Math 54: Midterm 1 Review

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Important Concepts

Systems of Equations

- Row equivalent augmented matrices have the same solution set.
- A system has infinitely many solutions if it has a free variable in its reduced row echelon form.
- A system has no solutions if it has a relation of 0 equals a nonzero number.
- k vectors can never span $\mathbb{R}^n$ if $k < n$. For example, two vectors cannot span $\mathbb{R}^3$. This is because a system with more equations than variables (coefficient matrix has more rows than columns) cannot have a pivot in every row.
- k vectors are never linearly independent in $\mathbb{R}^n$ if $k > n$. For example, four vectors cannot be linearly independent in $\mathbb{R}^3$. This is because a system with more variables than equations (coefficient matrix has more columns than rows) will always have a free variable.

Invertible Matrices

- A square matrix is invertible if and only if its row echelon form is the identity matrix.
- The product of invertible matrices is invertible. To get the inverse, invert all of the matrices "backwards". e.g. If A, B, and C are all invertible, then $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.
- An upper or lower triangular matrix is invertible if and only if its diagonal entries are all nonzero.
- Invertible Matrix Theorem: For a square matrix A, $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ if and only if A is invertible.
- Know why $(ABA^{-1})^n = AB^nA^{-1}$.
- If $A^2 = I$ for a matrix, then A is its own inverse. (Why?)

Determinant

- You can only take the determinant of square matrices.
- The determinant of an upper triangular, lower triangular, or diagonal matrix is the product of the diagonal entries.
- The determinant is multiplicative. In particular, $\det(AB) = \det(A)\det(B) = \det(BA)$. Warning: For trace, $\mathrm{tr}(AB) \neq \mathrm{tr}(A)\mathrm{tr}(B)$, but $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.
- A matrix with one row being identical to another row, or being a multiple of another row, has determinant 0. (Similar for columns)
- The determinant being nonzero is equivalent to the matrix being invertible.

Matrix Multiplication

- $AI = IA = A$.
- Every matrix commutes with the zero matrix, the identity matrix, and scalar multiples of the identity matrix, such as $3I$.
- Every matrix commutes with itself and its powers. So for example, A and A^3 commute.
- Diagonal matrices commute with each other. (To multiply or take powers of diagonal matrices, just apply the operations to the diagonal entries.) But a diagonal matrix and a non-diagonal matrix do not necessarily commute.

General Advice

- Problems 1 and 2 should be quick, if you approach the question the right way. If a question is taking you a long time, there is probably a simple way to think about it.
- Remember to go for the simplest examples: zero matrix, then identity matrix, then diagonal matrix, then upper/lower triangular matrix.
- Know your equivalences well.
- Review your homework!! Many of the problems are based on things that you have shown in the homework, or they are easier versions of homework problems.

Problem 1 Practice

Give an example of each of the following, or briefly explain why the object described cannot exist.

- A 3 by 3 matrix that is its own inverse.
- Four vectors in $\mathbb{R}^2$ that span $\mathbb{R}^2$.
- Two vectors in $\mathbb{R}^4$ that span $\mathbb{R}^4$.
- A matrix with trace zero and nonzero determinant.
- A vector $\mathbf{v}_4$ in $\mathbb{R}^3$ such that $(1,0,0)$, $(2,0,0)$, $(0,1,1)$, and $\mathbf{v_4}$ are linearly independent.
- A system of equations with more equations than variables that has a unique solution.
- A 4 by 4 matrix A such that $A + A^t = I$.
- Two non-diagonal matrices A and B that commute with $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.
- A matrix A with more columns than rows such that $Ax = \mathbf{b}$ is inconsistent for some $\mathbf{b}$.
- A matrix with all positive entries that has negative determinant.

Problem 2 Practice

- Find $\det(ABAB)$ where

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

- Values for c and d such that $A\mathbf{x} = 0$ has a unique solution.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -2 & c & -2 \\ 0 & d & 0 \end{bmatrix}$$

- Assume that for a coefficient matrix A, $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions. Is it necessarily true that for a given $\mathbf{b}$, $A\mathbf{x} = \mathbf{b}$ has no solutions?
- Find all real numbers λ for which the matrix

$$M = \begin{bmatrix} 7 & 5 & 10 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \lambda I$$

  is not invertible, where I is the the 3 by 3 identity matrix.

- Calculate the determinant of the matrix

$$A = \begin{bmatrix} 1 & 0 & 3 & 1 \\ 2 & 1 & 6 & 1 \\ 0 & 1 & 0 & 3 \\ 1 & 1 & 3 & 1 \end{bmatrix}$$

Calculations (Questions 3-5)

These are the calculations you might encounter on the exam.

- Find conditions on a, b, c (or more letters) so that (a, b, c) is in the span of given vectors.
- Find conditions on a, b, c (or more letters) so that given vectors and (a, b, c) are linearly independent or linearly dependent.
- Problems that can be solved with systems of equations: solving matrix equations, polynomials
- Finding a matrix inverse, and using the inverse and matrix multiplication to solve a system of equations.
- Matrix multiplication, and interpreting the matrix multiplication as some sort of linear algebraic operation.
- Word problems based on the equivalence theorems that might involve row reduction or determinant.
- A conceptual problem: proving a result that you have proven on the homework.