Review (Midterm 1)

1) True
2) False (dim $P_{2019} = 2020$)
3) True (3 free variables)
4) False ($\mathbb{R}^m$)
5) False $(AB)^t = B^tA^t$ when the matrix mult makes sense
6) False $(I + (-I) = 0)$
7) False (The nonzero rows do)
8) True
9) True $(1+x, 1, x^2)$
10) False $\left(\frac{1}{2}(1+x) = \frac{1}{2} + \frac{1}{2}x\right)$
11) True
12) False (does not contain zero vector $y=0$)
13) False $\left(\begin{bmatrix}1&0\\0&0\end{bmatrix} + \begin{bmatrix}0&0\\0&1\end{bmatrix} = \begin{bmatrix}1&0\\0&1\end{bmatrix}\right)$
14) True
15) True $(\det(AB) = (\det A)(\det B))$
16) True $(\det(A) = \det(A^t))$
17) True
18) False $(-1(1,1) = (-1,-1))$
19) True
20) False (Consider $T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

Problem 1

$(1,0) = \frac{1}{2}(1,1) + \left(-\frac{1}{2}\right)(-1,1) \qquad (0,1) = \frac{1}{2}(1,1) + \frac{1}{2}(-1,1)$

$$T(1,0) = T\left(\frac{1}{2}(1,1) + \left(-\frac{1}{2}\right)(-1,1)\right)$$
$$= \frac{1}{2}T(1,1) + \left(-\frac{1}{2}\right)T(-1,1) = \frac{1}{2}(1,-2) + \left(-\frac{1}{2}\right)(2,1)$$
$$= \left(-\frac{1}{2}, -\frac{3}{2}\right)$$

$$T(0,1) = T\left(\frac{1}{2}(1,1) + \frac{1}{2}(-1,1)\right)$$
$$= \frac{1}{2}T(1,1) + \frac{1}{2}T(-1,1) = \frac{1}{2}(1,-2) + \frac{1}{2}(2,1) = \left(\frac{3}{2}, -\frac{1}{2}\right)$$

$$\boxed{T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = \begin{bmatrix}-\frac{1}{2} & \frac{3}{2}\\ -\frac{3}{2} & -\frac{1}{2}\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}}$$

$\begin{bmatrix}-\frac{1}{2} & \frac{3}{2}\\ -\frac{3}{2} & -\frac{1}{2}\end{bmatrix}$ is invertible with inverse $\begin{bmatrix}-\frac{1}{5} & -\frac{3}{5}\\ \frac{3}{5} & -\frac{1}{5}\end{bmatrix}$

so T is bijective with (injective and surjective) $\boxed{T^{-1}\left(\begin{bmatrix}y_1\\y_2\end{bmatrix}\right) = \begin{bmatrix}-\frac{1}{5} & -\frac{3}{5}\\ \frac{3}{5} & -\frac{1}{5}\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix}}$

Problem 2

kernel(T): $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a+d = 0$

$$\underbrace{a + 0b + 0c}_{\text{free}} + d = 0$$

$$(a,b,c,d) = b\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\boxed{\text{Basis for ker}(T): \left\{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\right\}}$$

range $(T) = \mathbb{R}$ since given any $r \in \mathbb{R}$, $T\left(\begin{bmatrix} r & 0 \\ 0 & 0 \end{bmatrix}\right) = r$.

$$\boxed{\text{Basis for range}(T): \{1\}}$$

Problem 3:

$$T(x^2-x+1) = \begin{bmatrix} (-1)^2-(-1)+1 \\ 0^2-0+1 \\ 1^2-1+1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$$T(x^2) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$T(ax^2+bx+c) = \begin{bmatrix} a-b+c \\ c \\ a+b+c \end{bmatrix}$$

ker (T): $a-b+c=0$, $c=0$, $a+b+c=0$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so $(a,b,c) = 0$ $\boxed{\text{ker}(T) = \{0\} \text{ empty basis}}$

range (T): Note that nullity $(T) = 0$, so since nullity(T) + rank(T) = dim$(P_2) = 3$, rank(T) = dim(range(T)) = 3. But since range(T) is a subspace of $\mathbb{R}^3$ and dim$(\mathbb{R}^3) = 3$, we have $\boxed{\text{range}(T) = \mathbb{R}^3}$

Basis: e_1, e_2, e_3

T is bijective.
Calculate $T^{-1}\left(\begin{bmatrix} w \\ y \\ z \end{bmatrix}\right)$

$$T(ax^2+bx+c) = \begin{bmatrix} a-b+c \\ c \\ a+b+c \end{bmatrix}$$

$$a-b+c = w$$
$$c = y$$
$$a+b+c = z$$

$$c = y \Rightarrow \begin{matrix} a-b = y-w. \\ a+b = z-y \end{matrix}$$

$$a = \tfrac{1}{2}(z-w) \quad b = \tfrac{1}{2}(z-2y+w)$$

$$\boxed{T^{-1}\left(\begin{bmatrix} w \\ y \\ z \end{bmatrix}\right) = \left(\tfrac{1}{2}(z-w)\right)x^2 + \left(\tfrac{1}{2}(z-2y+w)\right)x + y}$$

4) When is $c_1(1,1,2)+c_2(-1,-2,1)+c_3(0,-1,3) = (a,b,c)$ consistent for (c_1,c_2,c_3)? $\Rightarrow \begin{cases} c_1 - c_2 = a \\ c_1 - 2c_2 - c_3 = b \\ 2c_1 + c_2 + 3c_3 = c \end{cases}$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & a \\ 1 & -2 & -1 & b \\ 2 & 1 & 3 & c \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & a \\ 0 & 1 & 1 & a-b \\ 0 & 3 & 3 & -2a+c \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2a-b \\ 0 & 1 & 1 & a-b \\ 0 & 0 & 0 & -5a+3b+c \end{array}\right]$$

$\boxed{-5a+3b+c=0}$ to be in span

No such conditions for (a,b,c) exist for all four vectors to be linearly independent since four vectors in $\mathbb{R}^3$ can never be linearly independent.

5) If $T:\mathbb{R}^n \to \mathbb{R}^n$ is injective, nullity$(T)=0$, so rank$(T)=n$ by Rank-Nullity. So range(T) is a subspace of $\mathbb{R}^n$ with dim n, hence range$(T) = \mathbb{R}^n$ so T is surjective. So T is bijective.

If $T:\mathbb{R}^n \to \mathbb{R}^n$ is surjective, rank$(T)=n$ since range$(T)=\mathbb{R}^n$. So by Rank-Nullity, nullity$(T)=0$ so ker(T) is the zero subspace, so T is injective.

6) $P_{\mathcal{B}_2 \leftarrow \mathcal{B}_1}([v]_{\mathcal{B}_1}) = [v]_{\mathcal{B}_2}$

$\leftarrow (1,1) \in \mathbb{R}^2$

$P_{\mathcal{B}_2 \leftarrow \mathcal{B}_1}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\mathcal{B}_1}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{\mathcal{B}_2}$

$(1,1) = c_1(-2,3) + c_2(1,1)$

$c_1 = 0, \ c_2 = 1$

$\leftarrow (2,1) \in \mathbb{R}^2$

$P_{\mathcal{B}_2 \leftarrow \mathcal{B}_1}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}_{\mathcal{B}_1}\right) = \begin{bmatrix} -\frac{1}{5} \\ \frac{8}{5} \end{bmatrix}_{\mathcal{B}_2}$

$(2,1) = c_1(-2,3) + c_2(1,1)$

$-2c_1 + c_2 = 2$

$3c_1 + c_2 = 1$

$c_1 = -\frac{1}{5} \quad c_2 = \frac{8}{5}$

So $\boxed{[v]_{\mathcal{B}_2} = \begin{bmatrix} 0 & -\frac{1}{5} \\ 1 & \frac{8}{5} \end{bmatrix} [v]_{\mathcal{B}_1}}$

$[P]_{\mathcal{B}_2 \leftarrow \mathcal{B}_1} = \begin{bmatrix} 0 & -\frac{1}{5} \\ 1 & \frac{8}{5} \end{bmatrix}$