Conceptual Questions

1) Note that $(\text{LNull}(A)) = (\text{Col}(A))^{\perp}$ and $\text{Col}(A)$ is a subspace of $\mathbb{R}^m$. So

$$\dim(\text{Col}(A)) + \dim(\text{LNull}(A))$$
$$= \dim(\text{Col}(A)) + \dim(\text{Col}(A))^{\perp} = \dim(\mathbb{R}^m) = m$$

2) False (real numbers are also complex)

3) False - not closed under addition

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{diagonalizable}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$\uparrow$ <u>not</u> diagonalizable

4) False: $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\quad \lambda = i, -i$

5) If $\langle v, v \rangle = 0$, then $v = 0$ by positive definiteness.

6) False

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{char}(x) = x^2 - 2x$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{char}(x) = x^2 - x$$

7) True

$$\text{char}_{MBM^{-1}}(x) = \det(MBM^{-1} - xI)$$
$$= \det(MBM^{-1} - xMIM^{-1})$$
$$= \det(M(B - xI)M^{-1})$$
$$= \det(M)\det(B - xI)\det(M^{-1})$$
$$= \cancel{\det(M)}\det(B - xI)\frac{1}{\cancel{\det(M)}}$$
$$= \det(B - xI) = \text{char}_B(x)$$

8) $(W^{\perp})^{\perp} = W$

Problems

1) Find a diagonalizable and a nondiagonalizable matrix with characteristic polynomial $(1-x)^2(2-x)$.

diagonalizable:
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

nondiagonalizable:
$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

2)
$$1 + \lambda_2 + \lambda_3 = -1 \Rightarrow \lambda_2 + \lambda_3 = -2$$
$$1 \cdot \lambda_2 \cdot \lambda_3 = -8 \Rightarrow \lambda_2\lambda_3 = -8$$

$$\lambda_2^2 + \lambda_2\lambda_3 = -2\lambda_2$$
$$\lambda_2^2 - 8 = -2\lambda_2$$
$$\lambda_2^2 + 2\lambda_2 - 8 = 0$$
$$(\lambda_2 - 2)(\lambda_2 + 4) = 0$$
$$\lambda_2 = 2, -4$$
$$\Downarrow$$
$$\lambda_3 = -4, 2$$

1, 2 and -4 are the eigenvalues.

Since A has three distinct real eigenvalues and is 3×3, A is diagonalizable over $\mathbb{R}$ and $\mathbb{C}$.

$$A = S \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -4 \end{bmatrix} S^{-1} \qquad \lambda^n A^n = S \begin{bmatrix} \lambda^n & 0 & 0 \\ 0 & (2\lambda)^n & 0 \\ 0 & 0 & (-4\lambda)^n \end{bmatrix} S^{-1}$$

So $\lim_{n\to\infty} \lambda^n A^n$ exists when $\lim_{n\to\infty} \lambda^n$, $\lim_{n\to\infty} (2\lambda)^n$, $\lim_{n\to\infty} (-4\lambda)^n$ all exist.

$-1 < \lambda \le 1$ $\quad -\frac{1}{2} < \lambda \le \frac{1}{2}$ $\quad -\frac{1}{4} \le \lambda < \frac{1}{4}$

$$\boxed{-\frac{1}{4} \le \lambda < \frac{1}{4}}$$

3)

$$A = \begin{bmatrix} 1 & a & c \\ 0 & 2 & 0 \\ 0 & b & 1 \end{bmatrix}$$

$\text{char}_A(x) = (1-x)^2(2-x)$

$\lambda = 1$ alg mult 2 $\quad \lambda = 2$ alg mult 1

$\lambda = 2$ automatically has geometric multiplicity 1.

So we would only be concerned about $\lambda = 1$.

$$A - I : \begin{bmatrix} 0 & a & c \\ 0 & 1 & 0 \\ 0 & b & 0 \end{bmatrix}$$

For A to be diagonalizable, we need $A - I$ to have two free variables. The second column is a pivot column already, so we would need $c = 0$ (a, b can be anything).

$$\boxed{c = 0,\ a, b \text{ any reals}}$$

4) (x_1, x_2, x_3, x_4) is in $W^\perp$ if

$$\begin{cases} \langle (x_1, x_2, x_3, x_4), (1, 2, 1, 0) \rangle = 0 \\ \langle (x_1, x_2, x_3, x_4), (-1, 0, 1, 1) \rangle = 0 \end{cases}$$

So find a basis for solutions to $\begin{cases} x_1 + 2x_2 + x_3 = 0 \\ -x_1 + x_3 + x_4 = 0 \end{cases}$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ -1 & 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & 2 & 1 \end{bmatrix} R_1 + R_2$$

$$\to \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & \frac{1}{2} \end{bmatrix} \begin{matrix} R_1 - R_2 \\ \frac{1}{2} R_2 \end{matrix}$$

$$\vec{x} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

$$\boxed{\text{Basis: } \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right\}}$$

5) W is all $(x_1, x_2, x_3) \in \mathbb{R}^3$ such that

$$ax_1 + bx_2 + cx_3 = 0$$

W is a subspace of $\mathbb{R}^3$, of dim 2.

Since $\dim(W) = 2$ and $\dim(W) + \dim(W^\perp) = 3$ ← $\dim(\mathbb{R}^3)$

we have that $\dim(W^\perp) = 1$.

W is all (x_1, x_2, x_3) such that

$$(a,b,c) \cdot (x_1, x_2, x_3) = 0$$

so (a,b,c) is in $W^\perp$ and (a,b,c) is nonzero by assumption.

So since (a,b,c) is a nonzero vector in $W^\perp$ and $\dim(W^\perp) = 1$,

we have that $\{(a,b,c)\}$ is a basis for $W^\perp$.