

# Differential Equations Review Packet (Math 1B)

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# Differential Equations

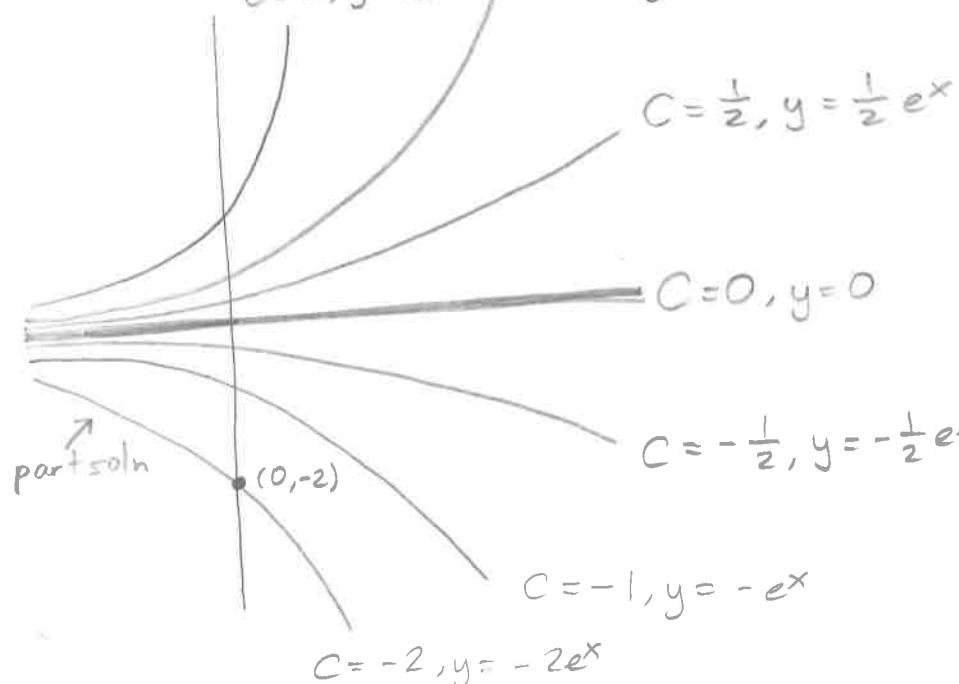
Remember that a differential equation is an equation involving  $y$  and its derivatives. The largest number of derivatives in the equation is called the order. Recall that the solution to a differential equation is a general solution involving a number of arbitrary constants equal to the order of the equation. Any choice of values for these arbitrary constants gives a valid solution. However, if we have enough initial conditions (the same number as the order), we can find a particular solution, where we can use the initial conditions to solve for the specific values of the arbitrary constants.

A Simple Example: Consider  $y' = y$ . This is first order.

The general solution is  $y = Ce^x$  since  $(Ce^x)' = Ce^x$  so  $y' = y$ . Notice there is only one arbitrary constant since this equation is first order.

$y = Ce^x$  is a solution, in the sense that for any value of  $C$ , this is a solution. e.g.  $y = 2e^x$ ,  $y = 0$  ( $0e^x$ ),  $y = -4e^x$  are all solutions. So the general solution is  $y = Ce^x$ , which is a family of solutions, as shown below.

$$C=2, y=2e^x \quad C=1, y=e^x$$



As shown in the diagram, if we have the initial condition  $y(0) = -2$ , we can pick out a single particular solution by solving for the arbitrary constant  $C$ .

$$y(0) = -2 \quad y = Ce^x$$

$$-2 = Ce^0$$

$$\text{so } C = -2.$$

So  $y = -2e^x$  is the particular solution satisfying  $y(0) = 2$ . initial condition.

## Section 1:

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### Graphing Solutions to Autonomous Differential Equations

Some problems will ask you to graph particular solutions to autonomous differential equations. These are equations of the form  $y' = h(y)$  for some function  $h$ .

We can visualize solutions of autonomous differential equations using slope fields. But these take a long time to draw. Here is another way of visualizing particular solutions.

Given  $y' = h(y)$ , to graph the particular solution with the initial condition  $y(a) = b$ ,

- ① First, find the constant solutions of the form  $y = k$  to the equation  $y' = h(y)$  and graph them.
- ② Plot the point  $(a, b)$  that the particular solution passes through. If  $(a, b)$  is on one of the constant solutions you plotted, then the particular solution through  $(a, b)$  is just  $y = b$ .
- ③ If not, find  $h(b)$ . Since  $y'|_{(a,b)} = h(y)|_{(a,b)} = h(b)$ ,  $h(b)$  is the slope of the particular solution at  $(a, b)$ . Draw a line segment of slope  $h(b)$  through  $(a, b)$ , as in a slope field.
- ④ Use the following two rules to graph the particular solution through  $(a, b)$ .

Rule 1: Different particular solutions cannot intersect.  
(In particular, your particular solution cannot intersect any constant solutions you drew or any other particular solution you drew.)

Rule 2: Any particular solution is either constant, strictly increasing, or strictly decreasing. (This is a property of autonomous differential equations.)

Draw more slope field lines as needed to draw an accurate particular solution.

Example: Draw the particular solutions to

$$y' = (y^2 - 16)(y^2 - 1)$$

satisfying the initial conditions

$$\begin{aligned} y(0) &= 1, \quad y(0) = 3, \quad y(2) = 5, \quad y(-2) = -2, \\ y(0) &= -5, \quad y(0) = 0. \end{aligned}$$

Solution: First find the constant solutions of the form  $y = k$  for  $k$  constant.  $y = k \Rightarrow \frac{dy}{dx} = 0$ .

$$0 = (k^2 - 16)(k^2 - 1)$$

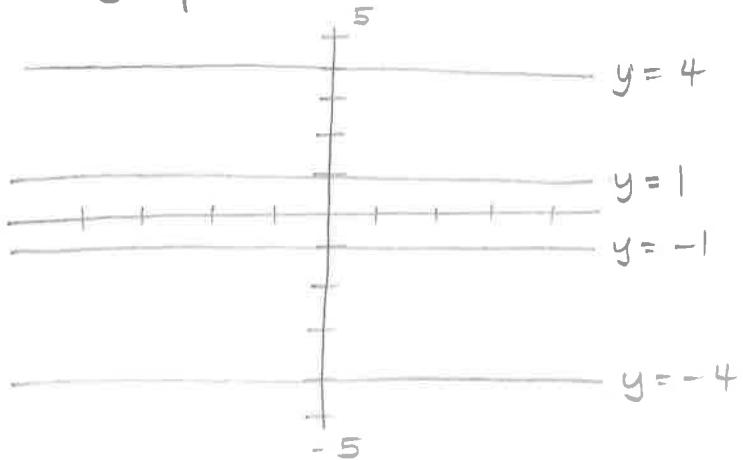
$$0 = (k-4)(k+4)(k-1)(k+1)$$

$$k = -4, -1, 1, 4$$

Constant solutions:

$$y = -4, y = -1, y = 1, y = 4$$

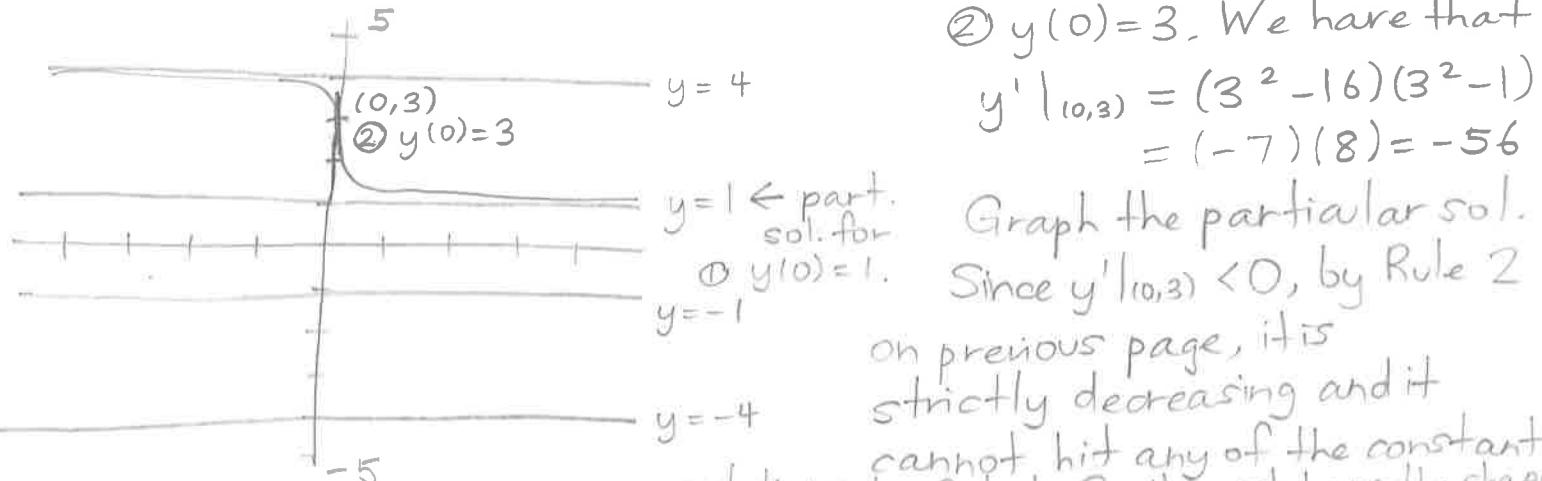
Then graph the constant solutions:



① Note that for  $y(0) = 1$ ,  $(0, 1)$  is on the constant solution  $y = 1$ . So  $y = 1$  is the particular solution passing through  $(0, 1)$ .

②  $y(0) = 3$ . We have that

$$\begin{aligned} y'(0,3) &= (3^2 - 16)(3^2 - 1) \\ &= (-7)(8) = -56 \end{aligned}$$



Graph the particular sol. Since  $y'(0,3) < 0$ , by Rule 2

on previous page, it is strictly decreasing and it cannot hit any of the constant solutions by Rule 1. So it must have the shape shown on the graph.

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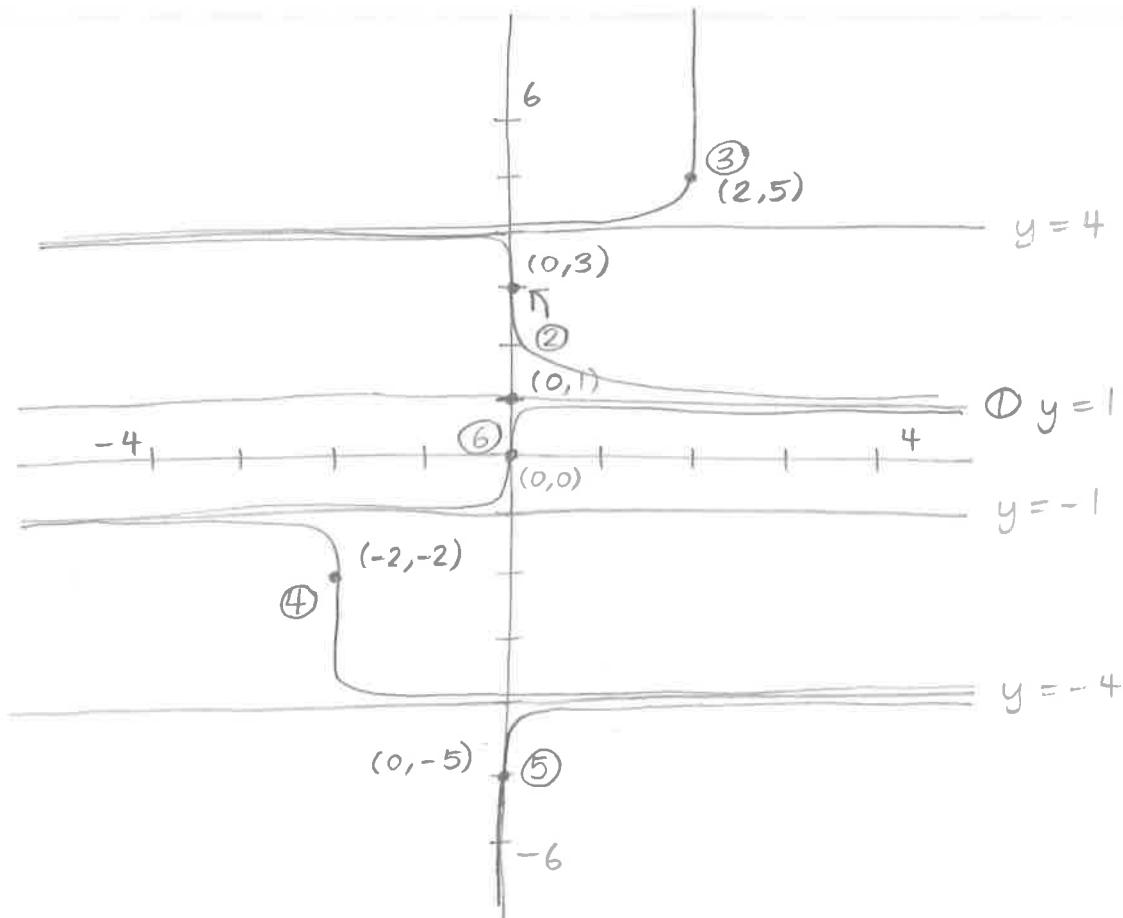
$$\textcircled{3} \quad y(2) = 5 \quad y'|_{(2,5)} = (5^2 - 16)(5^2 - 1) = 9(24) = 216$$

$$\textcircled{4} \quad y(-2) = -2 \quad y'|_{(-2,-2)} = ((-2)^2 - 16)((-2)^2 - 1) = (-12)(3) = -36$$

$$\textcircled{5} \quad y(0) = -5 \quad y'|_{(0,-5)} = ((-5)^2 - 16)((-5)^2 - 1) = 216$$

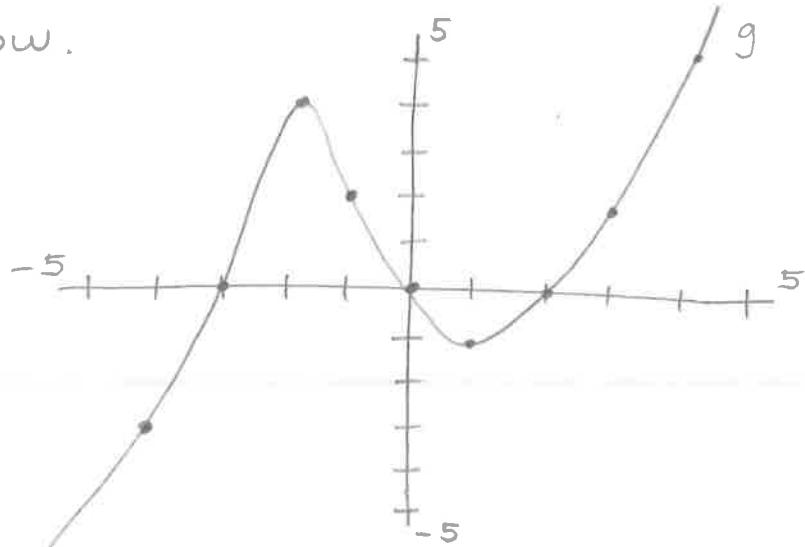
$$\textcircled{6} \quad y(0) = 0 \quad y'|_{(0,0)} = (0^2 - 16)(0^2 - 1) = 16$$

So by Rule 2, particular solutions  $\textcircled{3}, \textcircled{5}, \textcircled{6}$  are strictly increasing and particular solution  $\textcircled{4}$  is strictly decreasing. Then use Rule 1 to graph the solutions. We graph all particular solutions below.



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Example: Graph the particular solutions to  $y' = g(y)$  satisfying the initial conditions  $y(0) = 1$ ,  $y(2) = 2$ ,  $y(-1) = -1$ ,  $y(0) = -4$ ,  $y(0) = 4$ , where  $g$  is given by the graph below.



Solution: Find the constant solutions to  $y' = g(y)$ .

Consider  $y = k$  for  $k$  constant.  $\frac{dy}{dx} = 0 \Rightarrow 0 = g(k)$

Using the graph of  $g$ , we see  $k = -3, 0, 2$ .

So the constant solutions are  $y = -3, y = 0, y = 2$ .

$$y = 2 \quad \textcircled{1} \quad y(0) = 1.$$

$$y'|_{(0,1)} = g(1) = -1.$$

②  $y(2) = 2$ .  $(2, 2)$  is on the constant solution  $y = 2$  so  $y = 2$  is the particular solution for  $y(2) = 2$ .

$$\textcircled{3} \quad y(-1) = -1.$$

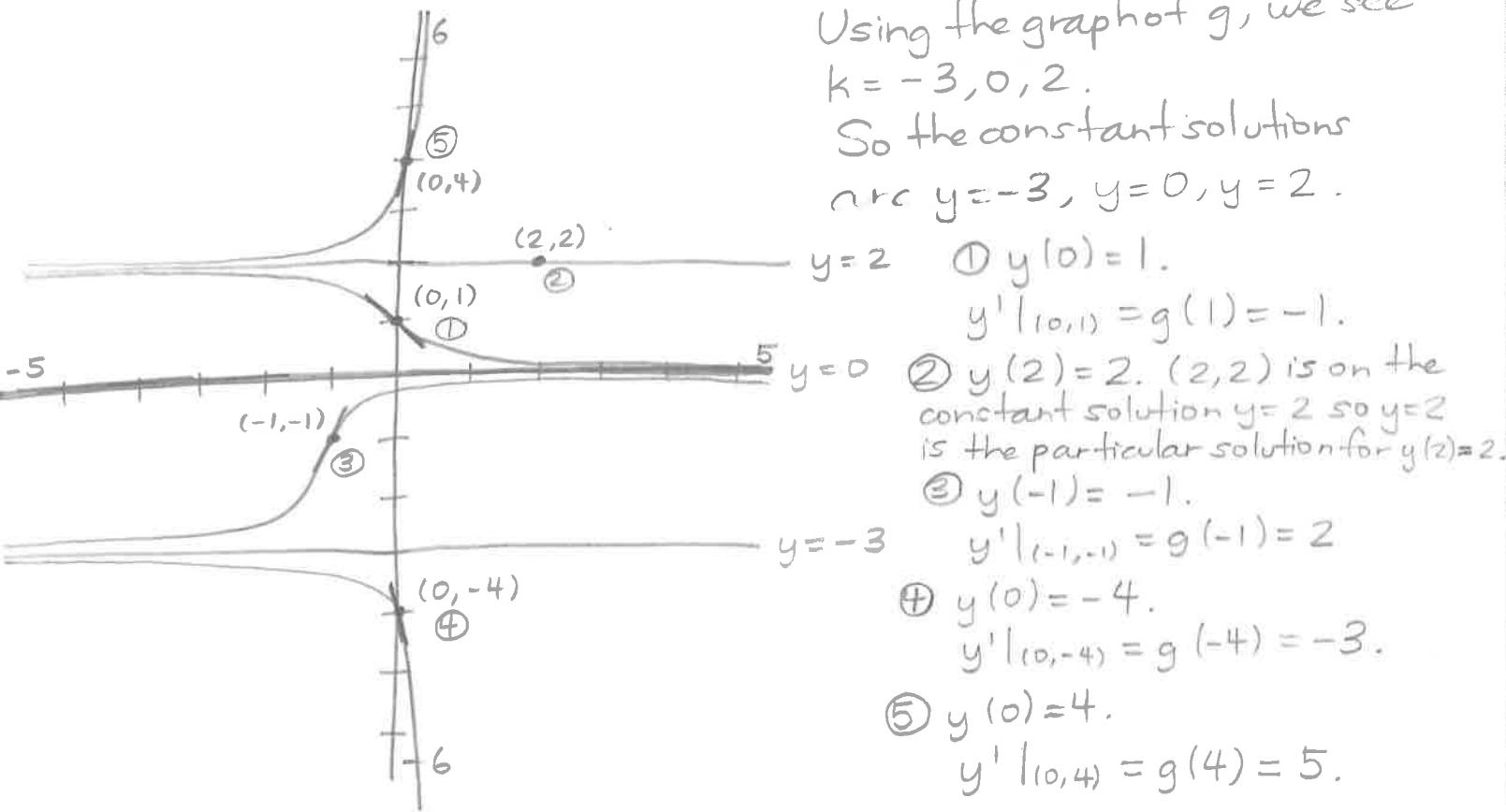
$$y'|_{(-1,-1)} = g(-1) = 2$$

$$\textcircled{4} \quad y(0) = -4.$$

$$y'|_{(0,-4)} = g(-4) = -3.$$

$$\textcircled{5} \quad y(0) = 4.$$

$$y'|_{(0,4)} = g(4) = 5.$$



Use Rules 1 and 2 to graph the particular solutions.

## Section 2: First-Order Differential Equations

There are two main techniques for solving first-order differential equations.

① Separation of variables (for separable equations)

② Integrating factor (for first-order linear differential equations)

### Method 1: Separation of Variables

The Idea: When we integrate, we cannot mix variables.

For example, the variable of integration and the variable in the

$\xrightarrow{\text{the } dx \text{ variable}}$

integrand in the integral

$$\int y^2 dx$$

do not match, so this integral does not make sense.

However, these two integrals make sense since the variable in the integrand and differential match:

$$\int \frac{1}{x} dx = \ln|x| + C \quad \int \frac{y^2}{1+y^3} dy = \frac{1}{3} \ln|1+y^3| + C$$

- \* So the idea of separation of variables is that if we treat  $\frac{dy}{dx}$  as a fraction, if we can move all of the  $y$ 's with  $dy$  to one side and if we can move all of the  $x$ 's with  $dx$  to the other side, we can integrate both sides in a way that makes sense. If an equation can be "separated" in this way, it is called a separable equation. Note: Not all first-order differential equations are separable!
- \* Whenever you separate variables, you must check separately for constant solutions of the form  $y=k$  for  $k$  a constant (e.g.  $y=0$ , or  $y=3$ , or  $y=-4$  for example) since separation of variables tends to miss constant solutions.

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Example: Find the general solution to

$$\frac{\csc x}{x} \frac{dy}{dx} = 1 + y^2.$$

Solution: We note that this equation is separable. If we think of  $\frac{dy}{dx}$  as a fraction, we can move all  $y$ 's to the left and all  $x$ 's to the right to get  $\frac{1}{1+y^2} dy = \frac{x}{\csc x} dx$ . Step 1: Separate variables to find nonconstant solutions.

$$\frac{1}{1+y^2} dy = x \sin x dx \quad (\text{since } \csc x = \frac{1}{\sin x})$$

$$\int \frac{1}{1+y^2} dy = \int x \sin x dx$$

Since the variables in the integrand and differential match on both sides, these integrals make sense!

$$\arctan y = -x \cos x + \sin x + C \quad <$$

$$\text{ASIDE: } \int x \sin x dx = -x \cos x + \int \cos x dx$$

$$= -x \cos x + \sin x + C$$

$$u = x \quad dv = \sin x dx \\ du = dx \quad v = -\cos x$$

$$y = \tan(-x \cos x + \sin x + C)$$

(nonconstant solution)

Step 2: Check separately for constant solutions.

Move  $x$  and  $\frac{dy}{dx}$  to one side and all  $y$ 's to the other side.

$$\Rightarrow \frac{\csc x}{x} \frac{dy}{dx} = 1 + y^2. \quad \text{Consider } y = k \text{ for } k \text{ constant.}$$

$$\Rightarrow \frac{dy}{dx} = 0.$$

Plug in and solve for allowable constants  $k$ .

$$\frac{\csc x}{x}(0) = 1 + k^2$$

$$0 = 1 + k^2 \Rightarrow \text{no solutions } k.$$

So there are no constant solutions.

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Example: Find a particular solution to

$$(2x+3) \frac{dy}{dx} + y^2 - 4y = 0$$

for each of the initial conditions:  $y(2) = -3$ ,  
 $y(2) = 4$ .

Solution: To find a particular solution to an initial value problem (IVP), first find a general solution, then use the initial condition to solve for the arbitrary constant to get the particular solution.

So first find the general solution. Note that this equation is separable.

Step 1: Separate variables to find nonconstant solutions.

$$\begin{aligned} (2x+3) \frac{dy}{dx} + y^2 - 4y &= 0 \\ (2x+3) \frac{dy}{dx} &= -y^2 + 4y \\ -\frac{1}{y^2-4y} dy &= \frac{1}{2x+3} dx \\ -\int \frac{1}{y(y-4)} dy &= \int \frac{1}{2x+3} dx \\ -\frac{1}{4} \ln \left| \frac{y-4}{y} \right| &= \frac{1}{2} \ln |2x+3| + C \\ \ln \left| 1 - \frac{4}{y} \right| &= -2 \ln |2x+3| - 4C \\ \ln \left| 1 - \frac{4}{y} \right| &= \ln \left| \frac{1}{(2x+3)^2} \right| - 4C \quad \uparrow \ln |(2x+3)^{-2}| \end{aligned}$$

$$\left| 1 - \frac{4}{y} \right| = e^{-4C} \left| \frac{1}{(2x+3)^2} \right|$$

$$1 - \frac{4}{y} = \pm e^{-4C} \left| \frac{1}{(2x+3)^2} \right|$$

ASIDE:

$$\begin{aligned} \frac{1}{y(y-4)} &= \frac{A}{y} + \frac{B}{y-4} \\ 1 &= A(y-4) + By \\ 1 &= (A+B)y - 4A \\ A+B &= 0, -4A = 1 \\ \text{so } A &= -\frac{1}{4}, B = \frac{1}{4} \\ \text{Thus, } \int \frac{1}{y(y-4)} dy &= \frac{1}{4} \int \frac{1}{y-4} dy - \frac{1}{4} \int \frac{1}{y} dy \\ &= \frac{1}{4} (\ln|y-4| - \ln|y|) + C \\ &= \frac{1}{4} \ln \left| \frac{y-4}{y} \right| + C \end{aligned}$$

$$y = \frac{4}{1 \pm \frac{e^{-4C}}{(2x+3)^2}}$$

nonconstant  
solution

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Step 2: Find constant solutions.

Move  $x$  and  $\frac{dy}{dx}$  to one side and  $y$  to the other.

$$(2x+3)\frac{dy}{dx} = 4y - y^2. \quad \text{Let } y = k \text{ for } k \text{ constant.}$$

$$\frac{dy}{dx} = 0.$$

Plug in to find allowable constants  $k$ .

$$(2x+3)(0) = 4k - k^2$$

$$0 = 4k - k^2$$

$$0 = k(4-k) \Rightarrow k=0, 4$$

So the constant solutions are

$$y=0, y=4$$

Now use the initial conditions to find particular solutions.

①  $y(2) = -3$ . The nonconstant solution is

$$y = \frac{4}{1 \pm \frac{e^{-4c}}{(2x+3)^2}} \quad -3 = \frac{4}{1 \pm \frac{e^{-4c}}{7^2}}$$

$$1 \pm \frac{e^{-4c}}{49} = -\frac{4}{3}$$

$$\pm \frac{e^{-4c}}{49} = -\frac{7}{3}$$

$$\pm e^{-4c} = -\frac{1}{21}$$

So particular solution is

$$y = \frac{4}{1 - \frac{1}{21} \left( \frac{1}{(2x+3)^2} \right)}$$

②  $y(2) = 4$ . Notice that  $y=4$  is a constant solution, so the particular solution for the initial condition  $y(2) = 4$  is the constant solution  $y=4$ .

# An Application of Separation of Variables:

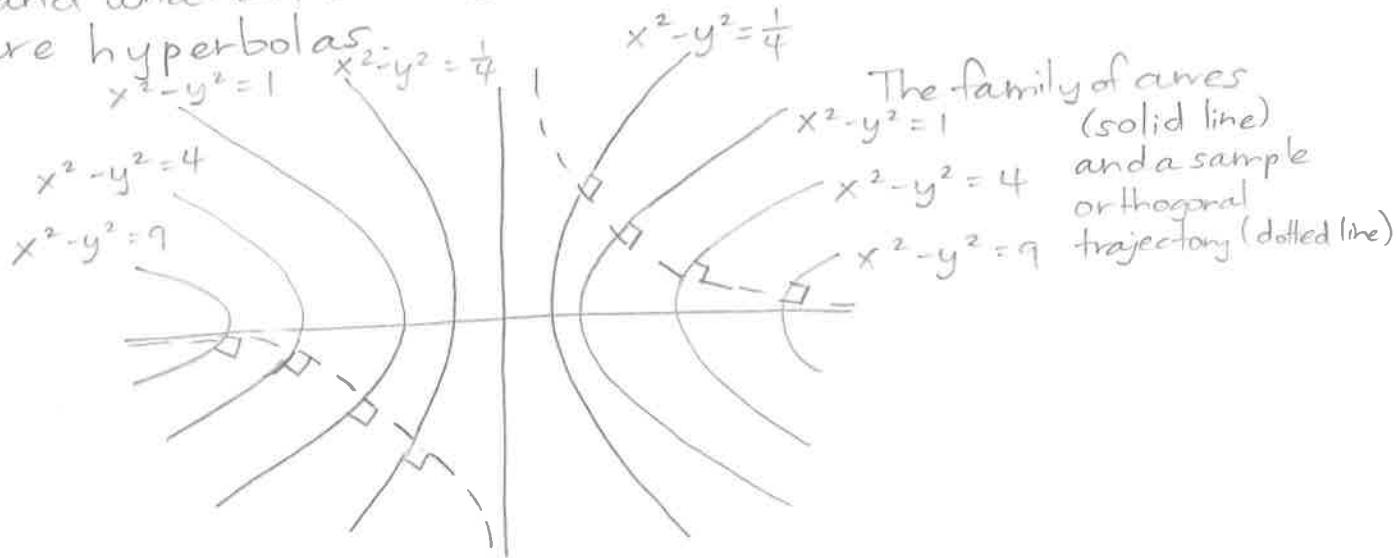
III

## Orthogonal Trajectories

First, what is an orthogonal trajectory to a family of curves? It's easiest to see this with a specific example.

- Find the orthogonal trajectories to the family of curves  $x^2 - y^2 = k$ . Then, find the orthogonal trajectory that passes through  $(1, 3)$  for this family of curves.

First, why is  $x^2 - y^2 = k$  a family of curves? It is a family of curves because for every possible value of  $k$ , we get a different curve. When you solve these questions, you don't need to graph the family of curves, but let's do this so we can understand what an orthogonal trajectory is. Note that these curves are hyperbolae.



We show two sample orthogonal trajectories. (dotted lines). They are perpendicular to every curve in the family of curves that they cross. Recall that curves are perpendicular when the slopes of their tangent lines at the intersection point are negative reciprocals of each other.

To find the orthogonal trajectory, first find  $\frac{dy}{dx}$  for the family of curves by implicit differentiation.

$$\frac{d}{dx}(x^2 - y^2) = \frac{d}{dx}(k)$$

$$2x - 2y \frac{dy}{dx} = 0 \leftarrow \text{since } k \text{ is a constant, so } \frac{d}{dx}(k) = 0.$$

$$2y \cdot \frac{dy}{dx} = 2x \quad \frac{dy}{dx} = \frac{x}{y} \quad \text{for the family of curves}$$

So since  $\frac{dy}{dx}$  for the orthogonal trajectory is the negative reciprocal of  $\frac{dy}{dx}$  for the family of curves, we have that for the orthogonal trajectory,

$$\frac{dy}{dx} = -\frac{y}{x} \quad (\text{for orthogonal trajectory})$$

We can solve this by separation of variables. Remember this is two steps. Step 1: Separate variables,  
Step 2: Solve for constant solutions  $y = A$  for  $A$  a constant.

Step 1: Separate variables.

$$\frac{dy}{dx} = -\frac{y}{x} \quad -\frac{1}{y} dy = \frac{1}{x} dx$$

$$\int -\frac{1}{y} dy = \int \frac{1}{x} dx$$

$$-\ln|y| = \ln|x| + C \leftarrow -\ln|y| = \ln|y^{-1}|$$

$$\ln|\frac{1}{y}| = \ln|x| + C$$

$$|\frac{1}{y}| = e^C |x|$$

$$|y| = e^{-C} |\frac{1}{x}|$$

$$y = \pm e^{-C} \left| \frac{1}{x} \right| \quad \text{nonconstant solutions}$$

$$= B \left| \frac{1}{x} \right| \quad \text{for } B \neq 0.$$

(since  $e^{-C}$  gives every nonzero real number as  $C$  traverses all reals)

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Step 2: Find constant solutions.

Move  $x$  and  $\frac{dy}{dx}$  to one side and  $y$  to the other.

$$\frac{dy}{dx} = -\frac{y}{x} \Rightarrow -x \frac{dy}{dx} = y \quad y = A \text{ for a constant}$$

$\downarrow \leftarrow \frac{dy}{dx} = 0$

$$-x(0) = A \Rightarrow A = 0 \quad \text{So the only constant solution is } \boxed{y=0}.$$

So the orthogonal trajectories are

$$\begin{cases} y = \pm e^{-c} \left| \frac{1}{x} \right| = B \left| \frac{1}{x} \right| \text{ for } B \neq 0 \text{ (nonconstant solutions)} \\ y = 0 \text{ (constant solution)} \end{cases}$$

To find the orthogonal trajectory through  $(1, 3)$ , note that since  $(1, 3)$  is not on the constant solution, this particular solution is nonconstant.

$$y = B \left| \frac{1}{x} \right| \Rightarrow 3 = B \left| \frac{1}{1} \right| \Rightarrow B = 3$$

$$\boxed{y = 3 \left| \frac{1}{x} \right|}$$

Example: Find the orthogonal trajectories for the family of curves

$$y = \frac{1}{x^2 + k} + 1 \quad \leftarrow \text{IMPORTANT NOTE!}$$

Solution: It helps to isolate  $k$  on one side.

$$y = \frac{1}{x^2 + k} + 1 \Rightarrow \frac{1}{x^2 + k} = y - 1 \Rightarrow \frac{1}{y-1} = x^2 + k$$

$\downarrow$

$$\frac{1}{y-1} - x^2 = k.$$

Now, we can implicitly differentiate.

$$\frac{d}{dx} \left( \frac{1}{y-1} - x^2 \right) = \frac{d}{dx}(k) \Rightarrow -\frac{1}{(y-1)^2} \frac{dy}{dx} - 2x = 0$$

$$\frac{1}{(y-1)^2} \frac{dy}{dx} = -2x \quad \frac{dy}{dx} = -2x(y-1)^2$$

for family of curves.

So  $\frac{dy}{dx} = -2x(y-1)^2$  for the family of curves.

Recall that  $\frac{dy}{dx}$  for an orthogonal trajectory is the negative reciprocal of  $\frac{dy}{dx}$  for the family of curves.

So for the orthogonal trajectory,

$$\frac{dy}{dx} = \frac{1}{2x(y-1)^2} \quad (\text{for orthogonal trajectory})$$

Step 1: Separate variables.

$$\frac{dy}{dx} = \frac{1}{2x(y-1)^2}$$

$$(y-1)^2 dy = \frac{1}{2x} dx$$

$$\int y^2 - 2y + 1 dy = \int \frac{1}{2x} dx$$

$$\frac{1}{3}y^3 - y^2 + y = \frac{1}{2}\ln|x| + C$$

$$\ln|x| = \frac{2}{3}y^3 - y^2 + 2y - 2C$$

$$|x| = e^{-2C} e^{\frac{2}{3}y^3 - 2y^2 + 2y}$$

$$x = \pm e^{-2C} e^{\frac{2}{3}y^3 - 2y^2 + 2y}$$

(nonconstant solution)

Step 2: Find constant solutions.

Move  $x$  and  $\frac{dy}{dx}$  to one side,  $y$  to the other side.

$$\frac{dy}{dx} = \frac{1}{2x(y-1)^2} \Rightarrow 2x \frac{dy}{dx} = \frac{1}{(y-1)^2}$$

Let  $y = A$  for  $A$  constant.  $\frac{dy}{dx} = 0$ .

$$2x(0) = \frac{1}{(A-1)^2} \Rightarrow \frac{1}{(A-1)^2} = 0$$

no solutions  $A \uparrow$

So there are no constant solutions.

## Method 2: Integrating Factor for First-Order Linear Diff Eq

We solve equations of the form

$$y' + f(x)y = g(x) \quad (*)$$

where  $f(x)$  and  $g(x)$  are arbitrary functions that depend only on  $x$ .

We solve these equations by multiplying the equation through by an integrating factor,

$$I(x) = e^{\int f(x) dx}$$

After multiplying through by the integrating factor  $I(x)$ , the left-hand side can be written as the derivative of a product by the Product Rule. Then, we can integrate to solve the differential equation.

Some important remarks (\*)

- ① If the equation is not in the form (\*), you may need to manipulate it so that it is in that form.
- ② You will usually see that the equation (\*) is not separable.
- ③ When computing  $I(x) = e^{\int f(x) dx}$ , and ONLY when computing integrating factors, you can leave off absolute values in  $\ln$ , and omit  $+C$ . For example, for an integrating factor computation with  $f(x) = \frac{1}{x}$ , you can just do

$$I(x) = e^{\int \frac{1}{x} dx} = e^{\ln(x)} = x.$$

- ④ As always, to solve an IVP,

(initial value problem)

first find a general solution, then use the initial condition to find any arbitrary constants.

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Example: Solve the initial value problem

$$\frac{y'}{x^3} + y + x^4 = 0, \quad y(1) = 3.$$

Solution: We note that this equation is not separable.  
(Try this yourself to see this).

So let's try to solve it using an integrating factor. Before doing this, we need to get the equation into the form

$$y' + f(x)y = g(x).$$

We can do this as follows:

$$\frac{y'}{x^3} + y = -x^4$$

$$y' + x^3 y = -x^7$$

Now, find the integrating factor,

$$I(x) = e^{\int x^3 dx} = e^{\frac{1}{4}x^4}$$

Multiply both sides by the integrating factor.

$$e^{\frac{x^4}{4}} y' + x^3 e^{\frac{x^4}{4}} y = -x^7 e^{\frac{x^4}{4}}$$

When we multiply by the integrating factor, we can then write the left-hand side as the derivative of a product using product rule backward.

$$\frac{d}{dx}(e^{\frac{x^4}{4}} y) = -x^7 e^{\frac{x^4}{4}} \quad (\text{since } \frac{d}{dx}(e^{\frac{x^4}{4}} y) \\ = e^{\frac{x^4}{4}} y' + (e^{\frac{x^4}{4}})^1 y)$$

Now integrate!  $e^{\frac{x^4}{4}} y = \underbrace{\int -x^7 e^{\frac{x^4}{4}} dx}_{\text{ASIDE: Let } u = \frac{x^4}{4}, du = x^3 dx}$

$$e^{\frac{x^4}{4}} y = -x^4 e^{\frac{x^4}{4}} + 4e^{\frac{x^4}{4}} + C$$

$$y = -x^4 + 4 + \frac{C}{e^{\frac{x^4}{4}}} \quad \text{general soln}$$

$$\begin{aligned} &= \int (-x^4) x^3 e^{\frac{x^4}{4}} dx = \int \underbrace{-4u e^u du}_{-4u = -x^4} \\ &= -4u e^u + 4e^u + C \end{aligned}$$

Use initial condition  $y(1) = 3$  to find  $C$ .

$$3 = -(1)^4 + 4 + \frac{C}{e^{\frac{1^4}{4}}} \Rightarrow C = 0$$

So  $\boxed{y = -x^4 + 4} \leftarrow \text{particular soln}$

$$\begin{aligned} &= -x^4 e^{\frac{x^4}{4}} + 4e^{\frac{x^4}{4}} + C \\ &= -4(\frac{x^4}{4}) e^{\frac{x^4}{4}} + 4e^{\frac{x^4}{4}} + C \\ &= -x^4 e^{\frac{x^4}{4}} + 4e^{\frac{x^4}{4}} + C \end{aligned}$$

Example: Find the general solution to the differential equation

$$y' - \frac{y}{3x} = x^4 + \frac{1}{\sqrt{x}}$$

Solution: This is not separable. So use an integrating factor.

Note that this equation is

$$y' + \left(-\frac{1}{3x}\right)y = x^4 + \frac{1}{\sqrt{x}}$$

Careful! You need this to be a + sign, since the form  
is  $y' + f(x)y = g(x)$

So this is now in the desired form. We calculate the integrating factor.

$$I(x) = e^{\int -\frac{1}{3x} dx} = e^{-\frac{1}{3}\ln(x)} = e^{\ln(x^{-\frac{1}{3}})} = x^{-\frac{1}{3}}.$$

We can leave off abs. values in  $\ln(x)$   
for integrating factor computation.

Multiply both sides by the integrating factor  $x^{-\frac{1}{3}}$ .

$$x^{-\frac{1}{3}}y' + x^{-\frac{1}{3}}\left(-\frac{1}{3x}\right)y = x^{-\frac{1}{3}}(x^4 + \frac{1}{\sqrt{x}})$$

$$x^{-\frac{1}{3}}y' + \frac{-1}{3x^{4/3}}y = x^{11/3} + x^{-5/6}$$

Note that  $(x^{-\frac{1}{3}})' = -\frac{1}{3x^{4/3}}$ . So we can rewrite the LHS (left-hand-side) using the Product Rule backward.

$$\frac{d}{dx}(x^{-\frac{1}{3}}y) = x^{\frac{11}{3}} + x^{-\frac{5}{6}} \quad \begin{aligned} & \text{(since } \frac{d}{dx}(x^{-\frac{1}{3}}y) \\ &= (x^{-\frac{1}{3}})y' + (x^{-\frac{1}{3}})'y \\ &= x^{-\frac{1}{3}}y' + \frac{-1}{3x^{4/3}}y \end{aligned}$$

Now integrate!

$$x^{-\frac{1}{3}}y = \int x^{\frac{11}{3}} + x^{-\frac{5}{6}} dx$$

$$x^{-\frac{1}{3}}y = \frac{3}{14}x^{\frac{14}{3}} + 6x^{\frac{1}{6}} + C$$

$$y = \frac{3}{14}x^5 + 6x^{\frac{1}{6}} + Cx^{\frac{1}{3}}$$

Example: Find the solution to

$$\frac{dy}{dx} - (\tan x)y + \sin^3 x = 0$$

that passes through  $(0, 2)$ .

Solution: This is asking for the particular solution that satisfies the initial condition  $y(0) = 2$ . So let us find the general solution first. This equation is not separable, so write this in the form of a first-order linear ODE so that we can use an integrating factor. So we want to write this in the form  $y' + f(x)y = g(x)$ .

We have  $y' + (-\tan x)y = -\sin^3 x$ .

Find the integrating factor, recalling that  $\int \tan x dx = \int \frac{\sin x}{\cos x} dx = -\ln|\cos x| + C$ .

$$I(x) = e^{\int \tan x dx} = e^{\ln(\cos x)} = \cos x.$$

↑  
can leave off abs. value in  $\ln$   
and  $+C$  in integrating factor computation.

Multiply by the integrating factor.

$$y' \cos x + \underbrace{(\cos x)(-\tan x)y}_{-\sin x} = -\sin^3 x \cos x$$

$$y' \cos x + (-\sin x)y = -\sin^3 x \cos x$$

Use the Product Rule backwards.

$$\frac{d}{dx}(y \cos x) = -\sin^3 x \cos x \quad \begin{aligned} &\text{since } \frac{d}{dx}(y \cos x) \\ &= y(\cos x)' + y' \cos x \\ &= -y \sin x + y' \cos x \end{aligned}$$

Integrate!  $y \cos x = \int -\sin^3 x \cos x dx$

$$y \cos x = \frac{1}{2} \cos^2 x - \frac{1}{4} \cos^4 x + C$$

$$y = \frac{1}{2} \cos x - \frac{1}{4} \cos^3 x + \frac{C}{\cos x}$$

general soln

$$\text{Use } y(0) = 2 \Rightarrow 2 = \frac{1}{2} - \frac{1}{4} + C \Rightarrow C = \frac{7}{4}$$

$$\begin{aligned} &\hookrightarrow = \int -\sin x (1 - \cos^2 x) \cos x dx \\ &= \int (1 - u^2) u du \quad \begin{aligned} &\text{Let } u = \cos x \\ &du = -\sin x dx \end{aligned} \\ &= \int u - u^3 du = \frac{1}{2} u^2 - \frac{1}{4} u^4 + C \\ &= \frac{1}{2} \cos^2 x - \frac{1}{4} \cos^4 x + C \end{aligned}$$

$$y = \frac{1}{2} \cos x - \frac{1}{4} \cos^3 x + \frac{7}{4 \cos x}$$

ANSWER (particular soln)

## Section 3: Second-Order Differential Equations

### Homogeneous Equations

First, we solve the simple case of a homogeneous equation

$$ay'' + by' + cy = 0$$

The fact that the right-hand side is 0 is what makes this equation homogeneous.

To solve these, consider the auxiliary equation

$$ar^2 + br + c = 0.$$

(Recall that this is motivated by the guess  $y = Ce^{rx}$ ).

Find the zeros of  $ar^2 + br + c = 0$ .

Case 1: If the zeros are distinct and real,  $r_1, r_2$ ,

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

Case 2: If the zeros are identical (so they are repeated and real),

$$y = C_1 e^{rx} + C_2 x e^{rx} \quad \text{where } r \text{ is the repeated root.}$$

Case 3: If the zeros are imaginary, they must be complex conjugate roots  $a+ib$  and  $a-ib$ .

$$\text{Then, } y = e^{ax} (C_1 \cos bx + C_2 \sin bx)$$

$$= C_1 e^{ax} \cos bx + C_2 e^{ax} \sin bx.$$

Note that there are two arbitrary constants  $C_1, C_2$  now, since we are considering equations of order 2.

Examples:

Find the general solutions to the following equations.

$$4y'' - 4y' + y = 0$$

$$3y'' - 2y' + 4y = 0$$

$$y'' - 5y' + 6y = 0$$

$$y'' + y = 0$$

$$2y'' - 7y' + y = 0$$

Solutions: ①  $4y'' - 4y' + y = 0$

$$4r^2 - 4r + 1 = 0$$

$$(2r-1)^2 = 0 \quad r = \frac{1}{2}, \frac{1}{2} \text{ (repeated root)}$$

$$y = C_1 e^{\frac{1}{2}x} + C_2 x e^{\frac{1}{2}x}$$

②  $3y'' - 2y' + 4y = 0$

$$3r^2 - 2r + 4 = 0$$

$$r = \frac{-(-2) \pm \sqrt{4 - 4(3)(4)}}{2(3)} = \frac{2 \pm \sqrt{-44}}{6} = \frac{1}{3} \pm \frac{\sqrt{11}}{3} i$$

$$a = \frac{1}{3}, b = \frac{\sqrt{11}}{3}$$

$$y = e^{\frac{1}{3}x} (C_1 \cos(\frac{\sqrt{11}}{3}x) + C_2 \sin(\frac{\sqrt{11}}{3}x))$$

③  $y'' - 5y' + 6y = 0$

$$r^2 - 5r + 6 = 0$$

$$(r-2)(r-3) = 0, \quad r = 2, 3$$

$$y = C_1 e^{2x} + C_2 e^{3x}$$

④  $y'' + y = 0$

$$r^2 + 1 = 0 \quad r^2 = -1, \quad r = \pm i$$

$$r = 0 \pm i \quad \text{so } a = 0, b = 1$$

$$y = e^{0x} (C_1 \cos x + C_2 \sin x)$$

$$y = C_1 \cos x + C_2 \sin x$$

⑤  $2r^2 - 7r + 1 = 0$

$$r = \frac{7 \pm \sqrt{49-8}}{4} = \frac{7}{4} \pm \frac{\sqrt{41}}{4} \quad \leftarrow \text{real, distinct roots}$$

$$y = C_1 e^{\frac{7+\sqrt{41}}{4}x} + C_2 e^{\frac{7-\sqrt{41}}{4}x}$$

## Nonhomogeneous Equations

Now, we want to consider an equation such as

$$y'' - 3y' + 2y = \underline{3e^{-x}}$$

This is not 0, which is what makes this nonhomogeneous.

To solve this, there are a few steps.

Step 1: Solve the homogeneous equation.

$$y'' - 3y' + 2y = 0$$

$$r^2 - 3r + 2 = 0, \quad (r-1)(r-2) = 0, \quad r=1,2$$

$$y_{\text{hom}} = C_1 e^x + C_2 e^{2x}$$

Step 2: Find a particular solution, any solution to

$$y'' - 3y' + 2y = 3e^{-x}.$$

Then, the full solution is  $y = y_{\text{hom}} + y_{\text{part}}$ .

We can find the particular solution by the method of undetermined coefficients.

Step 3: If solving an initial value problem, find the value of arbitrary constants using initial conditions.

Step 2 is the hardest. For the method of undetermined coefficients, make a guess using the form of the function on the other side. Here are some general rules.

$$(\text{const.})e^{rx} \Rightarrow \text{Guess: } y_p = Ae^{rx}$$

$$\text{polynomial of degree } d \Rightarrow \text{Guess: } y_p = \text{polynomial of degree } d.$$

$$\begin{cases} (\text{const.})\sin(rx) \\ (\text{const.})\cos(rx) \end{cases} \Rightarrow \text{Guess: } y_p = A\cos(rx) + B\sin(rx)$$

These rules give us what I call a basic guess. (We will worry about when to multiply by  $x$  later)

Example: (Basic Example 1)

$$y'' - 3y' + 2y = 3e^{-x}.$$

Solution:  $y_{\text{hom}} = C_1 e^x + C_2 e^{2x}$

We guess  $y_{\text{part}} = Ae^{-x}$  for the basic guess for our particular solution.

$$y = Ae^{-x}, y' = -Ae^{-x}, y'' = Ae^{-x}.$$

$$y'' - 3y' + 2y = 3e^{-x}$$

$$Ae^{-x} - 3(-Ae^{-x}) + 2Ae^{-x} = 3e^{-x}$$

$$6Ae^{-x} = 3e^{-x}$$

$$6A = 3 \quad (\text{Match coefficients})$$

$$A = \frac{1}{2}$$

$$\text{So } y_p = \frac{1}{2}e^{-x}.$$

Thus,  $y = y_{\text{hom}} + y_{\text{part}}$

$$= \frac{1}{2}e^{-x} + C_1 e^x + C_2 e^{2x}$$

Example: (Basic Example 2)

$$y'' - 4y' + 4y = x^2 - 3$$

Sol:  $y_{\text{hom}} = C_1 e^{2x} + C_2 x e^{2x}$ .

$x^2 - 3$  is a degree 2 polynomial. So guess  $y_p = Ax^2 + Bx + C$ .

$$y = Ax^2 + Bx + C \Rightarrow y' = 2Ax + B \Rightarrow y'' = 2A.$$

$$(2A) - 4(2Ax + B) + 4(Ax^2 + Bx + C) = x^2 - 3$$

$$4Ax^2 + (-8A + 4B)x + (2A - 4B + 4C) = x^2 + 0x - 3$$

$$\text{Match coefficients: } 4A = 1 \Rightarrow A = \frac{1}{4}$$

$$-8A + 4B = 0 \Rightarrow 4B = 8\left(\frac{1}{4}\right) = 2 \quad B = \frac{1}{2}$$

$$2A - 4B + 4C = -3 \Rightarrow 2\left(\frac{1}{4}\right) - 4\left(\frac{1}{2}\right) + 4C = -3$$

$$4C = -\frac{3}{2} \quad C = -\frac{3}{8}$$

$$y_p = \frac{1}{4}x^2 + \frac{1}{2}x - \frac{3}{8}$$

$$\text{So } y = \frac{1}{4}x^2 + \frac{1}{2}x - \frac{3}{8} + C_1 e^{2x} + C_2 x e^{2x}$$

Example: (Basic Example 3)

$$y'' + 2y' + 2y = -3\cos(2x)$$

Solution:  $r^2 + 2r + 2 = 0$

$$r = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$$

$$y_h = e^{-x}(C_1 \cos x + C_2 \sin x)$$

Guess:  $y_p = A \cos(2x) + B \sin(2x)$

$$y' = -2A \sin(2x) + 2B \cos(2x)$$

$$y'' = -4A \cos(2x) - 4B \sin(2x)$$

$$(-4A \cos(2x) - 4B \sin(2x)) + 2(-2A \sin(2x) + 2B \cos(2x)) + 2(A \cos(2x) + B \sin(2x)) = -3 \cos(2x)$$

$$(-2A + 4B) \cos(2x) + (-4A - 2B) \sin(2x) = -3 \cos(2x)$$

Match coefficients:  $-2A + 4B = -3$

$$-4A - 2B = 0 \Rightarrow B = -2A$$

$$-2A + 4(-2A) = -3$$

$$-10A = -3 \quad A = \frac{3}{10}, \quad B = -\frac{3}{5}$$

$$y_p = \frac{3}{10} \cos(2x) - \frac{3}{5} \sin(2x)$$

$$y = \frac{3}{10} \cos(2x) - \frac{3}{5} \sin(2x) + C_1 e^{-x} \cos x + C_2 e^{-x} \sin x$$

When the other side is a product of basic functions, multiply the basic guesses together, and expand (FOIL and distribute) and combine any constants together to get the basic guess.

e.g. For  $x \cos(2x)$ , the basic guess is gotten by

$$(Ax + B)(C \cos(2x) + D \sin(2x)) = ACx \cos(2x) + ADx \sin(2x) + BC \cos(2x) + BD \sin(2x)$$

↑                      ↑                      ↑                      ↓  
 first degree poly.    guess for cos/sin    Expand    Combine AC, AD, BC, BD  
 into their own constants.

$$y_p = Ax \cos(2x) + Bx \sin(2x) + C \cos(2x) + D \sin(2x)$$

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Example: (Basic Guess for a Product)

Solve  $y'' + 2y' = 2e^{-x} \cos x$

Solution:  $r^2 + 2r = 0$   
 $r(r+2) = 0 \quad r=0, r=-2$

$$\begin{aligned} y_{\text{hom}} &= C_1 e^{0x} + C_2 e^{-2x} \\ &= C_1 + C_2 e^{-2x} \end{aligned}$$

The basic guess is the product of our two basic guesses.

$$\begin{aligned} y_p &= Ae^{-x}(B \cos x + C \sin x) \\ &= AB e^{-x} \cos x + AC e^{-x} \sin x \\ &\stackrel{\text{Expand}}{\qquad\qquad\qquad} \downarrow \text{Combine AB and AC into constants.} \end{aligned}$$

$$\begin{aligned} y_p &= Ae^{-x} \cos x + Be^{-x} \sin x \\ y' &= -Ae^{-x} \cos x - Ae^{-x} \sin x - Be^{-x} \sin x + Be^{-x} \cos x \\ &= (-A+B)e^{-x} \cos x + (-A-B)e^{-x} \sin x \\ y'' &= (A-B)e^{-x} \cos x + (A-B)e^{-x} \sin x \\ &\quad + (A+B)e^{-x} \sin x + (-A-B)e^{-x} \cos x \\ &= -2Be^{-x} \cos x + 2Ae^{-x} \sin x \\ &\quad (-2Be^{-x} \cos x + 2Ae^{-x} \sin x) + 2(-A+B)e^{-x} \cos x + 2(-A-B)e^{-x} \sin x \\ &= 2e^{-x} \cos x \end{aligned}$$

$$-2Ae^{-x} \cos x - 2Be^{-x} \sin x = 2e^{-x} \cos x$$

$$-2A = 2, -2B = 0$$

$$\underline{A = -1, B = 0} \quad y_p = -e^{-x} \cos x$$

$$y = -e^{-x} \cos x + C_1 + C_2 e^{-2x}$$

Now, for the big question:

When do we multiply our guess by  $x$ ?

The Rule: Make a basic guess using the previous rules without considering  $y_{\text{hom}}$ . Expand (distribute, FOIL) it out. Keep multiplying the entire guess by  $x$  until every part (every summand) is not in  $y_{\text{hom}}$ .

Example: Solve  $y'' - 4y = 2e^{2x}$ .

$$y_{\text{hom}} = C_1 e^{2x} + C_2 e^{-2x}$$

For the particular solution, our basic guess would be

$$Ae^{2x}$$



But this is in  $y_{\text{hom}}$ ! So we need to multiply by  $x$ .



$$Ax e^{2x}$$

↙ This works since it is not in  $y_{\text{hom}}$  anymore!  
So we have

$$y_p = Axe^{2x}$$

$$y' = 2Axe^{2x} + Ae^{2x}$$

$$y'' = 4Axe^{2x} + 2Ae^{2x} + 2Ae^{2x} = 4Axe^{2x} + 4Ae^{2x}$$

$$(4Axe^{2x} + 4Ae^{2x}) - 4Axe^{2x} = 2e^{2x}$$

$$4Ae^{2x} = 2e^{2x}$$

$$4A = 2 \quad A = \frac{1}{2}$$

$$y_p = \frac{1}{2}xe^{2x}$$

$$y = \frac{1}{2}xe^{2x} + C_1 e^{2x} + C_2 e^{-2x}$$

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Example:  $y'' + 4y = (2x-1)\sin(2x)$

Solution:  $r^2 + 4 = 0$

$$r = \pm 2i = 0 \pm 2i$$

$$y_{\text{hom}} = C_1 \cos(2x) + C_2 \sin(2x)$$

Our first basic guess for  $y_p$  would be

$$(Ax+B)(C\cos(2x) + D\sin(2x)) = ACx\cos(2x) + ADx\sin(2x) + BC\cos(2x) + BD\sin(2x)$$

↑ Expand

$$y_p = Ax\cos(2x) + Bx\sin(2x) + C\cos(2x) + D\sin(2x)$$

↓ Multiply by  $x$

There are in  $y_{\text{hom}}$ !  
So our first guess does not work!

$$y_p = Ax^2\cos(2x) + Bx^2\sin(2x) + Cx\cos(2x) + Dx\sin(2x)$$

All are not in  $y_{\text{hom}}$ , so this works!

$$y = Ax^2\cos(2x) + Bx^2\sin(2x) + Cx\cos(2x) + Dx\sin(2x)$$

$$= (Ax^2 + Cx)\cos(2x) + (Bx^2 + Dx)\sin(2x)$$

$$y' = -2(Ax^2 + Cx)\sin(2x) + (2Ax + C)\cos(2x)$$

$$+ 2(Bx^2 + Dx)\cos(2x) + (2Bx + D)\sin(2x)$$

$$= (-2Ax^2 + (2B - 2C)x + D)\sin(2x) + (2Bx^2 + (2A + 2D)x + C)\cos(2x)$$

$$y'' = 2(-2Ax^2 + (2B - 2C)x + D)\cos(2x) + (-4Ax + 2B - 2C)\sin(2x)$$

$$+ (-2)(2Bx^2 + (2A + 2D)x + C)\sin(2x) + (4Bx + (2A + 2D))\cos(2x)$$

$$= (-4Ax^2 + (4B - 4C)x + 2D + 4Bx + (2A + 2D))\cos(2x)$$

$$+ (-4Ax + 2B - 2C - 4Bx^2 + (-4A - 4D)x - 2C)\sin(2x)$$

$$= (-4Ax^2 + (8B - 4C)x + (2A + 4D))\cos(2x)$$

$$+ (-4Bx^2 + (-8A - 4D)x + (2B - 4C))\sin(2x)$$

$$\begin{aligned}
 & (-4Ax^2 + (8B - 4C)x + (2A + 4D))\cos(2x) \\
 & + (-4Bx^2 + (-8A - 4D)x + (2B - 4C))\sin(2x) \\
 & + 4Ax^2\cos(2x) + 4Bx^2\sin(2x) + 4Cx\cos(2x) + 4Dx\sin(2x) \\
 & = 2x\sin(2x) - \sin(2x)
 \end{aligned}$$

$$\begin{aligned}
 8Bx\cos(2x) - 8Ax\sin(2x) + (2A + 4D)\cos(2x) \\
 + (2B - 4C)\sin(2x) = 2x\sin(2x) - \sin(2x)
 \end{aligned}$$

$$\begin{aligned}
 8B = 0 & \quad -8A = 2 & \quad 2A + 4D = 0 & \quad 2B - 4C = -1 \\
 B = 0 & \quad A = -\frac{1}{4} & \quad D = \frac{1}{8} & \quad C = \frac{1}{4}
 \end{aligned}$$

$$y_p = -\frac{1}{4}x^2\cos(2x) + \frac{1}{4}x\cos(2x) + \frac{1}{8}x\sin(2x)$$

$$\boxed{
 \begin{aligned}
 y = & -\frac{1}{4}x^2\cos(2x) + \frac{1}{4}x\cos(2x) + \frac{1}{8}x\sin(2x) \\
 & + C_1\cos(2x) + C_2\sin(2x)
 \end{aligned}
 }$$

Finally, how do we deal with addition or subtraction?  
 Separate into separate problems. Note each problem will have the same  $y_{hom}$ .

Example:  $y'' + 2y' + y = -2xe^{-x} + 3\cos x$

Separate into  $y'' + 2y' + y = -2xe^{-x}$  and  $y'' + 2y' + y = 3\cos x$ .  
 Both have the same  $y_{hom} = C_1e^{-x} + C_2xe^{-x}$ .

$$\textcircled{1} \quad y'' + 2y' + y = -2xe^{-x}$$

Basic guess:  $y_p = (Ax + B)e^{-x} = ACxe^{-x} + BCe^{-x}$

$$y_p = Ax e^{-x} + B e^{-x} \quad X$$

Both in  $y_{hom}$  - doesn't work.

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So multiply by  $x$  to get our next guess.

$$y_p = Ax^2 e^{-x} + Bx e^{-x} \quad \times$$

↑  
still in  $y_{hom}$  so still does not work!

So multiply by  $x$  again.

$$y_p = Ax^3 e^{-x} + Bx^2 e^{-x} \quad \checkmark$$

Both not in  $y_{hom}$  so this works now!

$$y = (Ax^3 + Bx^2)e^{-x}$$

$$y' = (3Ax^2 + 2Bx)e^{-x} - (Ax^3 + Bx^2)e^{-x}$$

$$= (-Ax^3 + (3A - B)x^2 + 2Bx)e^{-x}$$

$$y'' = (-3Ax^2 + (6A - 2B)x + 2B)e^{-x} - (-Ax^3 + (3A - B)x^2 + 2Bx)e^{-x}$$

$$= (Ax^3 + (-6A + B)x^2 + (6A - 4B)x + 2B)e^{-x}$$

$$y'' + 2y' + y = -2xe^{-x}$$

~~$$(Ax^3 + (-6A + B)x^2 + (6A - 4B)x + 2B)e^{-x}$$~~

~~$$(-2Ax^3 + (6A - 2B)x^2 + 4Bx)e^{-x}$$~~

~~$$+ (Ax^3 + Bx^2)$$~~

$$\underline{6Axe^{-x} + 2Be^{-x}} = -2xe^{-x}$$

$$6A = -2, 2B = 0$$

$$A = -\frac{1}{3}, B = 0$$

$$y_{part} = -\frac{1}{3}x^3 e^{-x}$$

$$\textcircled{2} \quad y'' + 2y' + y = 3\cos x$$

Basic guess:  $y_p = A\cos x + B\sin x$  ✓  
 both not in  $y_{hom}$  so this works!

$$y = A\cos x + B\sin x$$

$$y' = -A\sin x + B\cos x$$

$$y'' = -A\cos x - B\sin x$$

$$(-A\cos x - B\sin x) + (-2A\sin x + 2B\cos x) + (A\cos x + B\sin x) \\ = 3\cos x$$

$$-2A = 0 \quad 2B = 3$$

$$A = 0 \quad B = \frac{3}{2}$$

$$y_p = \frac{3}{2}\sin x$$

In such a case, the final solution is

$$y = y_{hom} + \text{particular solution to each sub problem}$$

$$y = C_1 e^{-x} + C_2 x e^{-x} + \left(-\frac{1}{3} x^3 e^{-x}\right) + \frac{3}{2} \sin x$$

Note: Write subtraction as addition of the opposite and deal with these in the same way. For example, for

$$y'' + 3y' + 2y = 2e^{-x} - 3x$$

$$\text{find } y_{hom} = C_1 e^{-x} + C_2 e^{-2x}$$

Write as  $2e^{-x} + (-3x)$  so this is now addition!

$$\text{and consider the subproblems } \textcircled{1} \ y'' + 3y' + 2y = 2e^{-x}$$

$$\textcircled{2} \quad y'' + 3y' + 2y = -3x$$

and then answer is  $y = y_{hom} + \text{part.sln for } \textcircled{1} + \text{part.sln for } \textcircled{2}$ .

## Section 4: Series Solutions

Sometimes, we cannot explicitly solve a differential equation. But we can solve them approximately with a series solution, where we assume the solution has the form

$$y = \sum_{n=0}^{\infty} a_n x^n \text{ and we want to solve for the power series coefficients } a_n.$$

Some things to remember:

- ① When differentiating a series, take out the constant term before differentiating.

e.g. If  $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$ , to differentiate,

write  $f(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n!}$ , then do  
 ↑  
 $n=0$  term  
 (constant term)

$$\begin{aligned} f'(x) &= 0 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{n-1} n}{n!} \\ &\quad \text{der. of constant is zero.} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n x^{n-1}}{(n-1)!} \end{aligned}$$

- ② To add series together:

Step 1: Make all exponents on  $x$  into the same exponent on  $x$  by reindexing.

Step 2: Match starting indices to the largest starting index by pulling out finitely many terms.

## The Basic Steps for Series Solutions:

Step 1: Write  $y = \sum_{n=0}^{\infty} a_n x^n$  and find power series expressions for the necessary derivatives of  $y$ , remembering to take out the constant before differentiating.

Step 2: Substitute power series into the diff eq. Add series together and use the fact that if a power series is 0, all of its coefficients are 0 to get a recursive relation.

Step 3: Understand the recursion on  $a_n$ . What are the seeds of recursion and which coefficients depend on which seeds?

Step 4: Find formulas for coefficients  $a_n$  using patterns.

Substitute in.

Step 5: If the question is an initial value problem, use original series to find values of arbitrary constants.

Example: Solve  $y'' + 3y = 0$ ,  $y(0) = 2$ ,  $y'(0) = 0$  using series solution.

Solution: Let  $y = \sum_{n=0}^{\infty} a_n x^n$ .

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + \sum_{n=1}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + \sum_{n=2}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

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$$y'' + 3y = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} 3a_n x^n = 0$$

Make  $x^n$  (larger exp.)  $\rightarrow x^{n-2}$  (smaller exp.)  
by reindexing.

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} 3a_{n-2} x^{n-2} = 0$$

Starting indices already match.

Since starting indices and  $x$  exponents match, we can add!

$$\sum_{n=2}^{\infty} (n(n-1)a_n + 3a_{n-2}) x^{n-2} = 0$$

This is a power series equal to 0, so all coefficients must be 0.

$$\text{So } n(n-1)a_n + 3a_{n-2} = 0 \text{ for } n \geq 2$$

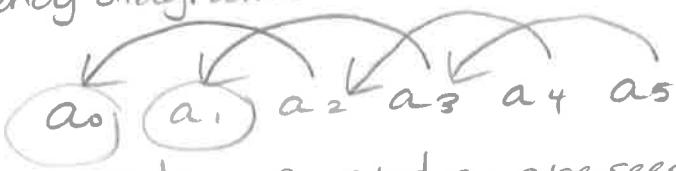
Write later  
coeffs in terms  
of earlier.

$$\text{e.g. } n=2: a_2 = -\frac{3}{2(1)} a_0$$

$$n=3: a_3 = -\frac{3}{3(2)} a_1$$

$$n=4: a_4 = -\frac{3}{4(3)} a_2$$

Dependency diagram:



$a_0$  and  $a_1$  are seeds of recursion (if we know them, we can find all coeffs.)

Note all even coeffs  $a_{2k}$  depend on  $a_0$  and all odd coeffs  $a_{2k+1}$  depend on  $a_1$ .

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Find a formula for  $a_{2k}$ ,  $a_{2k+1}$  by pattern recognition.

① Even  $a_{2k}$   $\xrightarrow{a_{2k} \text{ for } k=0} a_0$  seed for evens

$$a_{2k} \text{ for } k=1 \rightarrow a_2 = -\frac{3}{2(1)} a_0$$

$$a_{2k} \text{ for } k=2 \rightarrow a_4 = -\frac{3}{4(3)} a_2 = \frac{(-3)^2}{4 \cdot 3 \cdot 2 \cdot 1} a_0$$

$$a_{2k} \text{ for } k=3 \rightarrow a_6 = -\frac{3}{6(5)} a_4 = \frac{(-3)^3}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} a_0$$

So we see the pattern is

$$a_{2k} = \frac{(-3)^k}{(2k)!} a_0 \quad k \geq 0$$

② Odd  $a_{2k+1}$   $\xrightarrow{a_{2k+1} \text{ for } k=0} a_1$  seed for odds

$$a_{2k+1} \text{ for } k=1 \rightarrow a_3 = -\frac{3}{3(2)} a_1$$

$$a_{2k+1} \text{ for } k=2 \rightarrow a_5 = -\frac{3}{5(4)} a_3 = \frac{(-3)^2}{5 \cdot 4 \cdot 3 \cdot 2} a_1$$

$$a_{2k+1} \text{ for } k=3 \rightarrow a_7 = -\frac{3}{7(6)} a_5 = \frac{(-3)^3}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} a_1$$

$$\text{So } a_{2k+1} = \frac{(-3)^k}{(2k+1)!} a_1 \quad k \geq 0$$

Now, use  $y = \sum_{n=0}^{\infty} a_n x^n = \underbrace{\sum_{k=0}^{\infty} a_{2k} x^{2k}}_{a_0 x^0 + a_2 x^2 + a_4 x^4 + \dots} + \underbrace{\sum_{k=0}^{\infty} a_{2k+1} x^{2k+1}}_{a_1 x^1 + a_3 x^3 + a_5 x^5 + \dots}$

General solution  
(NOTE:  
 $a_0$  and  $a_1$  are the arbitrary constants!)  $\rightarrow y = \sum_{k=0}^{\infty} \frac{(-3)^k}{(2k)!} a_0 x^{2k} + \sum_{k=0}^{\infty} \frac{(-3)^k}{(2k+1)!} a_1 x^{2k+1}$

Find  $a_0$  and  $a_1$  from  $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \dots$$

$$y(0) = a_0 + a_1(0) + a_2(0)^2 + \dots = a_0 = 2 \quad \text{since } y(0) = 2$$

$$y'(0) = a_1 + 2a_2(0) + 3a_3(0)^2 + \dots = a_1 = 0 \quad \text{since } y'(0) = 0$$

$$y = \sum_{k=0}^{\infty} \frac{(-3)^k}{(2k)!} (2)x^{2k} + \sum_{k=0}^{\infty} \frac{(-3)^k}{(2k+1)!} (0)x^{2k+1} \Rightarrow y = \sum_{k=0}^{\infty} \frac{(-3)^k \cdot 2}{(2k)!} x^{2k}$$

particular solution

Example: (Section 17.4, Ex 8)

Solve  $y'' = xy$  with  $y(0) = 3$ ,  $y'(0) = -2$ , using power series.  
(I added the initial conditions)

Solution: Let  $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + \sum_{n=1}^{\infty} a_n x^n$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + \sum_{n=2}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$y'' - xy = 0$  (Put everything on the same side)

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

Change  $x^{n+1}$  into  $x^{n-2}$  by reindexing.  
larger exp  $\rightarrow$  smaller exp.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=3}^{\infty} a_{n-3} x^{n-2} = 0$$

Match starting indices to be  $n=3$ .

$$2a_2 + \sum_{n=3}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=3}^{\infty} a_{n-3} x^{n-2} = 0$$

$\uparrow$                              $\uparrow$   
( $n=2$  term in first sum)

starting indices and  $x$  exponents match, so we can combine!

$$2a_2 x^0 + \sum_{n=3}^{\infty} (n(n-1) a_n - a_{n-3}) x^{n-2} = 0$$

$\uparrow$                              $\uparrow$   
 $x^0$  term                     $x^{3-2} = x^1, x^{4-2} = x^2, x^3, x^4, \dots$  terms

$$\text{So } 2a_2 = 0 \Rightarrow$$

$$n(n-1)a_n - a_{n-3} = 0 \quad \text{for } n \geq 3$$

|   |
|---|
| $a_2 = 0$                                     |
| $a_n = \frac{a_{n-3}}{n(n-1)}$ for $n \geq 3$ |

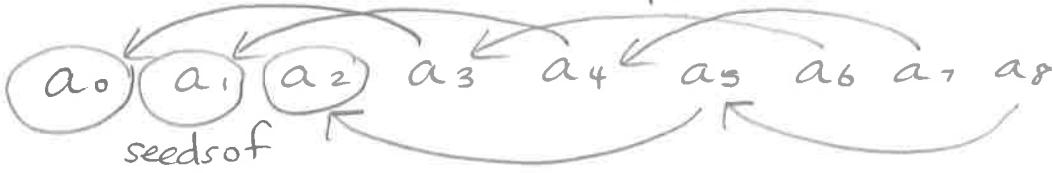
recursive relation

$$a_2 = 0 \quad (*) \quad a_n = \frac{a_{n-3}}{n(n-1)} \text{ for } n \geq 3$$

$$\text{e.g. } n=3 \quad a_3 = \frac{1}{3(2)} a_0 \quad n=4 \quad a_4 = \frac{1}{4(3)} a_1$$

$$n=5 \quad a_5 = \frac{1}{5(4)} a_2$$

Note every term starting with  $a_3$  depends on the term three before it.



All terms  $a_{3k}$  that are multiples of 3 depend on  $a_0$ .

All terms  $a_{3k+1}$  that are 1 more than a multiple of 3 depend on  $a_1$ .

All terms  $a_{3k+2}$  that are 2 more than a multiple of 3 depend on  $a_2$ .

①  $a_{3k}$  depend on  $a_0 \leftarrow a_{3k} \text{ for } k=0$

$$a_{3k} \text{ for } k=1 \rightarrow a_3 = \frac{1}{3(2)} a_0$$

$$a_{3k} \text{ for } k=2 \rightarrow a_6 = \frac{1}{6(5)} a_3 = \frac{1}{6 \cdot 5 \cdot 3 \cdot 2} a_0 = \frac{1}{6 \cdot 3} \frac{1}{5 \cdot 2} a_0 \quad 3 \cdot 6 = 3^2 (1 \cdot 2) \\ = 3^2 \cdot 2!$$

$$a_{3k} \text{ for } k=3 \rightarrow a_9 = \frac{1}{9(8)} a_6 = \frac{1}{9 \cdot 6 \cdot 3} \frac{1}{8 \cdot 5 \cdot 2} a_0 \quad 3 \cdot 6 \cdot 9 \\ = 3^3 (1 \cdot 2 \cdot 3) = 3^3 \cdot 3!$$

$$\text{So } a_{3k} = \frac{1}{3^k k!} \frac{1}{2 \cdot 5 \cdots (3k-1)} a_0, \quad k \geq 0$$

②  $a_{3k+1}$  depend on  $a_1 \leftarrow a_{3k+1} \text{ for } k=0$

$$a_{3k+1} \text{ for } k=1 \rightarrow a_4 = \frac{1}{4(3)} a_1$$

$$a_{3k+1} \text{ for } k=2 \rightarrow a_7 = \frac{1}{7(6)} a_4 = \frac{1}{7 \cdot 4 \cdot 1} \frac{1}{6 \cdot 3} a_1$$

$$a_{3k+1} \text{ for } k=3 \rightarrow a_{10} = \frac{1}{10(9)} a_7 = \frac{1}{10 \cdot 7 \cdot 4 \cdot 1} \frac{1}{9 \cdot 6 \cdot 3} a_1$$

$$\text{So } a_{3k+1} = \frac{1}{1 \cdot 4 \cdot 7 \cdots (3k+1)} \frac{1}{3^k k!} a_1, \quad k \geq 0$$

③  $a_{3k+2}$  depend on  $a_2 \leftarrow a_{3k+2} \text{ for } k=0 \quad \underline{a_2=0} \quad (*)$

$$a_5 = \frac{1}{5(4)} a_2 = 0$$

$$a_8 = \frac{1}{8(7)} a_5 = 0 \text{ etc. So all } a_{3k+2} = 0, \quad k \geq 0$$

$$\text{Use } y = \sum_{n=0}^{\infty} a_n x^n$$

$$\begin{aligned}
 &= \sum_{k=0}^{\infty} a_{3k} x^{3k} + \sum_{k=0}^{\infty} a_{3k+1} x^{3k+1} + \sum_{k=0}^{\infty} a_{3k+2} x^{3k+2} \\
 &= \boxed{\sum_{k=0}^{\infty} \left( \frac{1}{3^k k!} \frac{1}{2 \cdot 5 \cdots (3k-1)} \right) a_0 x^{3k} + \sum_{k=0}^{\infty} \left( \frac{1}{1 \cdot 4 \cdot 7 \cdots (3k+1)} \frac{1}{3^k k!} \right) a_1 x^{3k+1}}
 \end{aligned}$$

General solution ( $a_0, a_1$  are arbitrary constants)

Finally, use initial conditions.  $y(0) = 3, y'(0) = -2$

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \dots$$

$$y(0) = a_0 + a_1(0) + a_2(0)^2 + \dots = \underline{a_0 = 3}$$

$$y'(0) = a_1 + 2a_2(0) + 3a_3(0)^2 + \dots = \underline{a_1 = -2}$$

$$\begin{aligned}
 \text{So } y &= \boxed{\sum_{k=0}^{\infty} \left( \frac{1}{3^k k!} \frac{1}{2 \cdot 5 \cdots (3k-1)} \right) (3) x^{3k} + \sum_{k=0}^{\infty} \left( \frac{1}{1 \cdot 4 \cdot 7 \cdots (3k+1)} \frac{1}{3^k k!} \right) (-2) x^{3k+1}} \\
 &\quad \text{particular solution}
 \end{aligned}$$

Example: Solve  $y'' - xy' - y = 0, y(0) = 1, y'(0) = 0$   
(Section 17.4, Ex 9)

$$\text{Solution: Let } y = \sum_{n=0}^{\infty} a_n x^n = a_0 + \sum_{n=1}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + \sum_{n=2}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$y'' - xy' - y = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x \sum_{n=1}^{\infty} na_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} na_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0 \quad \text{Turn all exponents to } x^{n-2} \text{ by reindexing.}$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=3}^{\infty} (n-2)a_{n-2} x^{n-2} - \sum_{n=2}^{\infty} a_{n-2} x^{n-2} = 0 \quad \text{Pull out terms to match starting index}$$

$$2a_2 - a_0 + \sum_{n=3}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=3}^{\infty} (n-2)a_{n-2} x^{n-2} - \sum_{n=3}^{\infty} a_{n-2} x^{n-2} = 0 \quad \text{to } n=3.$$

$\uparrow$   
 $n=2$  for first sum     $n=2$  for last sum

$$(2a_2 - a_0) x^0 + \sum_{n=3}^{\infty} (n(n-1)a_n - (n-2)a_{n-2} - a_{n-2}) x^{n-2} = 0$$

combine

$$+(-(n-2)-1)a_{n-2} = (-n+1)a_{n-2} \\ = -(n-1)a_{n-2}$$

$$(2a_2 - a_0) x^0 + \sum_{n=3}^{\infty} (n(n-1)a_n - (n-1)a_{n-2}) x^{n-2} = 0$$

$x^0$  term       $\nwarrow$        $x^{3-2} = x^1, x^{4-2} = x^2, x^{3}, x^4, \dots$  terms

$$\begin{aligned} \text{So } 2a_2 - a_0 &= 0 \Rightarrow a_2 = \frac{1}{2} a_0 \\ n(n-1)a_n - (n-1)a_{n-2} &= 0 \Rightarrow a_n = \frac{1}{n} a_{n-2} \text{ for } n \geq 3 \end{aligned}$$

Recursive relation:

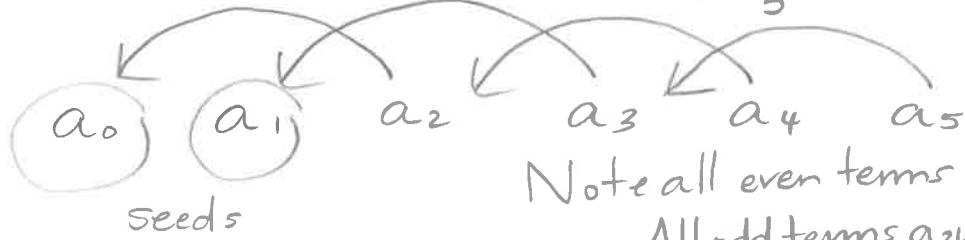
$$a_2 = \frac{1}{2} a_0 \quad a_n = \frac{1}{n} a_{n-2} \text{ for } n \geq 3$$

$\uparrow$   
 $a_2$  depends on  $a_0$   
 using this pattern.

$$\text{e.g. } n=3 \quad a_3 = \frac{1}{3} a_1$$

$$n=4 \quad a_4 = \frac{1}{4} a_2$$

$$n=5 \quad a_5 = \frac{1}{5} a_3$$



Note all even terms  $a_{2k}$  depend on  $a_0$ .  
 All odd terms  $a_{2k+1}$  depend on  $a_1$ .

① Even terms  $a_{2k}$   $\xrightarrow{a_0 \text{ seed for even terms}}$

$a_{2k} \text{ for } k=0$

$$a_2 = \frac{1}{2} a_0$$

$\leftarrow a_{2k} \text{ for } k=1$

(not in  $n \geq 3$  pattern)

$$n=4 \quad a_4 = \frac{1}{4} a_2 = \frac{1}{4} \left( \frac{1}{2} \right) a_0 \leftarrow a_{2k} \text{ for } k=2$$

$$n=6 \quad a_6 = \frac{1}{6} a_4 = \frac{1}{6 \cdot 4 \cdot 2} a_0 \leftarrow a_{2k} \text{ for } k=3$$

$$n=8 \quad a_8 = \frac{1}{8} a_6 = \frac{1}{8 \cdot 6 \cdot 4 \cdot 2} a_0 \leftarrow a_{2k} \text{ for } k=4$$

$$a_{2k} = \frac{1}{2 \cdot 4 \cdots (2k)} a_0 = \frac{1}{2^k k!} a_0, k \geq 0$$

② Odd terms  $a_{2k+1}$   $\xrightarrow{a_1 \text{ seed for odd terms}}$

$a_{2k+1} \text{ for } k=0$

$$n=3 \quad a_3 = \frac{1}{3} a_1 \leftarrow a_{2k+1} \text{ for } k=1$$

$$n=5 \quad a_5 = \frac{1}{5} a_3 = \frac{1}{5 \cdot 3} a_1 \leftarrow a_{2k+1} \text{ for } k=2$$

$$n=7 \quad a_7 = \frac{1}{7} a_5 = \frac{1}{7 \cdot 5 \cdot 3} a_1 \leftarrow a_{2k+1} \text{ for } k=3$$

$$a_{2k+1} = \frac{1}{1 \cdot 3 \cdot 5 \cdots (2k+1)} a_1, k \geq 0$$

$$\text{Use } y = \sum_{n=0}^{\infty} a_n x^n = \sum_{k=0}^{\infty} a_{2k} x^{2k} + \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1}$$

$$y = \sum_{k=0}^{\infty} \frac{1}{2^k k!} a_0 x^{2k} + \sum_{k=0}^{\infty} \frac{1}{1 \cdot 3 \cdot 5 \cdots (2k+1)} a_1 x^{2k+1}$$

General solution ( $a_0$  and  $a_1$  are arbitrary constants)

Use initial conditions  $y(0)=1, y'(0)=0$ .

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots \Rightarrow y(0) = a_0 + a_1(0) + a_2(0)^2 + \dots = \underline{a_0 = 1}$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \dots \Rightarrow y'(0) = a_1 + 2a_2(0) + 3a_3(0)^2 + \dots = \underline{a_1 = 0}$$

Substitute  $a_0=1, a_1=0$  into general solution.

$$y = \sum_{k=0}^{\infty} \frac{1}{2^k k!} x^{2k}$$

$\leftarrow$   
particular solution