Conceptual Questions

- 0 cannot be an eigenvalue of an invertible matrix since the only solution to $Av = 0v$ ($Av = 0$) is $v = 0$ and the zero vector cannot be an eigenvector.
- λ^{-1} is an eigenvalue of A^{-1} with v as an eigenvector.

$$Av = \underset{\uparrow}{\lambda} v \Rightarrow A^{-1}(\lambda v) = v \Rightarrow \lambda A^{-1}v = v \Rightarrow A^{-1}v = \lambda^{-1}v$$

($\lambda \neq 0$ since A is invertible)

- λ^k is an eigenvalue of A^k with v as an eigenvector.

$$A^k v = A^{k-1}(Av) = A^{k-1}(\lambda v) = \lambda A^{k-1}v = \cdots = \lambda^k v$$

- (Skip this for this week) No, every invertible matrix is not necessarily diagonalizable. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is invertible but it is not diagonalizable.
- A noninvertible matrix can be diagonalizable, e.g. $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is diagonalizable.

Problem 1

Jordan block with eigenvalue λ of size k

$$A = \underbrace{\begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & 1 & \\ & & \lambda & \ddots \\ & & & \ddots & 1 \\ & 0 & & & \lambda \end{bmatrix}}_{k} \Bigg\updownarrow k \qquad A - xI = \underbrace{\begin{bmatrix} \lambda - x & 1 & & 0 \\ & \lambda - x & 1 & \\ & & \ddots & \ddots \\ & & & \ddots & 1 \\ & 0 & & & \lambda - x \end{bmatrix}}_{k}$$

Since $A - xI$ is upper triangular, we easily see that

$$\det(A - xI) = (\lambda - x)^k = \mathrm{char}_A(x)$$

So λ is the only eigenvalue with algebraic multiplicity k and geometric multiplicity 1

$$A - \lambda I = \begin{bmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix} \quad \text{so } \underset{(A-\lambda I)}{\text{nullspace}} = \left\{ x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\} \ \dim = 1$$

Problem 2

- $(AMA^{-1})^k = (AMA^{-1})(AMA^{-1}) \cdots (AMA^{-1})$
  $= AM(\cancel{A^{-1}A})M(\cancel{A^{-1}A}) \cdots (\cancel{A^{-1}A})MA^{-1}$
  $= AM^kA^{-1}$

- $\det(AMA^{-1}) = \det(A)\det(M)\det(A^{-1})$
  $= \cancel{\det(A)}\det(M)\dfrac{1}{\cancel{\det(A)}} = \det(M)$

- $\text{char}_{AMA^{-1}}(x) = \det(AMA^{-1} - xI)$
  $= \det(AMA^{-1} - xAIA^{-1})$
  $= \det(A(M - xI)A^{-1})$
  $\uparrow$ x is a scalar
  $= \det(A)\det(M - xI)\det(A^{-1})$
  $= \cancel{\det(A)}\det(M - xI)\dfrac{1}{\cancel{\det(A)}}$
  $= \det(M - xI) = \text{char}_M(x)$

Problem 3

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\text{char}_A(x) = \det(A - xI)$$
$$= \begin{vmatrix} 1-x & -1 & 1 \\ -1 & 1-x & 1 \\ 0 & 0 & 3-x \end{vmatrix}$$
$$= (3-x)(\cancel{1} - 2x + x^2 - \cancel{1})$$
$$= (3-x)(-2+x)x$$

- $\lambda = 0$ alg mult 1, geom mult 1
  $U_{\lambda=0} = \text{nullspace}(A - 0I) = \text{nullspace}\left(\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}\right) = \left\{ x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

- $\lambda = 2$ alg mult 1, geom mult 1
  $U_{\lambda=2} = \text{nullspace}(A - 2I) = \text{nullspace}\left(\begin{bmatrix} -1 & -1 & 1 \\ -1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\right) = \left\{ x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

- $\lambda = 3$ alg mult 1, geom mult 1
  $U_{\lambda=3} = \text{nullspace}(A - 3I) = \text{nullspace}\left(\begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \left\{ x_3 \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix} \right\}$

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathrm{char}_A(x) = \det(A - xI) = \begin{vmatrix} 2-x & 1 & 0 & 0 \\ 0 & 2-x & 0 & 0 \\ 0 & 0 & -x & 1 \\ 0 & 0 & 0 & -x \end{vmatrix}$$

$$= (2-x)^2 x^2$$

$\lambda = 0$ alg mult 2
geom mult 1

$$U_{\lambda=0} = \text{nullspace}(A - 0I) = \text{nullspace}\left(\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right) = \left\{ x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$\lambda = 2$ alg mult 2
geom mult 1

$$U_{\lambda=2} = \text{nullspace}(A - 2I) = \text{nullspace}\left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}\right) = \left\{ x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Problem 4

(Skip for just this week)

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & k & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \qquad \mathrm{char}_A(x) = \det(A - xI) = \begin{vmatrix} 1-x & 0 & 1 & 0 & 0 \\ 0 & 3-x & 0 & 0 & 0 \\ 1 & 0 & 1-x & 0 & 0 \\ 0 & 0 & 0 & k-x & 1 \\ 0 & 0 & 0 & 0 & 2-x \end{vmatrix}$$

$$= (2-x)(k-x) \begin{vmatrix} 1-x & 0 & 1 \\ 0 & 3-x & 0 \\ 1 & 0 & 1-x \end{vmatrix}$$

$$= (2-x)(k-x)\big((1-x)(3-x)(1-x) + 1(-1)(3-x)\big)$$

$$= (2-x)(k-x)(3-x)(\cancel{1} - 2x + x^2 \cancel{-1})$$

$$= -x(2-x)^2(k-x)(3-x)$$

Two cases:

<u>Case 1:</u> $k = 2$. Then $\mathrm{char}_A(x) = -x(2-x)^3(3-x)$

$\lambda = 0$: alg mult 1
geom mult 1

$$U_0 = \text{nullspace}(A) = \left\{ x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$\lambda = 2$ alg mult 3
geom mult 2

$$U_2 = \text{nullspace}(A - 2I) = \text{nullspace}\left(\begin{bmatrix} -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\right) = \left\{ x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$\lambda = 3$ alg mult 1
geom mult 1

$$U_3 = \text{nullspace}(A - 3I) = \text{nullspace}\left(\begin{bmatrix} -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}\right) = \left\{ x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Case 2: $k \neq 2$ Then $\text{char}_A(x) = -x(2-x)^2(k-x)(3-x)$

- $\lambda = 0$ alg mult 1
  geom mult 1 $\quad U_0 = \text{nullspace}(A) = \text{nullspace}\left(\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & k & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}\right) = \left\{ x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

- $\lambda = 2$ alg mult 2
  geom mult 2 $\quad U_2 = \text{nullspace}(A - 2I) = \text{nullspace}\left(\begin{bmatrix} -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & k-2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\right)$

$$= \left\{ x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{k-2} \\ 1 \end{bmatrix} \right\}$$

($k-2 \neq 0$ since $k \neq 2$)

- $\lambda = k$ alg mult 1
  geom mult 1 $\quad U_k = \text{nullspace}(A - kI) = \text{nullspace}\left(\begin{bmatrix} 1-k & 0 & 1 & 0 & 0 \\ 0 & 3-k & 0 & 0 & 0 \\ 1 & 0 & 1-k & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2-k \end{bmatrix}\right)$

$$= \left\{ x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

- $\lambda = 3$ alg mult 1
  geom mult 1 $\quad U_3 = \text{nullspace}(A - 3I) = \text{nullspace}\left(\begin{bmatrix} -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & k-3 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}\right)$

$$= \left\{ x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$\boxed{\text{The matrix is diagonalizable whenever } k \neq 2.}$