

A Brief Review of Partial Fraction Decomposition

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Remember that partial fraction decomposition is a way of integrating rational functions (polynomials over polynomials) by writing rational function as sums of smaller functions that give what I like to call "snack-bite integrals."

First, let us learn how to compute different types of snack-bite integrals.

Type 1: Integrating a fraction with a linear factor in the denominator and a constant in the numerator.

Examples: $\int \frac{1}{x-3} dx, \int \frac{2}{(2x+1)^2} dx, \int \frac{7}{(x+2)^4} dx$

* The trick to these is to set u to be the linear term in the denominator
(u-sub)

Ex: $\int \frac{1}{x-3} dx = \int \frac{1}{u} du = \ln|u| + C = \ln|x-3| + C$

Let $u = x-3$
 $du = dx$

Ex: $\int \frac{2}{(2x+1)^2} dx = \int \frac{1}{u^2} du = -\frac{1}{u} + C = -\frac{1}{2x+1} + C$

Let $u = 2x+1$
 $du = 2dx$

Ex: $\int \frac{7}{(x+2)^4} dx = \int \frac{7}{u^4} du = -\frac{7}{3} u^{-3} + C = -\frac{7}{3} \left(\frac{1}{(x+2)^3} \right) + C$

Let $u = x+2$.
 $du = dx$

Type 2: Integrating a fraction with an irreducible quadratic to some power in the denominator and a linear term in the numerator. ②

Examples: $\int \frac{2x+5}{x^2+4} dx$, $\int \frac{3x-1}{x^2-4x+5} dx$,

$$\int \frac{x+2}{(x^2-2x+5)^2} dx$$

The trick to these:

- If the bottom quadratic has no linear term, break up any sum in the numerator and integrate each term.
- • If the bottom quadratic has a linear term, complete the square and set u to be the thing inside the square.

CAUTION: Only do these strategies if the quadratic in the bottom is irreducible. To check this, a quadratic is irreducible when

$$b^2 - 4ac < 0 \quad (ax^2 + bx + c)$$

In the examples above, indeed $x^2 + 4$, $x^2 - 4x + 5$, and $x^2 - 2x + 5$ are irreducible.

$$\begin{aligned} b^2 - 4ac &= 0^2 - 4(1)(4) & b^2 - 4ac &= (-4)^2 - 4(1)(5) & b^2 - 4ac \\ &= -16 < 0 \checkmark & &= -4 < 0 \checkmark & = (-2)^2 - 4(1)(5) \\ & & & & = -16 < 0 \checkmark \end{aligned}$$

* So for example, do not complete the square for $\int \frac{2}{x^2-1} dx$, since the denominator is not irreducible since $x^2-1 = (x-1)(x+1)$.

Ex: (Type 2 snack-bite integrals)

$$\int \frac{2x}{x^2+1} dx = \int \frac{1}{u} du = \ln|u| + C = \ln|x^2+1| + C.$$

$$\begin{aligned} \text{Let } u &= x^2 + 1 \\ du &= 2x dx \end{aligned}$$

$$\int \frac{3}{x^2+4} dx = 3 \int \frac{1}{x^2+4} dx = \frac{3}{4} \int \frac{1}{\frac{x^2}{4} + 1} dx$$

For integral of this form, make the constant 1 by pulling out a constant!

$$= \frac{3}{4} \int \frac{1}{(\frac{x}{2})^2 + 1} dx$$

$$\text{Let } u = \frac{x}{2}, du = \frac{1}{2} dx \\ 2du = dx$$

$$= \frac{3}{4} \int \frac{1}{u^2 + 1} 2 du$$

$$= \frac{3}{2} \arctan u + C$$

$$= \frac{3}{2} \arctan\left(\frac{x}{2}\right) + C$$

$$\int \frac{2x+5}{x^2+4} dx = \underbrace{\int \frac{2x}{x^2+4} dx}_\text{break up the sum} + \underbrace{\int \frac{5}{x^2+4} dx}_\text{②}$$

break up the sum

$$\text{① } \int \frac{2x}{x^2+4} dx \quad \begin{matrix} \text{Let } u = x^2 + 4. \\ du = 2x dx \end{matrix} \\ = \int \frac{1}{u} du = \ln|u| + C = \ln|x^2 + 4|$$

$$\text{② } \int \frac{5}{x^2+4} dx \\ = 5 \int \frac{1}{x^2+4} dx \\ = \frac{5}{4} \int \frac{1}{\frac{x^2}{4} + 1} dx$$

$$\boxed{\text{ANS: } \ln|x^2+4| + \frac{5}{2} \arctan\left(\frac{x}{2}\right) + C}$$

$$\begin{matrix} u = \frac{x}{2} \\ du = \frac{1}{2} dx \\ dx = 2du \end{matrix} \\ = \frac{5}{4} \int \frac{1}{(\frac{x}{2})^2 + 1} dx \\ = \frac{5}{4} \int \frac{1}{u^2 + 1} 2 du \\ = \frac{5}{2} \arctan u + C \\ = \frac{5}{2} \arctan\left(\frac{x}{2}\right) + C$$

$$\int \frac{3x-1}{x^2-4x+5} dx$$

The x term in the irreducible quadratic is nonzero so we need to complete the square.

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$$= \int \frac{3x-1}{(x-2)^2+1} dx$$

Now set u to the thing inside the square and it will become an integral we know how to do.

$$\begin{aligned}
 \text{Let } u &= x-2 & du &= dx & = \int \frac{3(u+2)-1}{u^2+1} du \\
 x &= u+2 & & & = \int \frac{3u+5}{u^2+1} du & \leftarrow \text{Now this is an integral of a type we have already done!} \\
 & & & & = \int \frac{3u}{u^2+1} du + \int \frac{5}{u^2+1} du \\
 & & & & \downarrow & \\
 & & v = u^2+1 & & 5 \arctan u + C \\
 & & dv = 2u du & & \\
 & & u du = \frac{1}{2} dv & & \\
 3 \int \frac{1}{v} \frac{1}{2} dv &= \frac{3}{2} \ln|v| + C & & & \\
 &= \frac{3}{2} \ln|u^2+1| + C & & & \\
 &= \frac{3}{2} \ln|(x-2)^2+1| + 5 \arctan(x-2) + C & & &
 \end{aligned}$$

$$\int \frac{x-2}{(x^2-2x+5)^2} dx$$

The irreducible quadratic in the denominator has a nonzero x term so we complete the square.

$$\begin{aligned}
 &= \int \frac{x-2}{((x-1)^2+4)^2} dx & \text{Set } u \text{ to be the thing in the square.} \\
 && u = x-1 & du = dx & \\
 && x = u+1 & & \\
 &= \int \frac{(u+1)-2}{(u^2+4)^2} du &= \int \frac{u-1}{(u^2+4)^2} du \\
 && \underbrace{\quad}_{\textcircled{1}} & & \underbrace{\quad}_{\textcircled{2}} \\
 &= \int \frac{u}{(u^2+4)^2} du - \int \frac{1}{(u^2+4)^2} du
 \end{aligned}$$

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$$\textcircled{1} \int \frac{u}{(u^2+4)^2} du \quad \begin{aligned} &\text{Let } v = u^2 + 4 \\ &dv = 2u du \\ &\frac{1}{2} dv = u du \end{aligned}$$

$$= \int \frac{1}{v^2} \frac{1}{2} dv = \int \frac{1}{2v^2} dv = -\frac{1}{2v} + C = -\frac{1}{2(u^2+4)} + C$$

$$\textcircled{2} \int \frac{1}{(u^2+4)^2} du \quad \text{Use trig sub!}$$

$$= \int \frac{1}{(\sqrt{u^2+4})^4} du \quad \begin{array}{c} \sqrt{u^2+4} \\ \diagdown \quad \downarrow \quad \diagup \\ u \\ 2 \end{array}$$

$$= \int \frac{1}{(2\sec\theta)^4} 2\sec^2\theta d\theta \quad \begin{aligned} \tan\theta &= \frac{u}{2} & \frac{\sqrt{u^2+4}}{2} &= \sec\theta \\ u &= 2\tan\theta & \sqrt{u^2+4} &= 2\sec\theta \\ du &= 2\sec^2\theta d\theta \end{aligned}$$

$$= \frac{1}{8} \int \cos^2\theta d\theta \quad \begin{array}{c} \uparrow \\ \cos^2\theta = \frac{1+\cos 2\theta}{2} \end{array} = \frac{1}{8} \int \frac{1+\cos 2\theta}{2} d\theta = \frac{1}{16} \int 1 + \cos 2\theta d\theta$$

$$\begin{aligned} &= \frac{1}{16} \theta + \frac{1}{32} \sin 2\theta + C \\ &\tan\theta = \frac{u}{2} \quad \rightarrow \quad \text{so } \theta = \arctan\left(\frac{u}{2}\right) = \frac{1}{16} \theta + \frac{1}{16} \sin\theta \cos\theta + C \\ &\sin 2\theta = 2\sin\theta \cos\theta \quad = \frac{1}{16} \arctan\left(\frac{u}{2}\right) + \frac{1}{16} \left(\frac{u}{\sqrt{u^2+4}}\right) \left(\frac{2}{\sqrt{u^2+4}}\right) + C \\ &= \frac{1}{16} \arctan\left(\frac{u}{2}\right) + \frac{u}{8(u^2+4)} + C \end{aligned}$$

$$\text{So ANS: } -\frac{1}{2(u^2+4)} - \frac{1}{16} \arctan\left(\frac{u}{2}\right) - \frac{u}{8(u^2+4)} + C$$

Recall
 $u = x+1$

$$= \boxed{-\frac{1}{2((x+1)^2+4)} - \frac{1}{16} \arctan\left(\frac{x+1}{2}\right) - \frac{x+1}{8((x+1)^2+4)} + C}$$

⑥

Now, we have covered how to do all types of snack-bite integrals.
 Note that these integrals are not easy! Try some yourself:

Exercise:

$$\int \frac{2}{(x-3)^4} dx$$

$$\int \frac{2x+1}{x^2+9} dx$$

$$\int \frac{3}{x-1} + \frac{4}{(x-1)^2} dx$$

$$\int \frac{x+4}{x^2-2x+2} dx$$

$$\int \frac{2x+1}{(x^2+4x+8)^2} dx$$

Now, let's do some examples of partial fraction decomposition.

Example: $\int \frac{x^2}{(x-1)^3} dx$

Solution: $\deg(\text{num}) = 2$ is less than $\deg(\text{den}) = 3$ so no need to long divide.
 So we can go straight to a partial fraction decomposition.

$$\frac{x^2}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}$$

Multiply both sides by $(x-1)^3$ to clear denominators.

$$x^2 = A(x-1)^2 + B(x-1) + C \quad \text{We want to solve for } A, B, C$$

There are two ways to do this:

Method 1: Expand and group like terms.

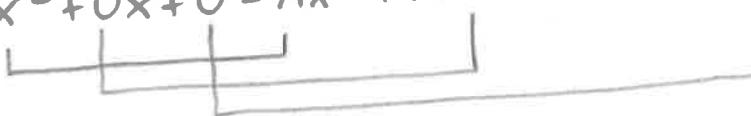
$$x^2 = A(x-1)^2 + B(x-1) + C$$

$$x^2 = A(x^2 - 2x + 1) + Bx - B + C$$

$$x^2 = Ax^2 - 2Ax + A + Bx - B + C$$

$$x^2 = Ax^2 + (-2A+B)x + (A-B+C)$$

$$x^2 + 0x + 0 = Ax^2 + (-2A+B)x + (A-B+C)$$



$$\text{So } A = 1$$

$$-2A + B = 0 \Rightarrow -2(1) + B = 0 \Rightarrow B = 2$$

$$A - B + C = 0 \Rightarrow 1 - 2 + C = 0 \Rightarrow C = 1$$

$A = 1$
$B = 2$
$C = 1$

Method 2: Plug in x values and solve a system of equations.

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$$x^2 = A(x-1)^2 + B(x-1) + C$$

We have three variables to solve for. So plug in three x -values.

$$x=1: \quad 1^2 = A(0)^2 + B(0) + C \Rightarrow C=1$$

$$x^2 = A(x-1)^2 + B(x-1) + 1$$

$$x=2: \quad 4 = A + B + 1 \Rightarrow A + B = 3 \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{Add eqs. together:}$$

$$x=0: \quad 0 = A - B + 1 \Rightarrow A - B = -1 \quad \left. \begin{array}{l} \\ \end{array} \right\} \quad 2A = 2 \quad A = 1$$

$$1 + B = 3, \quad B = 2$$

$$\boxed{A=1, B=2, C=1}$$

So $\frac{x^2}{(x-1)^3} = \frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{1}{(x-1)^3}$ Partial fractions gives us three snack-bite integrals!

$$\int \frac{x^2}{(x-1)^3} dx = \int \frac{1}{x-1} dx + \int \frac{2}{(x-1)^2} dx + \int \frac{1}{(x-1)^3} dx$$

$$\left(\begin{array}{l} u=x-1 \\ du=dx \end{array} \right)$$

$$\left(\begin{array}{l} u=x-1 \\ du=dx \end{array} \right)$$

$$\int \frac{1}{u} du = \ln|u| + C$$

$$= \ln|x-1| + C$$

$$\int \frac{2}{u^2} du$$

$$= -\frac{2}{u} + C$$

$$\int \frac{1}{u^3} du$$

$$= -\frac{1}{2u^2} + C = -\frac{1}{2(x-1)^2} + C$$

$$= -\frac{2}{x-1} + C$$

$$= \boxed{\ln|x-1| - \frac{2}{x-1} - \frac{1}{2(x-1)^2} + C} \quad //$$

Let's do one last example!

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Example:

$$\int \frac{2x^4 + 5x^3 + 22x^2 + 10x + 10}{x^3 + 2x^2 + 10x} dx$$

Solution: Notice that $\deg(\text{num}) = 4 \geq \deg(\text{den}) = 3$, so we need to long divide! Remember to fill in all missing terms!

$$\begin{array}{r}
 2x + 1 + \frac{10}{x^3 + 2x^2 + 10x} \\
 \hline
 x^3 + 2x^2 + 10x + 0 \overline{)2x^4 + 5x^3 + 22x^2 + 10x + 10} \\
 - (2x^4 + 4x^3 + 20x + 0x) \\
 \hline
 x^3 + 2x^2 + 10x + 10 \\
 - (x^3 + 2x^2 + 10x + 0) \\
 \hline
 0x^2 + 0x + 10 \leftarrow \text{remainder}
 \end{array}$$

* Review long division of polynomials!

So from long division,

$$\begin{aligned}
 & \int \frac{2x^4 + 5x^3 + 22x^2 + 10x + 10}{x^3 + 2x^2 + 10x} dx \\
 &= \int 2x + 1 + \frac{10}{x^3 + 2x^2 + 10x} dx \\
 &= \int 2x + 1 + \frac{10}{x(x^2 + 2x + 10)} dx
 \end{aligned}$$

Note that $x^2 + 2x + 10$ is irreducible since $b^2 - 4ac = 2^2 - 4(1)(10) = -36 < 0$

Do partial fraction decomposition.

$$\frac{10}{x(x^2 + 2x + 10)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 2x + 10}$$

Multiply by $x(x^2 + 2x + 10)$.

$$10 = A(x^2 + 2x + 10) + (Bx + C)x$$

$$10 = (A + B)x^2 + (2A + C)x + 10A$$

$$0x^2 + 0x + 10 = (A + B)x^2 + (2A + C)x + 10A$$



(You can also plug in three values of x and solve a system of equations.)

Match coefficients:

$$A + B = 0$$

$$2A + C = 0$$

$$10A = 10 \Rightarrow A = 1$$

$$1 + B = 0 \Rightarrow B = -1$$

$$2(1) + C = 0 \Rightarrow C = -2$$

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$$\text{So } \frac{10}{x(x^2+2x+10)} = \frac{1}{x} + \frac{-x-2}{x^2+2x+10}$$

$$\text{Thus } \int \frac{2x^4+5x^3+22x^2+10x+10}{x^3+2x^2+10x} dx$$

$$= \int 2x+1 + \frac{10}{x(x^2+2x+10)} dx \quad (\text{from long division})$$

$$= \underbrace{\int 2x dx}_{\textcircled{1}} + \underbrace{\int 1 dx}_{\textcircled{2}} + \underbrace{\int \frac{1}{x} dx}_{\textcircled{3}} + \underbrace{\int \frac{-x-2}{x^2+2x+10} dx}_{\textcircled{4}} \quad (\text{from partial fractions})$$

So long division and partial fractions broke our original integral into four manageable shack-bite integrals!

$$\textcircled{1} \int 2x dx = x^2 + C \quad \textcircled{2} \int 1 dx = x + C$$

$$\textcircled{3} \int \frac{1}{x} dx = \ln|x| + C$$

$$\textcircled{4} \int \frac{-x-2}{x^2+2x+10} dx$$

This is a shack-bite integral with an irreducible quadratic in the denominator. The x term is nonzero so complete the square

$$= \int \frac{-x-2}{(x+1)^2+9} dx$$

Set $u=x+1$, $x=u-1$, $dx=du$

$$= \int \frac{-(u-1)-2}{u^2+9} du = \int \frac{-u-1}{u^2+9} du$$

$$= - \int \frac{u}{u^2+9} du - \int \frac{1}{u^2+9} du$$

$$v = u^2 + 9 \\ dv = 2u du \\ u du = \frac{1}{2} dv$$

$$-\int \frac{1}{v} \left(\frac{1}{2} dv \right) = -\frac{1}{2} \ln|v| + C \\ = -\frac{1}{2} \ln|u^2+9| + C \\ = -\frac{1}{2} \ln|(x+1)^2+9| + C$$

since $u=x+1$

$$\frac{1}{9} \int \frac{1}{u^2+9} du \\ = \frac{1}{9} \int \frac{1}{(\frac{u}{3})^2+1} du \\ 3dv = du$$

$$= \frac{1}{9} \int \frac{1}{v^2+1} 3dv = \frac{1}{3} \arctan v + C \\ = \frac{1}{3} \arctan \left(\frac{u}{3} \right) + C$$

$$= \frac{1}{3} \arctan \left(\frac{x+1}{3} \right) + C$$

since $u=x+1$

So integral $\textcircled{4}$ is $-\frac{1}{2} \ln|(x+1)^2+9| - \frac{1}{3} \arctan \left(\frac{x+1}{3} \right)$.

Putting it all together, ANS:
$$\boxed{x^2+x+\ln|x|- \frac{1}{2} \ln|(x+1)^2+9| - \frac{1}{3} \arctan \left(\frac{x+1}{3} \right) + C}$$

Improper Integrals

There are two types of improper integrals.

- Limit of integration is ∞ or $-\infty$

e.g. $\int_0^\infty \frac{1}{1+x^2} dx$

- There is a discontinuity in the range of integration.

e.g. $\int_{-3}^2 \frac{1}{x} dx$ ← discontinuity at 0, where $-3 \leq x \leq 2$.

$\int_4^5 \frac{1}{\sqrt{x-4}} dx$ ← discontinuity at 4, where $4 \leq x \leq 5$.

(But note that

$\int_5^6 \frac{1}{\sqrt{x-4}} dx$ is NOT improper since the discontinuity at $x=4$ is not in the range of integration $[5, 6]$).

How do we handle these? By turning them into limits of definite integrals.

Example: (Limit of integration is ∞ or $-\infty$)

$$\int_0^\infty \frac{1}{1+x^2} dx$$

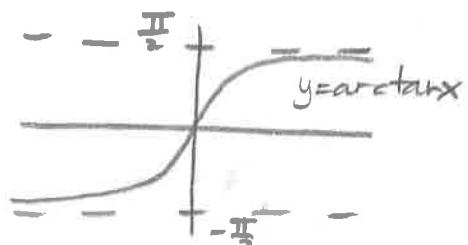
Solution: Write as a limit of a non-improper definite integral!

$$\int_0^\infty \frac{1}{1+x^2} dx = \lim_{N \rightarrow \infty} \left(\int_0^N \frac{1}{1+x^2} dx \right)$$

$$= \lim_{N \rightarrow \infty} (\arctan x) \Big|_0^N$$

$$= \lim_{N \rightarrow \infty} (\arctan N - \arctan 0)$$

\downarrow



$= 0$

$$\boxed{\frac{\pi}{2}}$$

So the integral is convergent!

// (has a finite value)

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Exercise: Check that

$$\int_{-\infty}^0 \frac{1}{1+x^2} dx = \frac{\pi}{2}$$

(You can write either

$$\int_{-\infty}^0 \frac{1}{1+x^2} dx$$

$$= \lim_{N \rightarrow -\infty} \int_N^0 \frac{1}{1+x^2} dx \text{ or}$$

$$= \lim_{N \rightarrow \infty} \int_{-N}^0 \frac{1}{1+x^2} dx$$

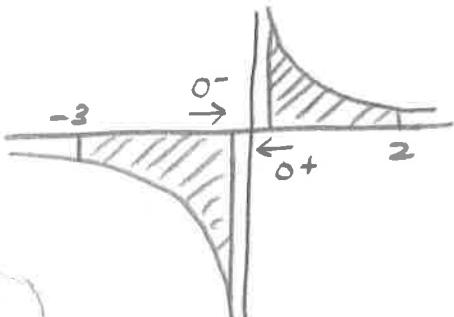
Example: (Discontinuity in range of integration)

$$\int_{-3}^2 \frac{1}{x} dx$$

Break up the integral at the bad point.

$$= \lim_{N \rightarrow 0^-} \int_{-3}^N \frac{1}{x} dx$$

$$+ \lim_{N \rightarrow 0^+} \int_N^2 \frac{1}{x} dx$$



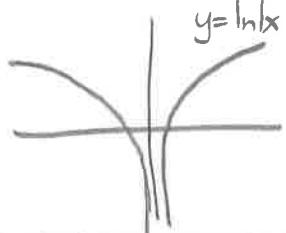
① When we have such sums in an improper integral, evaluate each part separately.

* I.E. if you do the first limit and it diverges (∞ or DNE), then the whole integral is divergent and you can stop there!

- If any part of the sum is infinity (∞ or $-\infty$), or DNE, the whole improper integral is divergent.
- If all parts of the sum are convergent, then the answer is just ① + ②.

Before doing ① and ②, recall the indefinite integral $\int \frac{1}{x} dx = \ln|x|$.

$$\text{Now, ① is } \lim_{N \rightarrow 0^-} \int_{-3}^N \frac{1}{x} dx = \lim_{N \rightarrow 0^-} \left[\ln|x| \right]_{-3}^N$$



$$= \lim_{N \rightarrow 0^-} \left(\underbrace{\ln(N)}_{\rightarrow -\infty} - \underbrace{\ln(-3)}_{=\ln 3} \right)$$

$$= "-\infty - \ln 3" \Rightarrow -\infty. \text{ (continued)}$$

So since ① is divergent (it is $-\infty$), the whole integral $\int_{-3}^2 \frac{1}{x} dx$ is divergent, and we can stop here without considering ②

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Example: (Another discontinuity example)

$$\int_4^5 \frac{1}{\sqrt{x-4}} dx$$



Since the discontinuity is at the endpoint, we only need one limit.

$$= \lim_{N \rightarrow 4^+} \int_N^5 \frac{1}{\sqrt{x-4}} dx$$

$$= \lim_{N \rightarrow 4^+} \left(2\sqrt{x-4} \right) \Big|_N^5 = \lim_{N \rightarrow 4^+} \frac{(2\sqrt{5-4} - 2\sqrt{N-4})}{2}$$

$$= 2 - 0 = \boxed{2}$$

//

For harder examples, we might need to break up our improper integral at points so that we get a sum of finite integrals that each only handle one point of impropriety at a time.

Example:

$$\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}} dx$$

Check: $1+e^{2x}$ is never zero since e^{2x} is always positive.

So there are no discontinuities on the range of integration.

There are two points of impropriety
- the limit of integration as and the limit of integration at $-\infty$.

Split the integral up at 0 (any point works) to get two integrals that are each improper in exactly one way.

$$= \int_{-\infty}^0 \frac{e^x}{1+e^{2x}} dx + \int_0^{\infty} \frac{e^x}{1+e^{2x}} dx$$

$$= \underbrace{\lim_{N \rightarrow \infty} \int_{-N}^0 \frac{e^x}{1+e^{2x}} dx}_{①} + \underbrace{\lim_{N \rightarrow \infty} \int_0^N \frac{e^x}{1+e^{2x}} dx}_{②}$$

Recall if ① and ② both finite, then answer is ① + ②.

If either ① or ② is ∞ , $-\infty$, or DNE, then the original improper integral is divergent.

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First, calculate the indefinite integral as an ASIDE.

ASIDE: $\int \frac{e^x}{1+e^{2x}} dx$ Let $u = e^x$
 $du = e^x dx$
 $= \int \frac{1}{1+u^2} du = \arctan(e^x) + C$

①: $\lim_{N \rightarrow \infty} \int_{-N}^0 \frac{e^x}{1+e^{2x}} dx = \lim_{N \rightarrow \infty} (\arctan(e^x)) \Big|_{-N}^0$
 $= \lim_{N \rightarrow \infty} (\arctan(1) - \arctan(e^{-N}))$
 $\quad \quad \quad \rightarrow \frac{\pi}{4} \quad \quad \quad \rightarrow \arctan(0) = 0$
 $= \frac{\pi}{4} - 0 = \frac{\pi}{4}$

②: $\lim_{N \rightarrow \infty} \int_0^N \frac{e^x}{1+e^{2x}} dx = \lim_{N \rightarrow \infty} (\arctan(e^x)) \Big|_0^N$
 $= \lim_{N \rightarrow \infty} (\underbrace{\arctan(e^N)}_{\text{"arctan}(\infty)\text{"}} - \underbrace{\arctan(1)}_{\frac{\pi}{4}})$
 $= \frac{\pi}{2}$
 $= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$

So $\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}} dx = \int_{-\infty}^0 \frac{e^x}{1+e^{2x}} dx + \int_0^{\infty} \frac{e^x}{1+e^{2x}} dx$
 $= \lim_{N \rightarrow \infty} \int_{-N}^0 \frac{e^x}{1+e^{2x}} dx + \lim_{N \rightarrow \infty} \int_0^N \frac{e^x}{1+e^{2x}} dx$
 $\quad \quad \quad \textcircled{1} \quad \quad \quad \textcircled{2}$
 $= \frac{\pi}{4} + \frac{\pi}{4} = \boxed{\frac{\pi}{2}}$

(We could have broken the integral up at any point, e.g. $\int_{-\infty}^1 \frac{e^x}{1+e^{2x}} dx + \int_1^{\infty} \frac{e^x}{1+e^{2x}} dx$, but why not choose an easy point like $x=0$?) //

Example:

$$\int_0^2 \frac{1}{x^2 - 4x + 3} dx$$

Solution: There are no infinite limits of integration.

So we just need to check for discontinuities.

$$x^2 - 4x + 3 = (x-1)(x-3) \text{ is zero at } x=1, x=3.$$

↖ denominator is zero at
 $x=1, x=3$

 $x=3$ is not a problem sinceit is not in $[0, 2]$. But $x=1$ is a problem!
(since $0 \leq 1 \leq 2$)

So break up the integral.

$$\begin{aligned} \int_0^2 \frac{1}{x^2 - 4x + 3} dx &= \int_0^2 \frac{1}{(x-1)(x-3)} dx \\ &= \underbrace{\lim_{N \rightarrow 1^-} \int_0^N \frac{1}{(x-1)(x-3)} dx}_{\textcircled{1}} + \underbrace{\lim_{N \rightarrow 1^+} \int_N^2 \frac{1}{(x-1)(x-3)} dx}_{\textcircled{2}} \end{aligned}$$

Compute $\textcircled{1}$ and $\textcircled{2}$. If any one of $\textcircled{1}$ and $\textcircled{2}$ is infinite or DNE, our integral is divergent.ASIDE: (Calculate indefinite integral)

$$\int \frac{1}{(x-1)(x-3)} dx$$

$$\begin{aligned} \frac{1}{(x-1)(x-3)} &= \frac{A}{x-1} + \frac{B}{x-3} \\ 1 &= A(x-3) + B(x-1) \end{aligned}$$

Method 1: Plug in.

$$\text{Plug in } x=3: \quad 1 = 2B \quad \boxed{B = \frac{1}{2}}$$

$$\text{Plug in } x=1: \quad 1 = -2A \quad \boxed{A = -\frac{1}{2}}$$

Method 2: Expand and match coefficients.

$$\begin{aligned} 1 &= Ax - 3A + Bx - B \\ 0x + 1 &= (A+B)x + (-3A - B) \end{aligned}$$

$$\begin{cases} A + B = 0 \\ -3A - B = 1 \end{cases} \xrightarrow{\text{Add}} \begin{cases} -2A = 1 \\ A = -\frac{1}{2} \end{cases} \Rightarrow \begin{cases} -\frac{1}{2} + B = 0 \\ B = \frac{1}{2} \end{cases}$$

$$\text{So } \int \frac{1}{(x-1)(x-3)} dx = -\frac{1}{2} \int \frac{1}{x-1} dx + \frac{1}{2} \int \frac{1}{x-3} dx \quad (15)$$

$$= -\frac{1}{2} \ln|x-1| + \frac{1}{2} \ln|x-3| + C.$$

Calculate ①:

$$\begin{aligned} ① \text{ is } & \lim_{N \rightarrow 1^-} \int_0^N \frac{1}{(x-1)(x-3)} dx \\ &= \lim_{N \rightarrow 1^-} \left(-\frac{1}{2} \ln|x-1| + \frac{1}{2} \ln|x-3| \right) \Big|_0^N \\ &= \lim_{N \rightarrow 1^-} \left(-\frac{1}{2} \ln|N-1| + \frac{1}{2} \ln|N-3| + \frac{1}{2} \ln|-1| - \frac{1}{2} \ln|-3| \right) \\ &\quad \downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow \\ &\quad N-1 \rightarrow 0 \text{ so} \qquad \frac{1}{2} \ln|1-2| = \frac{1}{2} \ln 2 \qquad \frac{1}{2} \ln(1) = 0 \qquad \frac{1}{2} \ln 3 \\ &\quad -\frac{1}{2} \ln|N-1| = -(-\infty) \qquad = \infty \\ &= "\infty + \frac{1}{2} \ln 2 + 0 - \frac{1}{2} \ln 3" \Rightarrow \infty \end{aligned}$$

So ① is ∞ and hence divergent. So we have that our original integral $\int \frac{1}{(x-1)(x-3)} dx$ is divergent. (We don't even need to calculate ②!) //

Final important note: Comparison Test can be used to tell that an integral converges or diverges.

Comparison Test: If $0 \leq g(x) \leq f(x)$ on $[a, b]$,

$$0 \leq \int_a^b g(x) dx \leq \int_a^b f(x) dx. * (a, b \text{ can be infinite limits of integration!})$$

So if $\int_a^b g(x) dx$ diverges, so does $\int_a^b f(x) dx$

If $\int_a^b f(x) dx$ is finite, so is $\int_a^b g(x) dx$.

Example: (Comparison Test)

(16)

Is $\int_0^3 \frac{2+\cos x}{\sqrt{x}} dx$ convergent or divergent?

Solution: Important fact:

$$-1 \leq \cos x \leq 1$$



$$\text{So } 1 \leq 2 + \cos x \leq 3.$$

also
 $-1 \leq \sin x \leq 1$

$$0 \leq \frac{1}{\sqrt{x}} \leq \frac{2+\cos x}{\sqrt{x}} \leq \frac{3}{\sqrt{x}}.$$

Thus, $\left[\int_0^3 \frac{1}{\sqrt{x}} dx \right]$

$$= \lim_{N \rightarrow 0^+} \int_N^3 \frac{1}{\sqrt{x}} dx$$

$$= \lim_{N \rightarrow 0^+} 2\sqrt{x} \Big|_N^3$$

$$= \lim_{N \rightarrow 0^+} 2\sqrt{3} - 2\sqrt{N}$$

$= 2\sqrt{3}$ convergent

can pull
constant
out of
improper
integrals too!

$$\rightarrow 3 \int_0^3 \frac{1}{\sqrt{x}} dx$$

$$= 3(2\sqrt{3})$$

$$= 6\sqrt{3}.$$

convergent

Therefore, $\int_0^3 \frac{2+\cos x}{\sqrt{x}} dx$



MUST BE FINITE

$$\text{since } \int_0^3 \frac{3}{\sqrt{x}} dx = 6\sqrt{3}.$$

so use Comparison Test.

$$\frac{2+\cos x}{\sqrt{x}} \leq \frac{3}{\sqrt{x}}.$$

(17)

Note that $\int_0^3 \frac{1}{\sqrt{x}} dx$ was convergent too, but that does not tell us anything about $\int_0^3 \frac{2 + \cos x}{\sqrt{x}} dx$ since $0 \leq \frac{1}{\sqrt{x}} \leq \frac{2 + \cos x}{\sqrt{x}}$. (The inequality needs to be the other way - look at the statement of Comparison Test).

Since you cannot tell ahead of time whether $\int_0^3 \frac{1}{\sqrt{x}} dx$ or $\int_0^3 \frac{3}{\sqrt{x}} dx$ will work with the Comparison Test, do them both. //

Example: (Another comparison test)

Is $\int_1^\infty \frac{4 + 3 \cos x}{x} dx$ convergent or divergent?

Solution: Use comparison test.

$$-1 \leq \cos x \leq 1$$

$$-3 \leq 3 \cos x \leq 3$$

$$1 \leq 4 + 3 \cos x \leq 7$$

$$0 \leq \frac{1}{x} \leq \frac{4 + 3 \cos x}{x} \leq \frac{7}{x}$$

Check: $\int_1^\infty \frac{1}{x} dx = \lim_{N \rightarrow \infty} \int_1^N \frac{1}{x} dx = \lim_{N \rightarrow \infty} \ln|x| \Big|_1^N$
 $= \lim_{N \rightarrow \infty} (\ln|N| - \ln|1|) = \infty$.

$$\int_1^\infty \frac{7}{x} dx = 7 \int_1^\infty \frac{1}{x} dx = 7(\infty) = \infty$$

(or you can just do it out)

So since $0 \leq \frac{1}{x} \leq \frac{4 + 3 \cos x}{x^2}$ and $\int_1^\infty \frac{1}{x} dx$ is divergent, by the Comparison Test, $\int_1^\infty \frac{4 + 3 \cos x}{x} dx$ is also divergent! //

Exercise: Show that $\int_1^\infty \frac{\arctan x}{x^2}$ is convergent using the Comparison Test. (Hint: $0 \leq \arctan x \leq \frac{\pi}{2}$ on $[1, \infty)$) (18)

Exercise: Calculate $\int_0^\infty xe^{-x} dx$ if it exists.

Exercise: Check that $\int_{-\infty}^\infty \frac{1}{x} dx$ is divergent.

(Hint: It is improper because of both infinite limits and the discontinuity at $x=0$. So express this as

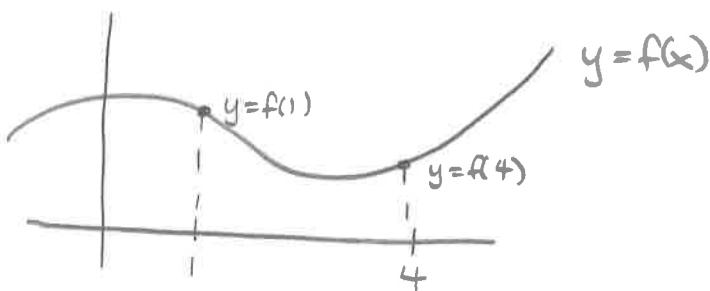
$$\int_{-\infty}^\infty \frac{1}{x} dx = \lim_{N \rightarrow \infty} \underbrace{\int_{-N}^{-1} \frac{1}{x} dx}_{\textcircled{1}} + \lim_{N \rightarrow 0^-} \underbrace{\int_{-1}^N \frac{1}{x} dx}_{\textcircled{2}} + \lim_{N \rightarrow 0^+} \underbrace{\int_N^1 \frac{1}{x} dx}_{\textcircled{3}} + \lim_{N \rightarrow \infty} \underbrace{\int_1^N \frac{1}{x} dx}_{\textcircled{4}}$$

where $x = -1$ and $x = 1$ are randomly chosen "splice" points.

If any of $\textcircled{1}, \textcircled{2}, \textcircled{3}, \textcircled{4}$ is divergent, then the whole integral is divergent.)

Arc Length

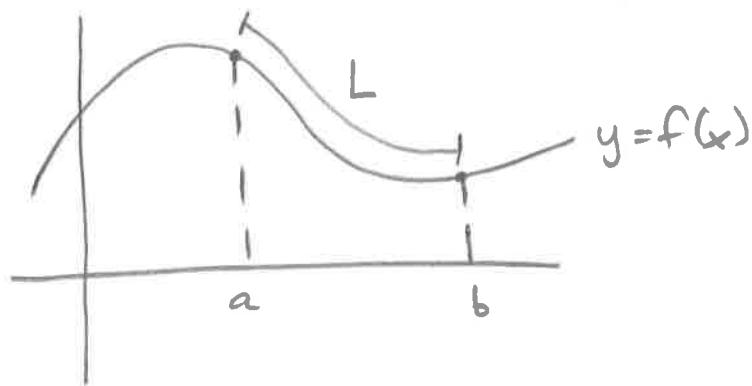
Given the graph of some curve $y = f(x)$, the graph is some function of x like this:



For each x , we have exactly one y -value, $f(x)$.

The arc length of f from $x=a$ to $x=b$ is the length of a string that we place along f from $x=a$ to $x=b$.

So visually, the arc length of $y = f(x)$ from $x=a$ to $x=b$ is



(19)

and is given by the formula

$$L = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

So arc length questions are easy in the sense that you just need to plug into the formula - but evaluating the integral can be hard! (since there is a huge square root).

A common strategy is to see if $1 + (f'(x))^2$ is a perfect square. If it is, then $\sqrt{1 + (f'(x))^2}$ is easy to compute (e.g. $\sqrt{x^2} = |x|$). If not, try to see if you can use a u-sub or trig sub.

Example: Find the arc length of $y = 2x^{3/2}$ from $x=1$ to $x=4$.

Solution: Use the formula!

$$a=1, b=4, f(x)=2x^{3/2} \text{ so } f'(x)=3x^{1/2}=3\sqrt{x}.$$

$$\begin{aligned} L &= \int_1^4 \sqrt{1 + (f'(x))^2} dx = \int_1^4 \sqrt{1 + (3\sqrt{x})^2} dx \\ &= \int_1^4 \sqrt{1 + 9x} dx \end{aligned}$$

We can immediately see that we can do a u-sub.

(20)

Do the indefinite integral first.

ASIDE

$$\int \sqrt{1+9x} dx$$

$$= \int \sqrt{u} \cdot \frac{1}{9} du$$

$$= \frac{1}{9} \int \sqrt{u} du = \frac{2}{27} u^{3/2} + C = \frac{2}{27} (1+9x)^{3/2} + C.$$

$$\text{Let } u = 1+9x$$

$$du = 9dx$$

$$dx = \frac{1}{9} du$$

$$\text{So } L = \int_1^4 \sqrt{1+9x} dx = \frac{2}{27} (1+9x)^{3/2} \Big|_1^4$$

$$= \boxed{\frac{2}{27} (37)^{3/2} - \frac{2}{27} (10)^{3/2}}$$

//

Example: Find the arc length of $y = \ln(\sec x)$ from $x=0$ to $x=\frac{\pi}{4}$. (From textbook exercises)

Solution: Use the formula!

Chain Rule
↓

$$a=0, b=\frac{\pi}{4}, f(x) = \ln(\sec x), f'(x) = \frac{1}{\sec x} \sec x \tan x = \tan x.$$

$$\text{So } L = \int_0^{\frac{\pi}{4}} \sqrt{1+(f'(x))^2} dx = \int_0^{\frac{\pi}{4}} \sqrt{1+\tan^2 x} dx.$$

Oof-now what? Well, if we remember our identity

$$1+\tan^2 x = \sec^2 x,$$

we see that the thing under the square root is a perfect square.
This saves us!

$$\sec^2 x = (\sec x)^2$$

$$L = \int_0^{\frac{\pi}{4}} \sqrt{1+\tan^2 x} dx = \int_0^{\frac{\pi}{4}} \sqrt{\sec^2 x} dx = \int_0^{\frac{\pi}{4}} |\sec x| dx$$

$$\text{Remember } \sec^2 x = (\sec x)^2$$

$$\text{and } \sqrt{A^2} = |A|, \text{ so}$$

$$\sqrt{(\sec x)^2} = |\sec x|.$$

(continued)

$$\text{So } L = \int_0^{\frac{\pi}{4}} |\sec x| dx = \int_0^{\frac{\pi}{4}} \sec x dx \quad (21)$$

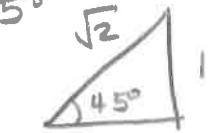
$|\sec x| = \sec x \text{ on } [0, \frac{\pi}{4}]$
 since $\sec x = \frac{1}{\cos x}$
 and $\cos x > 0$ on $[0, \frac{\pi}{4}]$,
 $\text{so } \sec x = \frac{1}{\cos x} > 0 \text{ on } [0, \frac{\pi}{4}]$.

From an integral table (provided in formula section on exam),

$$L = \int_0^{\frac{\pi}{4}} \sec x dx = \ln |\sec x + \tan x| \Big|_0^{\frac{\pi}{4}}$$

$$= \ln |\sec \frac{\pi}{4} + \tan \frac{\pi}{4}| - \ln |\sec 0 + \tan 0|$$

$$\frac{\pi}{4} = 45^\circ \quad = \ln |\sqrt{2} + 1| - \ln |1 + 0|$$



$$= \boxed{\ln |\sqrt{2} + 1|}$$

$$\sec 45^\circ = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{2}}{1} = \sqrt{2}$$

$$\tan 45^\circ = \frac{\text{opp}}{\text{adj}} = \frac{1}{1} = 1$$

//

Finally, let's do one more example where we can try to find a "hidden" perfect square.

← (A super important example!)

Example: Find the arc length of $y = \frac{1}{2}x^2 + 2x - \frac{\ln(x+2)}{4}$

from $x=0$ to $x=1$.

Solution: Use the formula! $a=0, b=1, f(x) = \frac{1}{2}x^2 + 2x - \frac{\ln(x+2)}{4}$

$$f'(x) = x+2 - \frac{1}{4(x+2)}$$



★ IMPORTANT: Notice that we see $x+2$ appearing in multiple places!

When this happens, group it together as such!

$$f'(x) = (x+2) - \frac{1}{4(x+2)}$$

Plugging into the formula,

$$\begin{aligned} L &= \int_0^1 \sqrt{1 + (f'(x))^2} dx \\ &= \int_0^1 \sqrt{1 + \left[(x+2) - \frac{1}{4(x+2)} \right]^2} dx \end{aligned}$$

Remember that $(A+B)^2 = A^2 + 2AB + B^2$
 $(A-B)^2 = A^2 - 2AB + B^2$.

Using $(A-B)^2 = A^2 - 2AB + B^2$ for
 $A = x+2, B = \frac{1}{4(x+2)}$, we get

$$\begin{aligned} L &= \int_0^1 \sqrt{1 + \left[(x+2) - \frac{1}{4(x+2)} \right]^2} dx \\ &= \int_0^1 \sqrt{1 + \left[(x+2)^2 - 2(x+2)\left(\frac{1}{4(x+2)}\right) + \frac{1}{16(x+2)^2} \right]} dx \\ &\quad \text{This is just } \frac{1}{2}. \\ &= \int_0^1 \sqrt{\underline{1} + \left[(x+2)^2 - \frac{1}{2} + \frac{1}{16(x+2)^2} \right]} dx \\ &= \int_0^1 \sqrt{(x+2)^2 + \frac{1}{2} + \frac{1}{16(x+2)^2}} dx \end{aligned}$$

Ugh this integral is NASTY! We could hope that

$$(x+2)^2 + \frac{1}{2} + \frac{1}{16(x+2)^2}$$

is a perfect square... Think $(A+B)^2 = \underline{A^2} + 2AB + \underline{B^2}$
 $(A-B)^2 = \underline{A^2} - 2AB + \underline{B^2}$

Look for a potential A^2 and B^2 in

$$(x+2)^2 + \frac{1}{2} + \frac{1}{16(x+2)^2}.$$

Hm, well $(x+2)^2$ is a square and
 $\frac{1}{16(x+2)^2}$ is $\left(\frac{1}{4(x+2)}\right)^2$. So maybe $(x+2)^2$ is A² and $\frac{1}{16(x+2)^2} = \left(\frac{1}{4(x+2)}\right)^2$ is B²

(23)

If we have that $\frac{1}{2} = 2(x+2)\left(\frac{1}{4(x+2)}\right)$, we would be good!

2 A B

$$\text{But indeed, } \frac{1}{2} = 2(x+2)\left(\frac{1}{4(x+2)}\right)$$

$$\begin{aligned} \text{So } (x+2)^2 + \frac{1}{2} + \frac{1}{16(x+2)^2} &= (x+2)^2 + 2(x+2)\left(\frac{1}{4(x+2)}\right) + \left(\frac{1}{4(x+2)}\right)^2 \\ &\quad A^2 \qquad \qquad \qquad + 2AB \qquad \qquad \qquad + B^2 \\ &= \left((x+2) + \frac{1}{4(x+2)}\right)^2. \end{aligned}$$

Thus, returning to our previous integral for L,

$$L = \int_0^1 \sqrt{(x+2)^2 + \frac{1}{2} + \frac{1}{16(x+2)^2}} dx$$

↙ But the thing inside the square root is a perfect square!

$$= \int_0^1 \sqrt{\left((x+2) + \frac{1}{4(x+2)}\right)^2} dx$$

$$= \int_0^1 \left| x+2 + \frac{1}{4(x+2)} \right| dx$$

$$= \int_0^1 x+2 + \frac{1}{4(x+2)} dx = \frac{1}{2}x^2 + 2x + \frac{1}{4}\ln|x+2| \Big|_0^1$$

↗

$$= \frac{1}{2} + 2 + \frac{1}{4}\ln(3)$$

$$-(0 + 0 + \frac{1}{4}\ln(2))$$

$$= \boxed{\frac{5}{2} + \frac{1}{4}(\ln 3 - \ln 2)}$$

can drop absolute values

since $x+2 + \frac{1}{4(x+2)} > 0$ for $0 \leq x \leq 1$

since on $0 \leq x \leq 1$,

$x+2 \geq 0$ and $\frac{1}{4(x+2)} \geq 0$.

//

Clearly, an important part of arc length is being able to recognize perfect squares. Here are some quick exercises.

(24)

Exercise: Write $9x^2 - 6x + 1$ and $9x^2 + 6x + 1$ as perfect squares.

Exercise: Write $x^4 - 4x^2 + 4$ as a perfect square.

(Hint: $(x^4) = (x^2)^2$. So $(x^4 - 4x^2 + 4)$
 $= (x^2)^2 - 4(x^2) + 2^2$)

Answer: $(x^2 - 2)^2$

Exercise: Write $(2x+1)^2 - \frac{1}{2} + \frac{1}{16(2x+1)^2}$ as

a perfect square. Write $\frac{1}{4}e^{2x} - \frac{1}{2} + \frac{1}{4}e^{-2x}$ as a perfect square. ↵ ANS:

Exercise: Expand out $((x+3)^2 - \frac{1}{3(x+3)^2})^2$. $(\frac{1}{2}e^x - \frac{1}{2}e^{-x})^2$

Answer: $(x+3)^4 - \frac{2}{3} + \frac{1}{9(x+3)^4}$.

Exercise: Write $(x+5)^4 - \frac{1}{2} + \frac{1}{16(x+5)^4}$ as a perfect square.

Answer: $\left((x+5)^2 - \frac{1}{4(x+5)^2} \right)^2$.

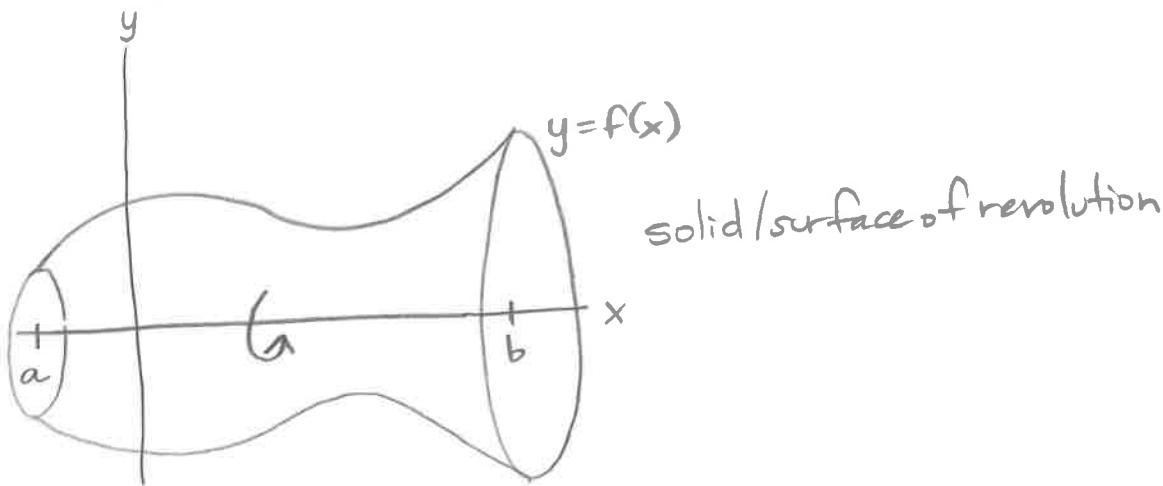


Please do these important exercises!

Surface area of surface of revolution

(25)

Remember that a solid of revolution is obtained by taking a curve and rotating it around some axis. For example, we can start with a curve $y = f(x)$ from $x = a$ to $x = b$ and rotate it around the x -axis to get a 3D solid.



The formula for the surface area of a surface of revolution obtained from rotating some curve $y = f(x)$ from $x = a$ to $x = b$ around the x -axis is:

$$A = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} dx$$

Again, in principle, these questions are easy since you just need to plug into the formula! But evaluating the integral can be tricky...

Some ideas: v -sub and trig sub usually work well.

If $1 + (f'(x))^2$ is a perfect square, again take advantage of this! (Since then $\sqrt{1 + (f'(x))^2}$ is easy).

Example: Find the surface area of the surface of revolution obtained by rotating $y = x^3$ from $x=1$ to $x=3$ around the x -axis.

(26)

Solution: Use the formula! $a=1, b=3, f(x)=x^3 \text{ so } f'(x)=3x^2$.

$$\begin{aligned} A &= \int_a^b 2\pi f(x) \sqrt{1+(f'(x))^2} dx \\ &= \int_1^3 2\pi x^3 \sqrt{1+(3x^2)^2} dx \\ &= \int_1^3 2\pi x^3 \sqrt{1+9x^4} dx \\ &= 2\pi \int_1^3 x^3 \sqrt{1+9x^4} dx \end{aligned}$$

How should we compute this integral? Notice that u-sub works well!

$$\begin{aligned} \text{ASIDE: } &\int x^3 \sqrt{1+9x^4} dx & \text{Let } u = 1+9x^4 \\ &= \int \sqrt{u} \frac{1}{36} du & du = 36x^3 dx \\ &= \frac{1}{36} \left(\frac{2}{3} \right) u^{3/2} + C = \frac{1}{54} u^{3/2} + C & x^3 dx = \frac{1}{36} du \\ &= \frac{1}{54} (1+9x^4)^{3/2} + C \end{aligned}$$

$$\begin{aligned} \text{So, } A &= 2\pi \int_1^3 x^3 \sqrt{1+9x^4} dx \\ &= 2\pi \left(\frac{1}{54} (1+9x^4)^{3/2} \right) \Big|_1^3 \\ &= \frac{\pi}{27} \left((1+9(81))^{3/2} - (1+9(1))^{3/2} \right) \\ &= \boxed{\frac{\pi}{27} \left((730)^{3/2} - (10)^{3/2} \right)} \quad // \end{aligned}$$

Example: Find the area of a surface of revolution obtained by rotating the curve

(27)

$$y = \frac{1}{6} e^{3x} + \frac{1}{6} e^{-3x} \text{ from } x = -1 \text{ to } x = 1$$

around the x-axis.

Solution: Use the formula! $a = -1, b = 1,$

$$f(x) = \frac{1}{6} e^{3x} + \frac{1}{6} e^{-3x}$$

$$f'(x) = \frac{1}{2} e^{3x} - \frac{1}{2} e^{-3x}.$$

$$\text{So } A = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} dx$$

$$= \int_{-1}^1 2\pi \left(\frac{1}{6} e^{3x} + \frac{1}{6} e^{-3x} \right) \sqrt{1 + \left(\frac{1}{2} e^{3x} - \frac{1}{2} e^{-3x} \right)^2} dx$$

$$= \int_{-1}^1 2\pi \left(\frac{1}{6} e^{3x} + \frac{1}{6} e^{-3x} \right) \sqrt{1 + \left(\left(\frac{1}{2} e^{3x} \right)^2 - 2 \left(\frac{1}{2} e^{3x} \right) \left(\frac{1}{2} e^{-3x} \right) + \left(\frac{1}{2} e^{-3x} \right)^2 \right)} dx$$

$$\text{Use } (A-B)^2$$

$$= A^2 - 2AB + B^2$$

$$\text{for } A = \frac{1}{2} e^{3x}, B = \frac{1}{2} e^{-3x}.$$

$$= \int_{-1}^1 2\pi \left(\frac{1}{6} e^{3x} + \frac{1}{6} e^{-3x} \right) \sqrt{1 + \left(\frac{1}{4} e^{6x} - \frac{1}{2} + \frac{1}{4} e^{-6x} \right)} dx$$

$$\begin{aligned} (\cancel{e^{3x}})^2 &= e^{6x} & 2 \left(\frac{1}{2} e^{3x} \right) \left(\frac{1}{2} e^{-3x} \right) \\ &= \frac{1}{2} e^{3x} e^{-3x} = \frac{1}{2} \end{aligned}$$

$$= \int_{-1}^1 2\pi \left(\frac{1}{6} e^{3x} + \frac{1}{6} e^{-3x} \right) \sqrt{\frac{1}{4} e^{6x} + \frac{1}{2} + \frac{1}{4} e^{-6x}} dx$$

Eek - looks hard. Maybe, we can hope $\frac{1}{4} e^{6x} + \frac{1}{2} + \frac{1}{4} e^{-6x}$ is a perfect square. Indeed, $\frac{1}{4} e^{6x} = \left(\frac{1}{2} e^{3x} \right)^2$ could be A^2 and $\frac{1}{4} e^{-6x} = \left(\frac{1}{2} e^{-3x} \right)^2$ could be B^2 . And

$$2AB = 2 \left(\frac{1}{2} e^{3x} \right) \left(\frac{1}{2} e^{-3x} \right) = \frac{1}{2} \checkmark$$

$$\begin{aligned}
 \text{So } & \frac{1}{4}e^{6x} + \frac{1}{2} + \frac{1}{4}e^{-6x} \\
 &= \left(\frac{1}{2}e^{3x}\right)^2 + 2\left(\frac{1}{2}e^{3x}\right)\left(\frac{1}{2}e^{-3x}\right) + \left(\frac{1}{2}e^{-3x}\right)^2 \\
 &= \left(\frac{1}{2}e^{3x} + \frac{1}{2}e^{-3x}\right)^2.
 \end{aligned}
 \tag{28}$$

$$\text{Thus, } A = \int_{-1}^1 2\pi \left(\frac{1}{6}e^{3x} + \frac{1}{6}e^{-3x}\right) \sqrt{\frac{1}{4}e^{6x} + \frac{1}{2} + \frac{1}{4}e^{-6x}} dx$$

$$= \int_{-1}^1 2\pi \left(\frac{1}{6}e^{3x} + \frac{1}{6}e^{-3x}\right) \sqrt{\left(\frac{1}{2}e^{3x} + \frac{1}{2}e^{-3x}\right)^2} dx$$

$$= \int_{-1}^1 2\pi \left(\frac{1}{6}e^{3x} + \frac{1}{6}e^{-3x}\right) \left| \frac{1}{2}e^{3x} + \frac{1}{2}e^{-3x} \right| dx$$



can drop the absolute value signs since $\frac{1}{2}e^{3x} + \frac{1}{2}e^{-3x}$ is always positive (e to any power is > 0)

$$= \int_{-1}^1 2\pi \left(\frac{1}{6}e^{3x} + \frac{1}{6}e^{-3x}\right) \left(\frac{1}{2}e^{3x} + \frac{1}{2}e^{-3x}\right) dx$$

$$= 2\pi \int_{-1}^1 \frac{1}{12}e^{6x} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12}e^{-6x} dx \leftarrow \text{F.O.I.L.}$$

and use
 $e^{3x}e^{-3x} = 1$

$$= 2\pi \int_{-1}^1 \frac{1}{12}e^{6x} + \frac{1}{6} + \frac{1}{12}e^{-6x} dx$$

$$= 2\pi \left(\frac{1}{72}e^{6x} + \frac{1}{6}x - \frac{1}{72}e^{-6x} \right) \Big|_{-1}^1$$

$$= 2\pi \left(\frac{1}{72}e^6 + \frac{1}{6} - \frac{1}{72}e^{-6} - \frac{1}{72}e^{-6} + \frac{1}{6} + \frac{1}{72}e^6 \right)$$

$$= \boxed{2\pi \left(\frac{1}{36}e^6 + \frac{1}{3} - \frac{1}{36}e^{-6} \right)}$$



Example: Find the area of the surface of revolution obtained by rotating the curve $y = e^x$ from $x = 0$ to $x = 1$ around the x -axis.

Solution: Use the formula! $a = 0, b = 1, f(x) = e^x, f'(x) = e^x$.

$$\begin{aligned} A &= \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} dx \\ &= \int_0^1 2\pi e^x \sqrt{1 + (e^x)^2} dx \\ &= \int_0^1 2\pi e^x \sqrt{1 + e^{2x}} dx = 2\pi \int_0^1 e^x \sqrt{1 + e^{2x}} dx \end{aligned} \quad (29)$$

ASIDE: Calculate the indefinite integral.

$$\begin{aligned} \int e^x \sqrt{1 + e^{2x}} dx &\quad \text{Use } u = e^x. \\ &\quad du = e^x dx. \\ &= \int \sqrt{1 + u^2} du \quad \nwarrow \text{Trig sub!} \\ &= \int \sec \theta \sec^2 \theta d\theta \\ &= \int \sec^3 \theta d\theta \\ &= \frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| \end{aligned}$$



$$\begin{aligned} \tan \theta &= u & du &= \sec^2 \theta d\theta \\ \sec \theta &= \sqrt{1+u^2} \end{aligned}$$

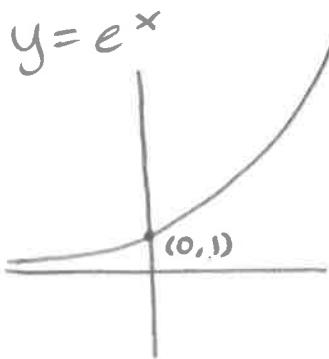
This integral would probably be given to you!

$$\begin{aligned} &= \frac{1}{2} \sqrt{1+u^2} \cdot u + \frac{1}{2} \ln |\sqrt{1+u^2} + u| + C \\ &= \frac{1}{2} e^x \sqrt{1+e^{2x}} + \frac{1}{2} \ln |\sqrt{1+e^{2x}} + e^x| + C. \end{aligned}$$

$$\begin{aligned} \text{So } A &= 2\pi \int_0^1 e^x \sqrt{1+e^{2x}} dx = 2\pi \left(\frac{1}{2} e^x \sqrt{1+e^{2x}} + \frac{1}{2} \ln |\sqrt{1+e^{2x}} + e^x| \right) \Big|_0^1 \\ &= 2\pi \left(\frac{1}{2} e \sqrt{1+e^2} + \frac{1}{2} \ln |\sqrt{1+e^2} + e| - \frac{1}{2} e^0 \sqrt{1+e^0} - \frac{1}{2} \ln |\sqrt{1+1}| \right) \\ &= \boxed{2\pi \left(\frac{1}{2} e \sqrt{1+e^2} + \frac{1}{2} \ln |\sqrt{1+e^2} - e| - \frac{1}{2} \sqrt{2} - \frac{1}{2} \ln |\sqrt{2} + 1| \right)} \end{aligned}$$

Some Graphs and Formulas

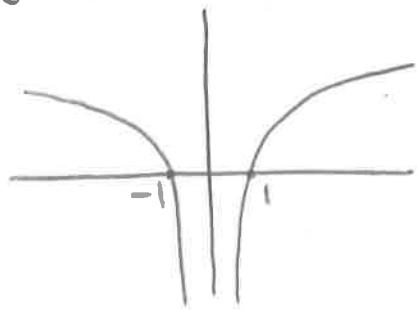
(30)



$$\lim_{x \rightarrow -\infty} e^x = 0 \quad \lim_{x \rightarrow \infty} e^{-x} = 0$$

$$\lim_{x \rightarrow \infty} e^x = \infty \quad \lim_{x \rightarrow -\infty} e^{-x} = \infty$$

$$y = \ln|x|$$



$$\ln(1) = 0. \text{ So } \ln|1| = \ln|-1| = 0.$$

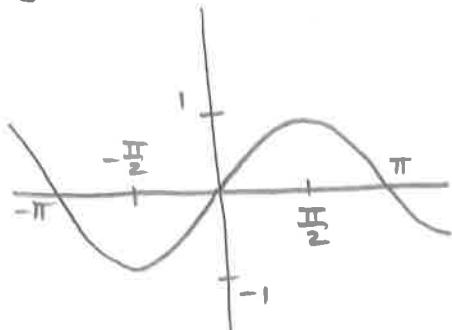
$$\lim_{x \rightarrow \infty} \ln|x| = \infty$$

$$\lim_{x \rightarrow 0^+} \ln|x| = -\infty$$

$$\lim_{x \rightarrow 0^-} \ln|x| = -\infty$$

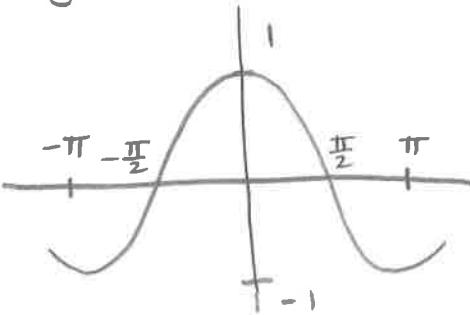
$$\lim_{x \rightarrow -\infty} \ln|x| = \infty$$

$$y = \sin x$$



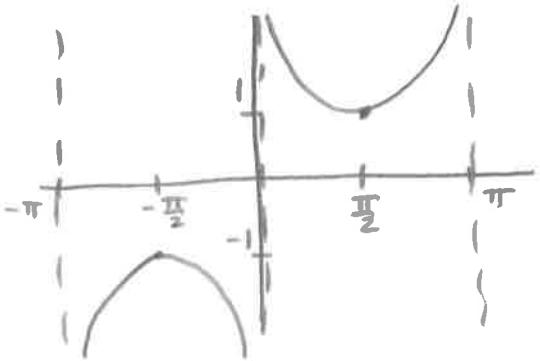
$$-1 \leq \sin x \leq 1$$

$$y = \cos x$$

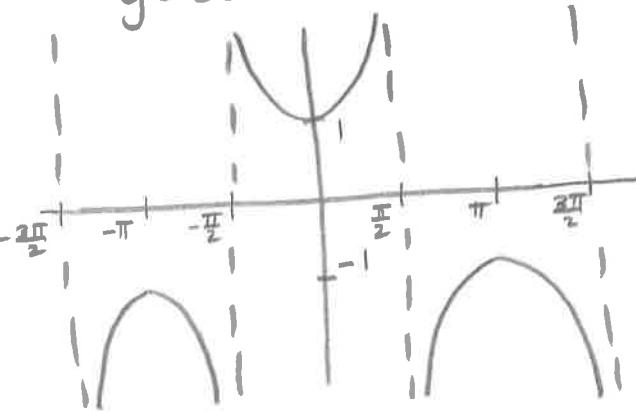


$$-1 \leq \cos x \leq 1$$

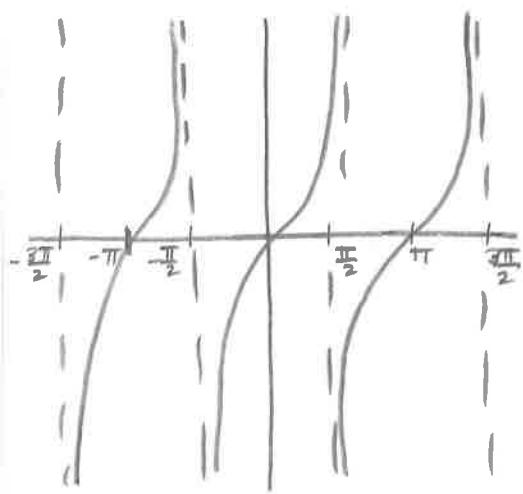
$$y = \csc x \quad (\csc x = \frac{1}{\sin x})$$



$$y = \sec x \quad (\sec x = \frac{1}{\cos x})$$

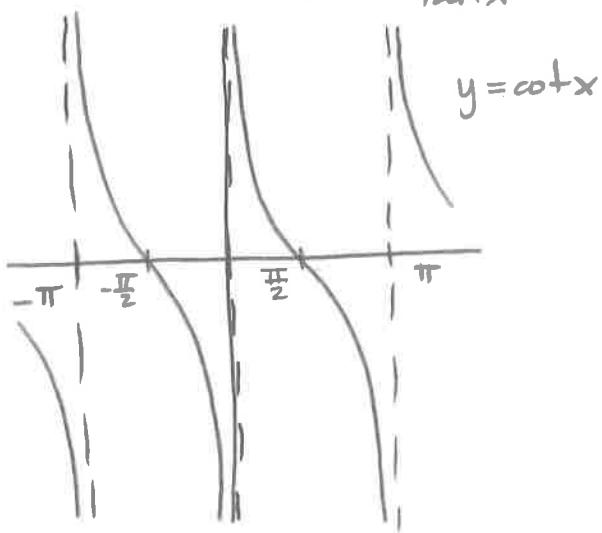


$$y = \tan x = \frac{\sin x}{\cos x}$$

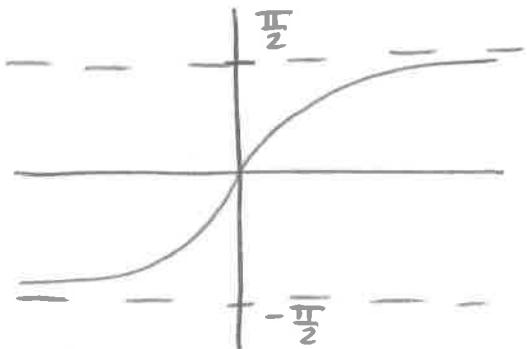


$$y = \cot x = \frac{\cos x}{\sin x} = \frac{1}{\tan x}$$

(31)



$$y = \arctan x$$



$$\lim_{x \rightarrow \infty} \arctan x = \frac{\pi}{2}$$

$$\lim_{x \rightarrow -\infty} \arctan x = -\frac{\pi}{2}$$

Derivative formulas:

$$\frac{d}{dx}(e^x) = e^x \quad \frac{d}{dx}(\ln|x|) = \frac{1}{x} \quad \frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}(\sin x) = \cos x \quad \frac{d}{dx}(\tan x) = \sec^2 x \quad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cos x) = -\sin x \quad \frac{d}{dx}(\cot x) = -\csc^2 x \quad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}} \quad \frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2} \quad \frac{d}{dx}(\text{arccot } x) = -\frac{1}{1+x^2}$$

Integral formulas:

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + C \text{ for } n \neq -1$$

$$\text{If } n = -1, \int x^{-1} dx = \int \frac{1}{x} dx = \ln|x| + C$$

↑
don't forget the absolute values!

$$\int \sin x dx = -\cos x + C$$

$$\int \cos x dx = \sin x + C$$

$$\int \sec^2 x dx = \tan x + C$$

$$\int \csc^2 x dx = -\cot x + C$$

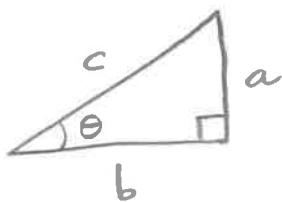
$$\int \sec x \tan x dx = \sec x + C$$

$$\int \csc x \cot x dx = -\csc x + C$$

$$\int e^x dx = e^x + C$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$$

$$\int \frac{1}{1+x^2} dx = \arctan x + C$$

Trigonometry:

$$a^2 + b^2 = c^2$$

$$c = \sqrt{a^2 + b^2} \quad \text{hypotenuse}$$

$$b = \sqrt{c^2 - a^2} \quad \text{legs}$$

$$a = \sqrt{c^2 - b^2}$$

$$\sin \theta = \frac{\text{opp}}{\text{hyp}} \quad \cos \theta = \frac{\text{adj}}{\text{hyp}} \quad \tan \theta = \frac{\text{opp}}{\text{adj}}$$

$$\csc \theta = \frac{\text{hyp}}{\text{opp}} \quad \sec \theta = \frac{\text{hyp}}{\text{adj}} \quad \cot \theta = \frac{\text{adj}}{\text{opp}}$$

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$1 + \cot^2 \theta = \csc^2 \theta$$

$$\begin{aligned} \sin(2\theta) &= 2 \sin \theta \cos \theta \\ \cos(2\theta) &= \cos^2 \theta - \sin^2 \theta \\ &= 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta \end{aligned}$$

$$\text{Power-reducing formulas: } \cos^2 \theta = \frac{1 + \cos(2\theta)}{2}$$

$$\sin^2 \theta = \frac{1 - \cos(2\theta)}{2}$$