does not matter whether we take $L = \lim_{n\to\infty} \frac{a_n}{b_n}$ or $L = \lim_{n\to\infty} \frac{b_n}{a_n}$, because as you may notice, these two limits are just reciprocals of each other. One is sometimes easier to evaluate than the other. If you choose one of the two limits (either b_n/a_n or a_n/b_n) to evaluate and you notice that the limit would be easier if the fraction you have is flipped, just take the reciprocal of your current limit and keep going with your calculation.

Just like for Direct Comparison, for the Limit Comparison Test, the trick to knowing what series to compare a given series to is finding the "essence" of a series. This consists of reducing the series to its barest parts, and thinking about the dominant terms only.

Example 4.6. Determine whether the series

$$\sum_{n=1}^{\infty} \frac{n^2 + n^3 \sin(1/n)}{n^3}$$

converges or diverges.

Guess. We first get an idea for what we think is going on. As $n \to \infty$, since sine is bounded between -1 and 1, we have that the dominant term in the numerator is n^2. Though n^3 is the bigger power here, remember that using a tangent line approximation at $x = 0$, $\sin(x) \sim x$, we have that $\sin(1/n)$ is basically $1/n$. So the series behaves like $\sum \frac{n^2+n^3 \cdot 1/n}{n^3} = \frac{2}{n}$ which diverges. (This might be hard to see, since one might think $n^3\sin(1/n)$ has the highest power, but we'll see this when we do the limit comparison rigorously). In particular, the "essence" of this series is $\sum \frac{1}{n}$ (we took away the constant because constants don't matter for convergence or divergence, but you could keep the constant there). So we guess that the original series diverges. $\square$

Solution. Because $\sin(1/n)$ changes signs, it is not possible to compare the series directly to $\sum \frac{1}{n}$. Since direct comparison fails here, we try limit comparison with $\sum \frac{1}{n}$. We calculate

$$L = \lim_{n\to\infty} \frac{\frac{n^2+n^3\sin(1/n)}{n^3}}{1/n} = \lim_{n\to\infty} \frac{n^3 + n^4 \sin(1/n)}{n^3} = \lim_{n\to\infty} 1 + n \cdot \sin(1/n)$$

The limit of $n \cdot \sin(1/n)$ upon plugging in gives the indeterminate form $\infty \cdot 0$. So we need to change this into a quotient. Once we do this, we get $\frac{0}{0}$ and we can apply L'Hopital's rule.

$$L = \lim_{n\to\infty} 1 + n \cdot \sin(1/n) = \lim_{n\to\infty} 1 + \frac{\sin(1/n)}{1/n} = \lim_{n\to\infty} 1 + \frac{(-1/n^2)\cos(1/n)}{-1/n^2}$$
$$= \lim_{n\to\infty} 1 + \cos(1/n) = 2$$

Since $L = 2$ is greater than 0 and finite, and since $\sum \frac{1}{n}$ diverges, we conclude by the limit comparison test that the original series diverges too. □

Example 4.7. Determine whether the series

$$\sum_{n=1}^{\infty} \frac{5^n}{2^n + 4^n}$$

converges or diverges.

Guess. The numerator acts like 5^n. The dominant term on the bottom is 4^n since 4^n grows faster than 2^n. So the "essence" of the series is $\sum \frac{5^n}{4^n} = \sum \left(\frac{5}{4}\right)^n$, which is a divergent geometric series. So we guess that the original series diverges. □

Proof. If we try direct comparison, we get $0 \le \frac{5^n}{2^n+4^n} \le \frac{5^n}{4^n}$. But the comparison is going the wrong way, so the direct comparison test is inconclusive. So in this case, we should try limit comparison instead. We saw that the essence of this series is $\sum \frac{5^n}{4^n}$, so use limit comparison with this series.

$$L = \lim_{n\to\infty} \frac{\frac{5^n}{4^n}}{\frac{5^n}{2^n+4^n}} = \lim_{n\to\infty} \frac{2^n + 4^n}{4^n} = \lim_{n\to\infty} \frac{2^n}{4^n} + 1 = \lim_{n\to\infty} \left(\frac{1}{2}\right)^n + 1 = 0 + 1 = 1$$

(Note this limit is easier than the other way, $\lim_{n\to\infty} \frac{\frac{5^n}{2^n+4^n}}{\frac{5^n}{4^n}}$, but both will give the same answer). Since $L = 1$ is > 0 and finite, both series converge or diverge the same. Since $\sum \frac{5^n}{4^n}$ is a divergent geometric series, the original series also diverges too.

□

Example 4.8 (Tangent line approximation)**.** Determine whether the series

$$\sum_{n=1}^{\infty} \tan\left(\frac{1}{n^3}\right)$$

converges or diverges.

Guess. Note that $1/n^3 \to 0$ as $n \to \infty$. So to get an idea for this series, we should understand how tangent behaves near 0. The best way to do this is to use the tangent line approximation. What is the tangent line for $y = \tan(x)$ at $x = 0$? The line must pass through $(0, y(0)) = (0, 0)$ and must have slope $y'(0) = \sec^2(0) = 1$. Thus, the tangent line at $x = 0$ to $y = \tan(x)$ is $y = x$. This means that for small x, $\tan(x)$ is approximately x. So as $n \to \infty$, since $1/n^3 \to 0$, we have that $\tan(1/n^3)$ behaves like $1/n^3$. So the "essence" of the series is $\sum \frac{1}{n^3}$ which converges. So we guess that the original series converges too. □

Solution. Now, we need to show this rigorously. It is not clear how to do direct comparison here, so use limit comparison. Let us limit compare to the essence of the series, $\sum \frac{1}{n^3}$. We calculate

$$L = \lim_{n\to\infty} \frac{\tan(1/n^3)}{1/n^3}$$

Plugging in, we get $\tan(0)/0 = 0/0$, which is an indeterminate quotient. So use L'Hopital's rule to get

$$L = \lim_{n\to\infty} \frac{(-3/n^4)\sec^2(1/n^3)}{-3/n^4} = \lim_{n\to\infty} \sec^2(1/n^3) = \sec^2(0) = 1$$

Since $L > 0$ is finite, both series converge or diverge the same. Since $\sum \frac{1}{n^3}$ converges, the original series converges too. □

5 Series Whose Terms Change Signs

The tests we learned in the past section only worked for nonnegative series (though note that the nth term test works for arbitrary series). What happens when our series change signs? Sometimes, series can change signs somewhat randomly, as in

$$\sum_{n=1}^{\infty} \sin(n) \cong 0.8415 + 0.9093 + 0.1411 - 0.7568 + 0.0872 + ...$$

However, the sign changes in some special series alternate back and forth between positive and negative. These are called **alternating series**. Examples of this are

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + ...$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + ...$$

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}} = -1 + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{5}} + ...$$

where in the last series, we use the fact that $\cos(n\pi) = (-1)^n$ (check this!). Usually, we will be able to say more about the convergence of alternating series, using the Alternating Series Test. In the next subsection, we will talk about a general strategy to handle general series with arbitrary signs and alternating series. After that, we will review the Ratio Test and the Root Test which also work for series with arbitrary signs.

5.1 Absolute Convergence and Conditional Convergence

The first strategy when you see any series with arbitrary signs (not just alternating series) is just to get rid of the signs. This is useful because of the following result.

If $\sum |a_n|$ converges, then $\sum a_n$ converges too.

However, **if $\sum |a_n|$ diverges, we know nothing about the convergence or divergence of $\sum a_n$**. For example, we know that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges by the fact above, since

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges. But what about a series like $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$? If we try to take away the signs,

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges, so we know nothing about the convergence or divergence of the original series, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$. So what do we do? When this happens, if we are working with an alternating series, we can use the Alternating Series Test.

Series Test 5.1 (Alternating Series Test)**.** Suppose that $\sum_{n=1}^{\infty} a_n$ is an alternating series (which will usually have $(-1)^n$, $(-1)^{n-1}$, or $(-1)^{n+1}$ in it). Let $b_n = |a_n|$ (so b_n is a_n without the alternating signs, so that $b_n \geq 0$. If

- b_n is (eventually) decreasing
- $\lim_{n\to\infty} b_n = 0$

then the original alternating series $\sum_{n=1}^{\infty} a_n$ converges. Note that this test says nothing about the type of converge (conditional vs. absolutely, which we will mention later). Also, if the conditions for the alternating series test do not hold, we know nothing about the convergence or divergence of the original series.

So if we go back to our previous example, $\sum (-1)^n \frac{1}{n}$, we see that this is an alternating series (because of the $(-1)^n$). Let's try to use the Alternating Series Test. Here, $b_n = 1/n$, and we can see that b_n is increasing and $\lim_{n\to\infty} b_n = 0$. So by the Alternating Series Test, the original alternating series $\sum (-1)^n \frac{1}{n}$ converges.

Note that the Alternating Series Test does not show divergence. In cases where we cannot use the Ratio or Root Test, we can get divergence from the nth term test. For example, $\sum (-1)^n n$ diverges, because

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} |(-1)^n n| = \lim_{n\to\infty} n = \infty \neq 0$$

Since $\lim_{n\to\infty} |a_n| \neq 0$, this is equivalent to $\lim_{n\to\infty} a_n \neq 0$, so the series $\sum (-1)^n n$ diverges by the nth term test/divergence test.

We see that we can be more specific about types of convergence, since $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ converged in "different" ways. In particular, $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$ converges since $\sum |a_n|$ converges, but for $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$, $\sum |a_n|$ diverges though the original series with the sign changes converges. We say the first series $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$ **converges absolutely**, and the second series $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ **converges conditionally**.

- A series $\sum a_n$ **converges absolutely** if $\sum |a_n|$ converges and hence $\sum a_n$ converges too.
- A series $\sum a_n$ **converges conditionally** if $\sum |a_n|$ diverges, but the original series $\sum a_n$ converges.

These are two more specific types of convergence for arbitrary series with general sign changes.

As an important note, why did we not have the notions of absolute vs. conditional convergence for nonnegative series? Well, for nonnegative series, $\sum |a_n| = \sum a_n$. So nonnegative series cannot converge conditionally. The only type of convergence for nonnegative series is absolute convergence.

When dealing with series with general signs,

- First try the nth term test to see if the series diverges. (Usually, you use the fact that $\lim_{n\to\infty}|a_n| \neq 0$ is equivalent to $\lim_{n\to\infty}a_n \neq 0$).
- Consider $\sum |a_n|$ BEFORE doing anything else. If $\sum |a_n|$ converges, then the original series $\sum a_n$ converges too and converges absolutely.
- If $\sum |a_n|$ diverges, if $\sum a_n$ is an alternating series, try to apply the Alternating Series Test to see if the series converges. If the series converges but $\sum |a_n|$ diverged, the original series $\sum a_n$ is conditionally convergent.
- Another way to approach this is just to use the Ratio or Root Test (see the next subsection).

Example 5.1. Determine whether the series

$$\sum_{n=2}^{\infty} (-1)^{n^2} \sqrt{\ln(n)}$$

converges absolutely, converges conditionally, or diverges.

Solution. Remember that we should just think about what the series is doing without the signs. If we take away the signs, we are left with $\sqrt{\ln(n)}$, which goes to infinity as n goes to infinity. So we get a feeling that this series diverges. How do we show this rigorously? Use the nth term test. In particular,

$$\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}|(-1)^{n^2}\sqrt{\ln(n)}| = \lim_{n\to\infty}\sqrt{\ln(n)} = \infty \neq 0$$

So since $\lim_{n\to\infty}|a_n| \neq 0$, we have that $\lim_{n\to\infty}a_n \neq 0$ too. So the series diverges by the nth term test. □

Example 5.2. Determine whether the series

$$\sum_{n=2}^{\infty} \frac{\cos(n)}{n^2 - n}$$

converges absolutely, converges conditionally, or diverges.

Solution. The first step is to consider the absolute value of the series. We have that

$$\sum |a_n| = \sum_{n=2}^{\infty} \frac{|\cos(n)|}{n^2 - n}$$

Note that

$$\frac{|\cos(n)|}{n^2 - n} \leq \frac{1}{n^2 - n}$$

We note that $\sum \frac{1}{n^2-n}$ converges by Limit Comparison with $\sum \frac{1}{n^2}$ (note that Direct Comparison Test on $\sum \frac{1}{n^2-n}$ is inconclusive in this case since $1/(n^2 - n) > 1/n^2$). Since $\sum \frac{1}{n^2-n}$ converges, the original series $\sum_{n=2}^{\infty} \frac{|\cos(n)|}{n^2-n}$ converges too by Direct Comparison. So since $\sum |a_n|$ converges, the original series $\sum_{n=2}^{\infty} \frac{\cos(n)}{n^2-n}$ converges too, and hence converges absolutely. □

Example 5.3. Determine whether the series

$$\sum_{n=2}^{\infty} \frac{(-1)^{n-1} \ln(n)}{n}$$

converges absolutely, converges conditionally, or diverges.

Solution. If we think of $|a_n| = \frac{\ln(n)}{n}$, one can see that $|a_n| \to 0$, and hence $a_n \to 0$, so the nth term test is inconclusive. So now consider $\sum |a_n|$. We have that

$$\sum |a_n| = \sum_{n=2}^{\infty} \frac{\ln(n)}{n}$$

Since $\ln(n)$ is increasing, we have that for $n \geq 2$,

$$\frac{\ln(n)}{n} \geq \frac{\ln(2)}{n} > 0$$

Since $\sum \frac{\ln(2)}{n} = \ln(2) \sum \frac{1}{n}$ diverges, we have that $\sum \frac{\ln(n)}{n}$ diverges by direct comparison. So $\sum |a_n|$ diverges. So we have to do more work.

We see that this series is an alternating series, with $b_n = |a_n| = \frac{\ln(n)}{n}$. We check the necessary conditions.

- b_n is decreasing. To see this, let $f(x) = \frac{\ln(x)}{x}$. Then,

  $$f'(x) = \frac{x \cdot \frac{1}{x} - \ln(x)}{x^2} = \frac{1 - \ln(x)}{x^2}$$

  So $f'(x) < 0$ for $x > e$ (since $1 - \ln(x) = 0$ at $x = e$, and $1 - \ln(x)$ is decreasing, since $\ln(x)$ is increasing). So f is eventually decreasing, and hence b_n is eventually decreasing.

- $\lim_{n\to\infty} b_n = 0$. To check this, we plug into the limit and get ∞/∞, so we use L'Hopital's Rule.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{\ln(n)}{n} = \lim_{n\to\infty}\frac{1/n}{1} = 0$$

So the conditions for the Alternating Series Test are met. So by the Alternating Series Test, the original series converges, but $\sum |a_n|$ diverges. So the original series $\sum_{n=2}^{\infty}\frac{(-1)^{n-1}\ln(n)}{n}$ converges conditionally. $\square$

5.2 Ratio Test and Root Test

The Ratio Test and the Root Test are tests for arbitrary series (where the terms can have any signs whatsoever). It only has three outcomes: the series is absolutely convergent, the series is divergent, and inconclusive. The Ratio Test is useful whenever there are factorials, or products with patterns of adding by a fixed number to get to the next number to multiply. The Root Test is useful whenever there are numbers or variables to powers involving n. We state the two tests below.

Series Test 5.2 (Ratio Test)**.** Consider the series $\sum a_n$ (where the a_n can have any sign). Define

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

- If $L < 1$, then $\sum a_n$ converges absolutely.
- If $L > 1$, then $\sum a_n$ diverges.
- If $L = 1$, the test is inconclusive.

Series Test 5.3 (Root Test)**.** Consider the series $\sum a_n$ (where the a_n can have any sign). Define

$$L = \lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}|a_n|^{1/n}$$

- If $L < 1$, then $\sum a_n$ converges absolutely.
- If $L > 1$, then $\sum a_n$ diverges.
- If $L = 1$, the test is inconclusive.

Before going into examples, let us just briefly recall what a factorial is. A factorial means, take that number, take all positive integers less than or equal to that number, and multiply them all together. For example,

$$5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$$

$$6! = 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 720$$

Observe for example that $6! = 6 \cdot (5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) = 6\dot{5}!$. So $\frac{6!}{5!} = 6$. And similarly, $\frac{6!}{4!} = 6 \cdot 5$. Check the following more general statements.

$$\frac{(n+1)!}{n!} = n+1, \quad \frac{(2n+2)!}{(2n)!} = \frac{(2n+2)(2n+1)(2n)!}{(2n)!} = (2n+2)(2n+1),$$

$$\frac{(3n+1)!}{(3n-2)!} = (3n+1)(3n)(3n-1)$$

Another important piece of notation that is confusing is illustrated by the following.

$$2 \cdot 5 \cdot 8 \cdot ... \cdot (3n-1)$$

What does this mean? This means that given $n \geq 1$, start at 2, keep adding 3 until you get to $3n-1$, and multiply everything together. For example, for $n = 1$, $3n-1$, so the above is just 2. For $n = 2$, $3n-1 = 5$ so the above becomes $2 \cdot 5$. For $n = 3$, $3n-1 = 8$, so the above is just $2 \cdot 5 \cdot 8$. For $n = 4$, $3n-1 = 11$, so the above is just $2 \cdot 5 \cdot 8 \cdot 11$. For $n = 5$, $3n-1 = 14$, so the above is just $2 \cdot 5 \cdot 8 \cdot 11 \cdot 14$. Let us try to find

$$\frac{2 \cdot 5 \cdot 8 \cdot ... \cdot (3(n+1)-1)}{2 \cdot 5 \cdot 8 \cdot ... \cdot (3n-1)}$$

How would we do this? Maybe, putting in numbers first, for example $n = 4$, we get

$$\frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 14}{2 \cdot 5 \cdot 8 \cdot 11} = 14$$

In particular, the top is the product of $n+1$ terms going up by threes, and the bottom is the product of n terms going up by threes. So the bottom n terms will cancel the corresponding terms on top. To make this clear, it helps to write the term before the last term, **just on the part that has the additional term, in this case the numerator**. So

$$\frac{2 \cdot 5 \cdot 8 \cdot ... \cdot (3(n+1)-1)}{2 \cdot 5 \cdot 8 \cdot ... \cdot (3n-1)} = \frac{2 \cdot 5 \cdot 8 \cdot ... \cdot (3n+2)}{2 \cdot 5 \cdot 8 \cdot ... \cdot (3n-1)}$$
$$= \frac{2 \cdot 5 \cdot 8 \cdot ... \cdot (3n-1) \cdot (3n+2)}{2 \cdot 5 \cdot 8 \cdot ... \cdot (3n-1)} = 3n+2$$

Now that we have covered the notation necessary for these problems, let's look at some examples.

Example 5.4. Determine whether the following series converges absolutely, converges conditionally, or diverges.

$$\sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{4n}{5n + \ln(n)} \right)^n$$

Solution. Here, we see n as a power, so the Root Test would be useful here. We calculate

$$L = \lim_{n\to\infty} \sqrt[n]{\left| (-1)^{n-1} \left(\frac{4n}{5n + \ln(n)} \right)^n \right|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{4n}{5n + \ln(n)} \right)^n} = \lim_{n\to\infty} \frac{4n}{5n + \ln(n)}$$

Plugging into this limit gives $\frac{\infty}{\infty}$, which is an indeterminate quotient. So using L'Hopital's Rule, we have that

$$L = \lim_{n\to\infty} \frac{4}{5+\frac{1}{n}} = \frac{4}{5+0} = \frac{4}{5} < 1$$

So since $L < 1$, we have by the Root Test that the given series converges absolutely. $\square$

Example 5.5. Determine whether the following series converges absolutely, converges conditionally, or diverges.

$$\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2 \cdot 7 \cdot 12 \cdot \ldots \cdot (5n-3)}$$

Proof. Since there are "inductive" products, use the Ratio Test. Remember in the $n+1$ term, write the term before the final term in the sum. So we have that

$$\begin{aligned} L = \lim_{n\to\infty} \left| \frac{\frac{1\cdot3\cdot5\cdot\ldots\cdot(2(n+1)-1)}{2\cdot7\cdot12\cdot\ldots\cdot(5(n+1)-3)}}{\frac{1\cdot3\cdot5\cdot\ldots\cdot(2n-1)}{2\cdot7\cdot12\cdot\ldots\cdot(5n-3)}} \right| &= \lim_{n\to\infty} \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n+1)}{2 \cdot 7 \cdot 12 \cdot \ldots \cdot (5n+2)} \cdot \frac{2 \cdot 7 \cdot 12 \cdot \ldots \cdot (5n-3)}{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)} \\ &= \lim_{n\to\infty} \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1) \cdot (2n+1)}{2 \cdot 7 \cdot 12 \cdot \ldots \cdot (5n-3) \cdot (5n+2)} \cdot \frac{2 \cdot 7 \cdot 12 \cdot \ldots \cdot (5n-3)}{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)} \\ &= \lim_{n\to\infty} \frac{2n+1}{5n+2} = \frac{2}{5} < 1 \end{aligned}$$

Since $L = 2/5 < 1$, we have by the Ratio Test that the given series converges absolutely. $\square$

Example 5.6. Determine whether the following series converges absolutely, converges conditionally, or diverges.

$$\sum_{n=1}^{\infty} \frac{(-1)^n (3n)!}{n! \cdot [1 \cdot 4 \cdot 7 \cdot \ldots \cdot (3n-2)]}$$

Solution. Since there are "inductive" products and factorials, use the Ratio Test. We calculate

$$\begin{aligned} L = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}(3(n+1))!}{(n+1)!\cdot[1\cdot4\cdot7\cdot\ldots\cdot(3(n+1)-2)]}}{\frac{(-1)^n(3n)!}{n!\cdot[1\cdot4\cdot7\cdot\ldots\cdot(3n-2)]}} \right| &= \lim_{n\to\infty} \frac{\frac{(3n+3)!}{(n+1)!\cdot[1\cdot4\cdot7\cdot\ldots\cdot(3n+1)]}}{\frac{(3n)!}{n!\cdot[1\cdot4\cdot7\cdot\ldots\cdot(3n-2)]}} \\ &= \lim_{n\to\infty} \frac{(3n+3)!}{(n+1)! \cdot [1 \cdot 4 \cdot 7 \cdot \ldots \cdot (3n-2) \cdot (3n+1)]} \cdot \frac{n! \cdot [1 \cdot 4 \cdot 7 \cdot \ldots \cdot (3n-2)]}{(3n)!} \\ &= \lim_{n\to\infty} \frac{(3n+3)(3n+2)(3n+1)}{(n+1)(3n+1)} = \infty > 1 \end{aligned}$$

where the limit is infinity, because the degree of the numerator is greater than the degree of the denominator. Since $L = \infty > 1$, the original series diverges by the Ratio Test. $\square$

Example 5.7. Determine whether the following series converges absolutely, converges conditionally, or diverges.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n^2} 2^n n^2 \ln(n)}{n!}$$

Solution. Use the Ratio Test. Consider $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$.

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{(n+1)^2}2^{n+1}(n+1)^2\ln(n+1)}{(n+1)!}}{\frac{(-1)^{n^2}2^n n^2\ln(n)}{n!}}\right| = \lim_{n\to\infty}\frac{\frac{2^{n+1}(n+1)^2\ln(n+1)}{(n+1)!}}{\frac{2^n n^2\ln(n)}{n!}}$$

$$= \lim_{n\to\infty}\frac{2^{n+1}(n+1)^2\ln(n+1)}{(n+1)!}\cdot\frac{n!}{2^n n^2\ln(n)}$$

Now group like terms together as separate fractions. For example, group 2^{n+1} and 2^n on the same fraction, since they both come from the same 2^n term in the original series. And group $(n+1)^2$ and n^2 together since they both come from the n^2 term. Doing all of these groupings, we get

$$\lim_{n\to\infty}\frac{2^{n+1}}{2^n}\cdot\frac{n!}{(n+1)!}\cdot\frac{(n+1)^2}{n^2}\cdot\frac{\ln(n+1)}{\ln(n)} = \lim_{n\to\infty} 2\cdot\frac{1}{n+1}\cdot\frac{(n+1)^2}{n^2}\cdot\frac{\ln(n+1)}{\ln(n)}$$

Now, consider each of the limits of the terms in the product.

$$\lim_{n\to\infty} 2 = 2$$

$$\lim_{n\to\infty}\frac{1}{n+1} = 0$$

$$\lim_{n\to\infty}\frac{(n+1)^2}{n^2} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^2 = \left(\lim_{n\to\infty}\frac{n+1}{n}\right)^2 = 1^2 = 1$$

where we are using the fact that squaring is a continuous operation ($y = x^2$ is a continuous function). Finally, for the last limit, we need to use L'Hopital's rule, since plugging in gives us an indeterminate quotient, $\frac{\infty}{\infty}$.

$$\lim_{n\to\infty}\frac{\ln(n+1)}{\ln(n)} = \lim_{n\to\infty}\frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{n+1} = 1$$

Putting it all together, we have that

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{(n+1)^2}2^{n+1}(n+1)^2\ln(n+1)}{(n+1)!}}{\frac{(-1)^{n^2}2^n n^2\ln(n)}{n!}}\right| = \lim_{n\to\infty} 2\cdot\frac{1}{n+1}\cdot\frac{(n+1)^2}{n^2}\cdot\frac{\ln(n+1)}{\ln(n)} = 2\cdot 0\cdot 1\cdot 1 = 0$$

Since $0 < 1$, we have by the Ratio Test that the original series converges absolutely. □

6 Power Series

At its root, a power series is just a function, where the values of the function are expressed as series. For example, if we consider the function

$$f(x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$$

this is a function of x, where we can find $f(1/2)$ for example, by plugging in $1/2$ for x. Then, $f(1/2)$ would be

$$f(1/2) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(1/2)^n}{n}$$

If this series on the right hand side converges, then it makes sense to say that $f(1/2)$ is equal to the value of that convergent series. However, if the series on the right hand side diverges, it doesn't make sense to consider $f(1/2)$ since we cannot reasonably assign values to divergent series.

So given a power series, the question we are interested in is - for what x does it make sense to think of the power series as a function of x? Equivalently, what is the domain of f? Or equivalently, which x values give us a convergent series when we plug them into the power series? To solve this, we use the Ratio or Root Test and treat the x variable as a constant relative to n. Let's do the example above in the following example.

Example 6.1. For what x values does the following power series converge?

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

Solution. Use the Ratio Test. We have that

$$\lim_{n\to\infty} \left| \frac{(-1)^n \frac{x^{n+1}}{n+1}}{(-1)^{n-1} \frac{x^n}{n}} \right| = \lim_{n\to\infty} \frac{|x|^{n+1}}{n+1} \cdot \frac{n}{|x|^n} = \lim_{n\to\infty} \frac{|x|^{n+1}}{|x|^n} \cdot \frac{n}{n+1} = \lim_{n\to\infty} |x| \cdot \frac{n}{n+1} = |x|$$

Note that we put absolute values around x since x could be negative. So by the Ratio Test, if $|x| < 1$, the power series converges. If $|x| > 1$, the power series diverges. But remember that if $|x| = 1$, the Ratio Test is inconclusive. So we need to test whether the power series is convergent for $x = 1$ and $x = -1$ explicitly. Let us first try $x = 1$. Plugging in $x = 1$, the power series becomes

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

which converges by the Alternating Series Test. So the power series converges at $x = 1$.

Now, consider $x = -1$. If we substitute $x = -1$ into the power series, we get

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{n}$$

When we have -1 to weird powers, it helps to think about whether the power is odd or even. Here, the power $2n - 1$ is odd, so $(-1)^{2n-1} = -1$. Thus, if we substitute $x = -1$ into the power series, we get

$$\sum_{n=1}^{\infty} -\frac{1}{n}$$

which diverges (it is negative the harmonic series, which diverges).

So we have that our series diverges for all $|x| < 1$ and also $x = 1$. So the power series converges for x in $(-1, 1]$. The center of this **interval of convergence** is $x = 0$, and the radius of convergence (half the length of the interval of convergence) is 1.

□

Not all intervals of convergence of power series are centered at $x = 0$, as the next example shows.

Example 6.2. For what x values does the following power series converge?

$$\sum_{n=1}^{\infty} \left(\frac{2n}{n+3}\right)^n (2x+1)^n$$

Solution. This looks tricky, but the key here is to notice that there is an exponent with an n. So we want to use the Root Test. We have that

$$\begin{aligned}\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{2n}{n+3}\right)^n (2x+1)^n\right|} &= \lim_{n\to\infty} \sqrt[n]{\left(\frac{2n}{n+3}\right)^n |2x+1|^n} \\ &= \lim_{n\to\infty} \frac{2n}{n+3} \cdot |2x+1| = 2 \cdot |2x+1|\end{aligned}$$

So by the Root Test, we have three cases.

- If $2 \cdot |2x+1| < 1$ (so $|2x+1| < 1/2$, so $-3/4 < x < -1/4$), then the series converges by the Root Test.
- If $2 \cdot |2x+1| > 1$ (so $|2x+1| > 1/2$, so $x < -3/4$ or $x > -1/4$), then the series diverges by the Root Test.
- If $2 \cdot |2x+1| = 1$ (so $|2x+1| = 1/2$, so $x = -3/4, -1/4$), then the Root Test is inconclusive. So we test these endpoints manually. If we plug in $x = -3/4$, we get

  $$\sum_{n=1}^{\infty} \left(\frac{2n}{n+3}\right)^n (2(-3/4)+1)^n = \sum_{n=1}^{\infty} \left(\frac{2n}{n+3}\right)^n \left(-\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} (-1)^n \left(\frac{n}{n+3}\right)^n$$

  If we plug in $x = -1/4$, we get

  $$\sum_{n=1}^{\infty} \left(\frac{2n}{n+3}\right)^n (2(-1/4)+1)^n = \sum_{n=1}^{\infty} \left(\frac{2n}{n+3}\right)^n \left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \left(\frac{n}{n+3}\right)^n$$

  We can show that both of these series diverge in the same way, using the nth term test. This is because

  $$\lim_{n\to\infty} \left(\frac{n}{n+3}\right)^n \neq 0$$

  To calculate this limit, we plug in and get 1^∞, which is an indeterminate power. So the trick here is to take a natural logarithm.

  $$\lim_{n\to\infty} \ln\left(\left(\frac{n}{n+3}\right)^n\right) = \lim_{n\to\infty} n \cdot \ln\left(1 - \frac{3}{n+3}\right)$$

This is an indeterminate product, $\infty \times 0$, but we turn it into an indeterminate quotient $0/0$ and then apply L'Hopital's rule.

$$= \lim_{n\to\infty} \frac{\ln\left(1 - \frac{3}{n+3}\right)}{1/n} = \lim_{n\to\infty} \frac{\frac{1}{1-\frac{3}{n+3}} \cdot \frac{3}{(n+3)^2}}{-\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{1-\frac{3}{n+3}} \cdot \frac{-3n^2}{n^2+6n+9}$$
$$= \frac{1}{1-0} \cdot -3 = -3$$

Since we took a natural logarithm to get here, we have that for the original limit,

$$\lim_{n\to\infty} \left(\frac{n}{n+3}\right)^n = e^{-3} \neq 0$$

Thus, $\sum_{n=1}^{\infty} (-1)^n \left(\frac{n}{n+3}\right)^n$ (which we got from plugging in $x = -3/4$) and $\sum_{n=1}^{\infty} \left(\frac{n}{n+3}\right)^n$ (which we got from plugging in $x = -1/4$) both diverge by the nth term test. (We are using the fact that $\lim_{n\to\infty} |a_n| \neq 0$ is equivalent to $\lim_{n\to\infty} a_n \neq 0$ for the first series).

So the power series converges for x in $(-3/4, -1/4)$. So the interval of convergence is centered at $x = -1/2$ and the radius of convergence is $1/4$. □

Sometimes, the interval of convergence can be all real numbers, or it can be a single point. In both cases, there are no endpoints to test, as the following examples show.

Example 6.3 (Interval of convergence is all real numbers)**.** For what values of x does the following power series converge?

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Solution. Since there are factorials here, use the Ratio Test.

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} \frac{x^{2(n+1)+1}}{(2(n+1)+1)!}}{(-1)^n \frac{x^{2n+1}}{(2n+1)!}} \right| = \lim_{n\to\infty} \frac{|x|^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{|x|^{2n+1}}$$
$$= \lim_{n\to\infty} |x|^2 \frac{1}{(2n+1)(2n+2)} = |x|^2 \cdot \lim_{n\to\infty} \frac{1}{(2n+1)(2n+3)} = |x|^2 \cdot 0 = 0 < 1$$

Note that no matter what x is, the ratio here is 0 which is less than 1. So the Ratio Test tells us that this power series converges for all x. So the power series converges for any real number x. Note that we can pull $|x|^2$ out of the limit since x is a constant relative to n. □

Example 6.4 (Interval of convergence is just a single point)**.** For what values of x does the following power series converge?

$$\sum_{n=1}^{\infty} \frac{n!}{2^n} x^n$$

Solution. There is a factorial, so use the Ratio Test. We have that

$$\lim_{n\to\infty}\left|\frac{\frac{(n+1)!}{2^{n+1}}x^{n+1}}{\frac{n!}{2^n}x^n}\right| = \lim_{n\to\infty}\frac{(n+1)!|x|^{n+1}}{2^{n+1}}\cdot\frac{2^n}{n!|x|^n} = \lim_{n\to\infty}\frac{(n+1)!}{n!}\cdot\frac{|x|^{n+1}}{|x|^n}\cdot\frac{2^n}{2^{n+1}}$$
$$= \lim_{n\to\infty}(n+1)\cdot|x|\cdot\frac{1}{2} = |x|\cdot\lim_{n\to\infty}\frac{1}{2}(n+1)$$

Now this is somewhat confusing. When we plug in, we get $|x|\cdot\infty$. We might think this is infinity, and it is, whenever $x \neq 0$. So when $x \neq 0$, since $\infty > 1$, we have by the Ratio Test that the power series diverges. But when $x = 0$, we get $0\cdot\infty$ which is indeterminate. But the easy solution to this is that at $x = 0$, if we plug $x = 0$ into the power series, the series just becomes $\sum_{n=1}^{\infty}\frac{n!}{2^n}0^n = \sum_{n=1}^{\infty}0 = 0$ which converges. So the power series converges just at the single point $x = 0$.

In particular, power series must converge at least at their center. (Here the center is $x = 0$).

□

7 Taylor Series and Maclaurin Series

Before we start talking about Taylor series, let us note the following weird notational quirks. First, we often will take about the "zeroth" derivative of a function f. **We will define the zeroth derivative of f, denoted by $f^{(0)}$, to be f itself.** So if $f = \sin(x)$, then $f^{(0)}(\pi) = f(\pi) = \sin(\pi) = 0$. Also, we recall the factorial notation from before, where for example $5! = 5\cdot4\cdot3\cdot2\cdot1$. **As a weird quirk of factorials, we define $0! = 1$.**

Now that we have the notation down, let's begin! First, let's begin by recalling what a tangent line to a function f is. Recall that the tangent line to a function f at the point $x = a$ is a line with slope $f'(a)$ that passes through the point $(a, f(a))$. So the equation of the tangent line to f at $x = a$ is

$$y = f(a) + f'(a)(x-a)$$

Let us call the tangent line P_1. So the equation for our tangent line to f at $x = a$ is

$$P_1(x) = f(a) + f'(a)(x-a)$$

Recall that the tangent line to f at $x = a$ approximates f well near $x = a$. So $P_1(x)$ is approximately equal to $f(x)$ for x near a. What makes the tangent line approximate f so well near $x = a$? Well, calculate $P_1(a)$ and $P_1'(a)$.

$$P_1(x) = f(a) + f'(a)(x-a)$$
$$P_1'(x) = f'(a)$$

so we see that

$$P_1^{(0)}(a) = f^{(0)}(a)(\text{or equivalently } P_1(a) = f(a)),\quad P_1'(a) = f'(a)$$

So the reason that P_1 approximates f so well near $x = a$ is that its zeroth and first derivatives at $x = a$ match the zeroth and first derivatives of f at $x = a$!

So the tangent line $P_1(x)$ (the first degree Taylor polynomial) is a degree 1 polynomial that matches the zeroth and first derivatives of f at $x = a$. This is what makes it a good approximation to f.

If we think about it, there's nothing stopping us from matching more derivatives of f. Maybe, we could try to find a polynomial function that matches not just the zeroth and first derivatives of f at $x = a$, but also the second derivative. Consider the quadratic polynomial

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

Then, computing some derivatives, we have that

$$P_2'(x) = f'(a) + f''(a)(x-a), \quad P_2''(x) = f''(a)$$

So we see that

$$P_2(a) = f(a), \quad P_2'(a) = f'(a), \quad P_2''(a) = f''(a)$$

So $P_2(x)$ (the second degree Taylor polynomial) is a quadratic polynomial that matches the zeroth, first, and second derivative of f at $x = a$.

Going one step further, we could maybe find a third degree polynomial $P_3(x)$ that matches the zeroth, first, second, and third derivative of f at $x = a$ to get an even better approximation to f near $x = a$. For this, consider

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3$$

The factor of 6 in the denominator for $f'''(a)$ is odd, since we might expect a 3 there. But the 6 is there because it is 3!. Every time we do the power rule, the exponent comes down and decreases by 1, so after three differentiations, a 3, a 2, and a 1 will be pulled down from the exponent, which is why we need 6 in the denominator to cancel all of these. We can check manually that

$$P_3'(x) = f'(a) + f''(a)(x-a) + \frac{f'''(a)}{2}(x-a)^2, \quad P_3''(x) = f''(a) + f'''(a)(x-a), \quad P_3'''(x) = f'''(a)$$

So then

$$P_3(a) = f(a), \quad P_3'(a) = f'(a), \quad P_3''(a) = f''(a), \quad P_3'''(a) = f'''(a)$$

So P_3 (the third degree Taylor polynomial) is a cubic function that matches the zeroth, first, second, and third derivatives of f at $x = a$.

We could imagine doing this forever. The more derivatives of f that we match at $x = a$, the better our Taylor polynomial (of higher and higher degree) will approximate f near $x = a$. We could imagine taking an "infinite degree" Taylor polynomial to get the best possible approximation for f. In particular, we could consider what is called a **Taylor series**, which - informally speaking - is an infinite degree polynomial (power series) that matches every single derivative of f at $x = a$. To see why the formula for the Taylor series makes sense, we will rewrite $P_1(x), P_2(x), P_3(x)$ from before as follows, using the fact that $f^{(0)}(a) = f(a)$ and $0! = 1$, $1! = 1$, $2! = 2$, $3! = 6$, and $f^{(k)}$ refers to the kth derivative of f.

$$P_1(x) = \frac{f^0(a)}{0!}(x-a)^0 + \frac{f^{(1)}(a)}{1!}(x-a)^1$$

$$P_2(x) = \frac{f^0(a)}{0!}(x-a)^0 + \frac{f^{(1)}(a)}{1!}(x-a)^1 + \frac{f^{(2)}(a)}{2!}(x-a)^2$$

$$P_3(x) = \frac{f^0(a)}{0!}(x-a)^0 + \frac{f^{(1)}(a)}{1!}(x-a)^1 + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3$$

So we can see that the formula for an infinite degree **Taylor series** would logically be

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

To check your understanding, do you see that the partial sums of a Taylor series are exactly the Taylor polynomials?

Remember that **power series do not converge everywhere**. The Taylor series $T(x)$ definitely converges at $x = a$ because if we plug in $x = a$, it becomes $\sum 0$. However, we need to be careful when using $T(x)$ to approximate f because we need to make sure we are only consider x values where $T(x)$ actually converges. So often, when we get a Taylor series, we are interested in finding its interval of convergence. This is done just as in the previous section on power series.

The general rule for this class is that **for any x for which a given Taylor series $T(x)$ centered at $x = a$ for $f(x)$ converges, we will have** $T(x) = f(x)$, where $T(x)$ here is really referring to the value that the resulting convergent series converges to when the given value for x is plugged into the power series.

Taylor series problems are of two types.

- Given a function, find its Taylor series by finding a general pattern for its derivatives $f^{(n)}(a)$ at the point a and plugging into the Taylor series formula for $T(x)$.
- Use manipulation of known Taylor series to get a Taylor series for f, and then use the coefficients of the Taylor series to find a formula for the derivatives of f at the point a at which the Taylor series is centered.

Note that you will be given some Taylor series formula in the formula section for the exam. Use these!

By the way, what is a **Maclaurin series**? A **Maclaurin series** is just a Taylor series where we expand around $x = 0$ (so $a = 0$). But a Maclaurin series is just a Taylor series, but we just expand at $x = 0$. It's nothing new!

As a final note, we will often have to manipulate power series. We can add/subtract power series termwise, multiply them by single monomials of the form $(x-a)^k$, differentiate them, and integrate them. Let's do some examples.

Example 7.1. Calculate the Maclaurin series for $\sin(x)$, find where it converges, and then use this Taylor series to find the Taylor series for $\sin(x^3)$ and $3x^2\cos(x^3)$. Then, find, for $g(x) = 3x^2\cos(x^3)$, the values of $g^{(122)}(0)$ and $g^{(100)}(0)$.

Solution. A Maclaurin series is just a Taylor series centered at $a = 0$. So the formula for the Taylor series of $f(x) = \sin(x)$ centered at $a = 0$ is

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

We can see that

$$f^{(0)}(x) = \sin(x), \quad f^{(1)}(x) = \cos(x), \quad f^{(2)}(x) = -\sin(x), \quad f^{(3)}(x) = -\cos(x)$$

and then $f^{(4)}(x)$ cycles back to $\sin(x)$ and we repeat. So the derivatives cycle in groups of four. Plugging in $x = 0$, we get that

$$f^{(0)}(0) = 0, \quad f^{(1)}(0) = 1, \quad f^{(2)}(0) = 0, \quad f^{(3)}(0) = -1$$

then at $f^{(4)}(0)$, we go back to 0 and repeat. So the derivatives cycle from 0, 1, 0, -1, and then go back to 0 and repeat. Plugging into the Taylor series formula, write out enough terms to find a pattern. We get

$$\begin{aligned} T(x) = \frac{0}{0!}x^0 + \frac{1}{1!}x^1 + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^0 + \frac{1}{5!}x^5 + \frac{0}{6!}x^6 + \frac{-1}{7!}x^7 + ... \\ = \frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... \end{aligned}$$

So we see that the pattern is that we go up in exponent by twos, changing signs each time. In addition, the number in front of the factorial is the same as the exponent. So we can write this generally as

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}(-1)^n$$

where **on the exam, you will have to give the general formula**. This takes practice, and the way to do this is to write something down and check on the first few n's to see if it matches the explicit expansion you wrote down before. Here, the $(-1)^n$ makes the alternating behavior (and we know it is $(-1)^n$ rather than $(-1)^{n-1}$ since the $n = 0$ term is positive, not negative), and the $2n + 1$ in the exponent reflects that the exponent starts at 1 and increases by twos.

So we have that the Maclaurin expansion for $y = \sin(x)$ is

$$\sin(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}(-1)^n$$

Now, we have to find where this Taylor series (which is a power series) converges. But we did this in Example 6.3 using the Ratio Test, and found that this Taylor series converges for all real numbers x. So for every real number x, this series converges and the value it converges to is $\sin(x)$.

How do we find the Taylor series for $\sin(x^3)$? Well, we have the Taylor series for $\sin(x)$, so to get $\sin(x^3)$, we just need to plug in x^3 for x. So since

$$\sin(x) = \frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}(-1)^n$$

we get that

$$\sin(x^3) = \frac{x^3}{1!} - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \frac{x^{21}}{7!} + ... = \sum_{n=0}^{\infty} \frac{(x^3)^{2n+1}}{(2n+1)!}(-1)^n = \sum_{n=0}^{\infty} \frac{x^{6n+3}}{(2n+1)!}(-1)^n$$

Finally, we need the Taylor expansion for $3x^2\cos(x^3)$, but we see that this function is just the derivative of $\sin(x^3)$. So differentiate the Taylor series term by term.

$$\sin(x^3) = \frac{x^3}{1!} - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \frac{x^{21}}{7!} + ... = \sum_{n=0}^{\infty} \frac{(x^3)^{2n+1}}{(2n+1)!}(-1)^n = \sum_{n=0}^{\infty} \frac{x^{6n+3}}{(2n+1)!}(-1)^n$$

So differentiating and using the power rule,

$$3x^2\cos(x^2) = 3\frac{x^2}{1!} - 9\frac{x^8}{3!} + 15\frac{x^{14}}{5!} - 21\frac{x^{20}}{7!} + ... = \sum_{n=0}^{\infty}(6n+3)\frac{x^{6n+2}}{(2n+1)!}(-1)^n$$

Finally, we need to find for $g(x) = 3x^2\cos(x^2)$ the values of $g^{(122)}(0)$ and $g^{(100)}(0)$. This seems scary, but we are definitely not going to do this by hand. The trick is to remember that the Taylor series formula says that the coefficient on x^n in the Taylor series for g centered at 0 is exactly

$$\frac{g^{(n)}(0)}{n!}$$

So to find $g^{(122)}(0)$, we just need to look at the coefficient on x^{122} in the Taylor series centered at 0,

$$g(x) = 3x^2\cos(x^2) = 3\frac{x^2}{1!} - 9\frac{x^8}{3!} + 15\frac{x^{14}}{5!} - 21\frac{x^{20}}{7!} + ... = \sum_{n=0}^{\infty}(6n+3)\frac{x^{6n+2}}{(2n+1)!}(-1)^n$$

Which index n gives us an x^{122} term? Well, this happens when $6n+2 = 122$, so when $n = 20$. So the $n = 20$ index gives us

$$\frac{123}{41!}x^{122}$$

since $(-1)^{20} = 1$. The coefficient here is exactly $\frac{g^{(122)}(0)}{122!}$ by the Taylor series expansion formula. So we have that

$$\frac{123}{41!} = \frac{g^{(122)}(0)}{122!}$$

Thus,

$$g^{(122)}(0) = 123 \cdot \frac{122!}{41!}$$

To find $g^{(100)}(0)$, we need the coefficient on the x^{100} term in the Taylor series. We might try to set $6n+2 = 100$ by we get $n = 16\frac{2}{3}$. Weird, because n must be an integer! What does this mean? Well, it means that the series expansion does not have an explicit x^{100} term in it! To see this, if you look at the explicit terms that we wrote out,

$$g(x) = 3x^2\cos(x^2) = 3\frac{x^2}{1!} - 9\frac{x^8}{3!} + 15\frac{x^{14}}{5!} - 21\frac{x^{20}}{7!} + ... = \sum_{n=0}^{\infty}(6n+3)\frac{x^{6n+2}}{(2n+1)!}(-1)^n$$

we see that the only terms with nonzero coefficients are numbers $2, 8, 14, 20, ...$ that are two more than multiples of 6. But 100 is not two more than a multiple of 6. So the coefficient of the x^{100} term in the Taylor series is actually zero. Thus,

$$\frac{g^{(100)}(0)}{100!} = 0$$

and so $g^{(100)}(0) = 0$. □

We can also integrate Taylor series. Let's see a quick example of this.

Example 7.2 (A quick but instructive example)**.** Use the Maclaurin series from $\cos(x)$ centered at $x = 0$ to find the Maclaurin series for $\sin(x)$, by first differentiating and then integrating.

Proof. One can calculate the Maclaurin series for $\cos(x)$ in the same way, but it is given in the formula sheet as

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6} + ... = \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}(-1)^n$$

We can get the Maclaurin series from $\sin(x)$ from a formula sheet, but let's get it by differentiating/integrating $\cos(x)$ to illustrate these techniques. Let's start with differentiating. Note that

$$\frac{d}{dx}(\cos(x)) = -\sin(x)$$

Differentiating the above Maclaurin series for $\cos(x)$ term by term, we get

$$\sin(x) = 0 - \frac{x}{1!} + \frac{x^3}{3!} - \frac{x^5}{5!} + ... = \sum_{n=0}^{\infty}(2n)\frac{x^{2n-1}}{(2n)!}(-1)^n$$

We might want to simplify this as $\sum_{n=0}^{\infty}\frac{x^{2n-1}}{(2n-1)!}(-1)^n$ but if we plug in $n = 0$, we get $(-1)!$ in the bottom, which makes no sense. This example illustrates an important point! **If your Taylor series has a constant term in it, separate it out first before differentiating.**

Otherwise you will get weird results. Note that we did not have a constant term in the Taylor series for $\sin(x^3)$ in the previous example, so we did not need to do this.

Separating out the constant term, we get

$$\cos(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}(-1)^n = 1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!}(-1)^n$$

Now differentiating, and using the fact that the derivative of 1 is 0, we get

$$-\sin(x) = \sum_{n=1}^{\infty}(2n)\frac{x^{2n-1}}{(2n)!}(-1)^n = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!}(-1)^n$$

Multiplying through each term by -1, we get

$$\sin(x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n-1}}{(2n-1)!}$$

Before we move on to integration, we had from the previous example that

$$\sin(x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$$

Why are these two series for the same function different? The answer is that **they are not!** There is just a shifted index, where in the first series, we start at $n = 1$ and in the second series, we start at $n = 0$. If you write out a few terms of each, you'll see that they are the same. **Be comfortable with shifting indices**. To shift indices yourself, you want to consider the first few terms of each series and change the necessary parts of the general term to get the first few terms to match.

Now, let us get $\sin(x)$ from integration. Recall that

$$\int \cos(x)dx = \sin(x) + C$$

So $\sin(x) = \int \cos(x)dx + C$, where we changed $C \to -C$ since it's just an arbitrary constant. Recall,

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}(-1)^n$$

Integration works nicely with Taylor series, so we do not need to worry about having a constant term in front like we did with differentiation. Just integrate each term separately, and tack on a final big $+C$.

$$\int \cos(x)dx = C + x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\int x^{2n}dx$$
$$= C + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\frac{x^{2n+1}}{2n+1}dx = C + \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$$

So

$$\int \cos(x)dx = C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

This is the indefinite integral, and would be the answer if the question asked for the indefinite integral.

If we wanted to get $\sin(x)$, we would note that $\sin(x)$ is the indefinite integral for a specific choice of C. So we would set

$$\sin(x) = C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

To find C, plug in $x = 0$. The left hand side is 0, and the right hand side is $C + \sum 0 = C$. So $C = 0$. This gives us,

$$\sin(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

as we already found before. $\square$

Example 7.3 (Be careful when adding power series!)**.** Find the Maclaurin expansion for

$$f(x) = \frac{x+4x^3}{(1-x)^2}$$

Then, find $f^{(100)}(0)$.

Solution. From a formula sheet, we have that the Maclaurin series for $\frac{1}{1-x}$ is

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ... = \sum_{n=0}^{\infty} x^n$$

Our first instinct is to try to get a Maclaurin series first for $\frac{1}{(1-x)^2}$. How can we get there? **Beware of this common mistake!** The first gut instinct we have is to note that

$$\left(\frac{1}{1-x}\right)^2 = \frac{1}{(1-x)^2}$$

(Remember to use the chain rule, so the signs work out.) **However, multiplying power series together is a very bad idea, since it is a very intractable calculation (think infinite FOILing)!** We can only multiply power series by single monomials (not even by finite polynomials - because even this is really tough). Let's instead note that

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right)$$

So lets differentiate the Maclaurin series for $\frac{1}{1-x}$. **Remember that when we differentiate, we need to separate out the constant term of the series if it exists, otherwise things will get weird quickly.** So we write

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ... = \sum_{n=0}^{\infty} x^n = 1 + \sum_{n=1}^{\infty} x^n$$

Now, we can differentiate term by term to get

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = 0 + \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + ...$$

Great! Now our next gut instinct is to multiply the Maclaurin series for $\frac{1}{(1-x)^2}$ by $(x+4x^3)$. **But again, this is a very bad idea!** It is only easy to multiply power series by **single monomials**, otherwise we risk the headache of infinite FOILing. Instead, let us find the Maclaurin series for

$$\frac{x}{(1-x)^2} \text{ and } \frac{4x^3}{(1-x)^2}$$

and then add them together! (Adding together power series is possible, by doing it term by term, just like for polynomials).

Recall that our Maclaurin series for $\frac{1}{(1-x)^2}$ at $x = 0$ is

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + ... = \sum_{n=1}^{\infty} nx^{n-1}$$

Multiplying termwise by x, we get

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + ... = \sum_{n=1}^{\infty} nx^{n}$$

and multiplying termwise by $4x^3$, we get

$$\frac{4x^3}{(1-x)^2} = 4x^3 + 8x^4 + 12x^5 + 16x^6 + ... = \sum_{n=1}^{\infty} 4nx^{n+2}$$

Now, we just need to add these series together, **but adding series together is a tricky business!** Why? Because to add two series together, we need to first

- Shift indices to match the exponents on x of the two series we are adding together. (So here, the exponents are not the same, since we have x^n and x^{n+2}). Do this by shifting the series with the larger exponent to the smaller exponent.
- Set aside finitely many terms so that the starting indices of the n on the bottom of the sum are the same after you match the x exponents.

Let me demonstrate this for our current example.

First, we have different exponents on x in the two series we want to add together, x^n and x^{n+2}. So since x^{n+2} has the larger exponent, we shift the series index so that x^{n+2} turns into x^n, the smaller exponent. To do this, we note that

$$\frac{4x^3}{(1-x)^2} = 4x^3 + 8x^4 + 12x^5 + 16x^6 + ... = \sum_{n=1}^{\infty} 4nx^{n+2} = \sum_{n=3}^{\infty} 4(n-2)x^{n}$$

where we do the shift by changing $n \to n-2$ from the first to second series. So now, we want to add together

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + ... = \sum_{n=1}^{\infty} nx^n$$

$$\frac{4x^3}{(1-x)^2} = 4x^3 + 8x^4 + 12x^5 + 16x^6 + ... = \sum_{n=3}^{\infty} 4(n-2)x^n$$

but as noted in the list above, we now need to match the starting indices, which do not match right now (they are $n = 1$ and $n = 3$). To get them to match, separate out the $n = 1$ and $n = 2$ term from the first series.

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + ... = \sum_{n=1}^{\infty} nx^n = x + 2x^2 + \sum_{n=3}^{\infty} nx^n$$

Now, we can add together the series, because they have the same exponent and the same starting index, $n = 3$. When we add series, keep the same indexing (the indexing should be the same for both series before you add them, as they are now) and add termwise. So we have that

$$\frac{x+4x^3}{(1-x)^2} = \frac{x}{(1-x)^2} + \frac{4x^3}{(1-x)^2} = x + 2x^2 + \sum_{n=3}^{\infty} nx^n + \sum_{n=3}^{\infty} 4(n-2)x^n$$

$$= x + 2x^2 + \sum_{n=3}^{\infty}(n + 4(n-2))x^n = x + 2x^2 + \sum_{n=3}^{\infty}(5n-8)x^n$$

Note that we could only add the two series going from the first to second line above because the exponents on x were the same, and the starting index was the same for both series (in this case, $n = 3$).

So we have our Maclaurin series for $f(x) = \frac{x+4x^3}{(1-x)^2}$.

$$\frac{x+4x^3}{(1-x)^2} = x + 2x^2 + \sum_{n=3}^{\infty}(5n-8)x^n$$

To find $f^{(100)}(0)$, we note that the coefficient on x^{100} in the Maclaurin series (Taylor series centered at 0) for $f(x)$ is

$$\frac{f^{(100)}(0)}{100!}$$

How do we get the coefficient in x^{100} in the Maclaurin series above? We note that we get x^{100} for the index $n = 100$, since the term in the series has x^n. **So plugging in $n = 100$, we see that the coefficient on x^{100} in the Maclaurin series is** $5(100) - 8 = 492$. So therefore,

$$492 = \frac{f^{(100)}(0)}{100!}$$

so we have that $f^{(100)}(0) = 100! \cdot 492$.

$\square$

As a final note, not all Taylor series are centered at $x = 0$, as we can see from the following simple example.

Example 7.4. What is the Taylor series for $y = xe^x$ centered at 1? Find all x where it converges.

Solution. We might be tempted to use the Maclaurin series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

given on the formula sheet for $y = e^x$, but this is wrong, since Maclaurin series are Taylor series centered at $a = 0$, and we need a Taylor series centered at $x = 1$. So we cannot manipulate the Maclaurin series for e^x to xe^x to find the Taylor series centered at $a = 1$. Also it is wrong to just replace x by $x - 1$ in the formula above, since the coefficients of the Taylor series change depending on where you center the Taylor series (since the coefficients depend on the derivatives of f at the centering point).

Instead, use the formula

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

for $a = 1$. So we need a general formula for $f^{(n)}(1)$ where $f(x) = e^x$. Trying some derivatives,

$$f(x) = xe^x, \quad f'(x) = xe^x + e^x, \quad f''(x) = xe^x + 2e^x, \quad f'''(x) = xe^x + 3e^x$$

So we see that $f^{(n)}(x) = xe^x + ne^x$. Thus, $f^{(n)}(1) = (n+1)e$. Plugging this into the Taylor series formula, we have that the Taylor series centered at $a = 1$ for $y = xe^x$ is

$$T(x) = \sum_{n=0}^{\infty} \frac{(n+1)e}{n!}(x-1)^n$$

To find all x where the Taylor series converges, use the Ratio Test.

$$\lim_{n\to\infty} \left| \frac{\frac{(n+2)e}{(n+1)!}(x-1)^{n+1}}{\frac{(n+1)e}{n!}(x-1)^n} \right| = \lim_{n\to\infty} \frac{(n+2)e}{(n+1)!}|x-1|^{n+1} \cdot \frac{1}{|x-1|^n} \frac{n!}{(n+1)e}$$
$$= \lim_{n\to\infty} \frac{n!}{(n+1)!} \cdot \frac{n+2}{n+1} \cdot \frac{e}{e} \cdot \frac{|x-1|^{n+1}}{|x-1|^n}$$
$$= \lim_{n\to\infty} \frac{n+2}{(n+1)^2}|x-1| = |x-1| \cdot \lim_{n\to\infty} \frac{n+2}{(n+1)^2} = |x-1| \cdot 0 = 0 < 1$$

So the Taylor series centered at $x = 1$ for $y = xe^x$ converges for all real numbers x. ☐