Quiz 6 Study Guide Part 2

Conceptual Questions

1) Let A be $n \times n$ where n is odd. Then $\text{char}_A(x)$ has degree n and has n complex roots. Since any nonreal roots come as complex conjugate pairs and n is odd, $\text{char}_A(x)$ must have at least one real root. So A has at least one real eigenvalue.

2) No. Consider $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

3) False. $D = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ has $\lambda = 2$ as a repeated eigenvalue but it is clearly diagonalizable (it is already diagonal).

4) Let all rows sum to c. Then $\lambda = c$ is an eigenvalue with $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = v$ as an eigenvector.

5) Let A be a 3×3 matrix with a non-real eigenvalue, λ_0. Then $\bar{\lambda}_0$ is also an eigenvalue. $\text{char}_A(x)$ has degree 3 and hence has three complex roots, two of which are λ_0 and $\bar{\lambda}_0$. Since nonreal roots come as complex conjugate pairs, the last root must be real. So all three roots of $\text{char}_A(x)$ (which are the eigenvalues of A) are distinct; hence, A is diagonalizable.

Problems

1) $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$

$$\text{char}_A(x) = \begin{vmatrix} 1-x & 2 & 1 \\ 1 & -1-x & 0 \\ -1 & -2 & -1-x \end{vmatrix}$$

$$= 1(-2-1-x) + (-1-x)(-1+x^2-2)$$

$$= -3 - x + (-1-x)(x^2-3)$$

$$= \cancel{-3} - x \cancel{+3} - x^3 - x^2 + 3x$$

$$= -x^3 - x^2 + 2x$$

$$= x(-x^2 - x + 2) = x(1-x)(2+x)$$

$\lambda = 0$: $U_{\lambda=0} = \left\{ x_3 \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -1 \end{pmatrix} \right\}$

$\lambda = 1$: $U_{\lambda=1} = \text{nullspace} \begin{pmatrix} 0 & 2 & 1 \\ 1 & -2 & 0 \\ -1 & -2 & -2 \end{pmatrix} = \left\{ x_3 \begin{pmatrix} 1 \\ \frac{1}{2} \\ -1 \end{pmatrix} \right\}$

$\lambda = -2$: $U_{\lambda=-2} = \text{nullspace} \begin{pmatrix} 3 & 2 & 1 \\ 1 & 1 & 0 \\ -1 & -2 & 1 \end{pmatrix} = \left\{ x_3 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$

A diagonalizable over real and complex numbers as

$$A = \begin{bmatrix} \frac{1}{3} & 1 & -1 \\ \frac{1}{3} & \frac{1}{2} & 1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 1 & -1 \\ \frac{1}{3} & \frac{1}{2} & 1 \\ -1 & -1 & 1 \end{bmatrix}^{-1}$$

$$B = \begin{bmatrix} 1 & -3 & 0 \\ 3 & -5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{char}_B(x) = \begin{vmatrix} 1-x & -3 & 0 \\ 3 & -5-x & 0 \\ 0 & 0 & 1-x \end{vmatrix}$$

$$= (1-x)(-5+4x+x^2+9)$$
$$= (1-x)(x^2+4x+4)$$
$$= (1-x)(x+2)^2$$

$\lambda = 1$: $U_{\lambda=1} = \text{nullspace} \begin{pmatrix} 0 & -3 & 0 \\ 3 & -6 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \left\{ x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\lambda = -2$: $U_{\lambda=-2} = \text{nullspace} \begin{pmatrix} 3 & -3 & 0 \\ 3 & -3 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \left\{ x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$

B not diagonalizable over complex or reals since geometric multiplicities (over complex/real) only sum to 2, not 3.

$$C = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$$

$$\text{char}_C(x) = \begin{vmatrix} -1-x & 0 & 1 \\ 1 & -1-x & 0 \\ -1 & 0 & -1-x \end{vmatrix}$$

$$= (-1-x)(-1-x)(-1-x) + 1(-1-x)$$

$$= (-1-x)(1 + 2x + x^2 + 1)$$

$$= (-1-x)(x^2 + 2x + 2)$$

$$\lambda = -1 : \quad U_{\lambda=-1} = \text{nullspace}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \left\{ x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$\lambda = -1+i : \quad U_{\lambda=-1+i} = \text{nullspace}\begin{pmatrix} -i & 0 & 1 \\ 1 & -i & 0 \\ -1 & 0 & -i \end{pmatrix}$$

$$\rightarrow \text{nullspace}\begin{pmatrix} 1 & -i & 0 \\ 0 & 1 & 1 \\ 0 & -i & -i \end{pmatrix} \begin{matrix} iR_2 + R_1 \\ R_2 + R_3 \end{matrix}$$

$$\rightarrow \text{nullspace}\begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} R_1 + iR_2 \\ \\ R_3 + iR_2 \end{matrix}$$

$$= \left\{ x_3 \begin{pmatrix} -i \\ -1 \\ 1 \end{pmatrix} \right\}$$

complex conjugates

$$\lambda = -1-i : \quad U_{\lambda=-1-i} = \left\{ x_3 \begin{pmatrix} i \\ -1 \\ 1 \end{pmatrix} \right\}$$

C is diagonalizable over complex numbers, but not over real numbers.

$$C = \begin{bmatrix} 0 & -i & i \\ 1 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1+i & 0 \\ 0 & 0 & -1-i \end{bmatrix} \begin{bmatrix} 0 & -i & i \\ 1 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix}^{-1}$$

2) $1-2i, 1+2i$ are eigenvalues.

trace = sum of eigenvalues, so the last eigenvalue is 2.

$\det(A) = 2(1-2i)(1+2i)$ (product of eigenvalues)

$= 2(1+4) = \boxed{10}$

$\text{char}_A(x)$ has $1-2i, 1+2i, 2$ as roots and the coefficient on the highest degree term is $(-1)^3 = -1$.

So $\text{char}_A(x) = -1(x-2)(x-(1-2i))(x-(1+2i))$

$= -1(x-2)(x^2-(1+2i+1-2i)x+5)$

$= (2-x)(x^2-2x+5)$

$= 2x^2 - x^3 - 4x + 2x^2 + 10 - 5x$

$= \boxed{-x^3+4x^2-9x+10}$

3) $M = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ $\quad \text{char}_M(x) = \begin{vmatrix} \frac{1}{2}-x & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -x & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2}-x \end{vmatrix}$

$= \left(\frac{1}{2}-x\right)\left(-\frac{1}{2}x + x^2 - \frac{1}{4}\right) + \frac{1}{2}\left(-\frac{1}{4}+\frac{1}{2}x\right)$

$= \left(\frac{1}{2}-x\right)\left(-\frac{1}{2}x+x^2-\frac{1}{4}\right) - \frac{1}{4}\left(\frac{1}{2}-x\right)$

$= \left(\frac{1}{2}-x\right)\left(x^2-\frac{1}{2}x-\frac{1}{2}\right)$

$= \left(\frac{1}{2}-x\right)\left(\frac{1}{2}+x\right)(-1+x)$

$\lambda = \frac{1}{2}$ $\quad U_{\lambda=\frac{1}{2}} = \text{nullspace}\begin{pmatrix} 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix} = \left\{ x_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\lambda = -\frac{1}{2}$ $\quad U_{\lambda=-\frac{1}{2}} = \text{nullspace}\begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & 1 \end{pmatrix} = \left\{ x_3 \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} \right\}$

$\lambda = 1:$ $\quad U_{\lambda=1} = \text{nullspace}\begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -1 & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = \left\{ x_3 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$

So $M = \begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-1}$

$= SDS^{-1}$

Note $M^n = SD^nS^{-1}$

so $$\lim_{n\to\infty} M^n = \lim_{n\to\infty} \begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} (\frac{1}{2})^n & 0 & 0 \\ 0 & (-\frac{1}{2})^n & 0 \\ 0 & 0 & 1^n \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} = \boxed{\begin{bmatrix} \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}}$$

4) $$A = \begin{bmatrix} \frac{1}{3} & 1 & -1 \\ \frac{1}{3} & \frac{1}{2} & 1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 1 & -1 \\ \frac{1}{3} & \frac{1}{2} & 1 \\ -1 & -1 & 1 \end{bmatrix}^{-1}$$

$$(\lambda A) = \begin{bmatrix} \frac{1}{3} & 1 & -1 \\ \frac{1}{3} & \frac{1}{2} & 1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & -2\lambda \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 1 & -1 \\ \frac{1}{3} & \frac{1}{2} & 1 \\ -1 & -1 & 1 \end{bmatrix}^{-1}$$

$$(\lambda A)^n = \begin{bmatrix} \frac{1}{3} & 1 & -1 \\ \frac{1}{3} & \frac{1}{2} & 1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \lambda^n & 0 \\ 0 & 0 & (-2\lambda)^n \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 1 & -1 \\ \frac{1}{3} & \frac{1}{2} & 1 \\ -1 & -1 & 1 \end{bmatrix}^{-1}$$

So $\lim_{n\to\infty} \lambda^n A^n = \lim_{n\to\infty} (\lambda A)^n$ only exists when $\lim_{n\to\infty} \lambda^n$ and $\lim_{n\to\infty} (-2\lambda)^n$ exist which happens when $-1 < \lambda <= 1$ and $-1/2 <= \lambda < 1/2$ respectively.

So $\lim_{n\to\infty} \lambda^n A^n$ exists whenever $\boxed{-1/2 <= \lambda < 1/2}$.

5) $(10,15,19) = c_1(1,1,0) + c_2(1,0,1) + c_3(0,1,-2)$

$$\begin{aligned} c_1 + c_2 \quad\quad &= 10 \\ c_1 \quad + c_3 &= 15 \\ c_2 - 2c_3 &= 19 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 10 \\ 1 & 0 & 1 & 15 \\ 0 & 1 & -2 & 19 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 15 \\ 0 & 1 & -1 & -5 \\ 0 & 1 & -2 & 19 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 15 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & 1 & -24 \end{array}\right] R_2 - R_3$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 39 \\ 0 & 1 & 0 & -29 \\ 0 & 0 & 1 & -24 \end{array}\right]$$

$A^{2019}((10,15,19))$

$= A^{2019}(39(1,1,0) + (-29)(1,0,1) + (-24)(0,1,-2))$

$= 39A^{2019}(1,1,0) + (-29)A^{2019}(1,0,1) + (-24)A^{2019}(0,1,-2)$

$= 39 \cdot (-1)^{2019}(1,1,0) + \cancel{(-29)\,0^{2019}(1,0,1)} + (-24)1^{2019}(0,1,-2)$

$= -39(1,1,0) + (-24)(0,1,-2)$

$= \boxed{(-39,-63,48)}$