Quiz 4 Study Guide Solutions

Conceptual Questions

1) e.g. if $k=3, n=2$, consider

$$T: \mathbb{R}^3 \to \mathbb{R}^2 \text{ given by } T(x,y,z)=(x,y)$$
(restriction map)

2) e.g. if $k=2, n=3$, consider

$$T: \mathbb{R}^2 \to \mathbb{R}^3 \text{ given by } T(x,y)=(x,y,0)$$
(inclusion map)

3) An isomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$ is given by

$$T(x_1, x_2, \dots, x_n) = (x_1, x_2, \dots, x_n)$$
$$= Ix \quad \text{(identity transformation)}$$

The zero transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ given by

$$T(x_1, x_2, \dots, x_n) = (0, 0, \dots, 0)$$

is not an isomorphism.

4) Such a transformation would be given by matrix multiplication by a non-square matrix A.

Since A is not square, it either has a free column or a zero row in its RREF. So either $Ax=0$ does not have just the trivial soln (so T is not injective) or $Ax=b$ is not always consistent (so T is not surjective). So T cannot be an isomorphism.

Problem 1

$$\boxed{A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ -1 & 3 \end{bmatrix}} \overset{\text{RREF}}{\Rightarrow} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

no free var, zero row

$Ax = 0$ only has soln $x = 0 \Rightarrow T$ is injective.

$Ax = b$ is not consistent for every $b \in \mathbb{R}^3$

$\Rightarrow T$ is NOT surjective

So T is not an isomorphism.

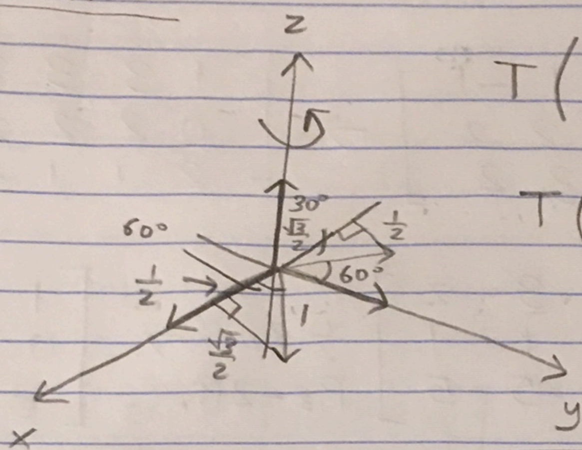
Problem 2

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

So $\boxed{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{2019} = \begin{bmatrix} 1 & 2019 \\ 0 & 1 \end{bmatrix}}$

Problem 3



$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}\frac{1}{2}\\\frac{\sqrt{3}}{2}\\0\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}-\frac{\sqrt{3}}{2}\\\frac{1}{2}\\0\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\boxed{A=\begin{bmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0\\\frac{\sqrt{3}}{2} & \frac{1}{2} & 0\\0 & 0 & 1\end{bmatrix}}$$

$$\Rightarrow\begin{bmatrix}1 & -\sqrt{3} & 0\\\sqrt{3} & 1 & 0\\0 & 0 & 1\end{bmatrix}\Rightarrow\begin{bmatrix}1 & -\sqrt{3} & 0\\0 & 4 & 0\\0 & 0 & 1\end{bmatrix}R_2-\sqrt{3}R_1$$

$$\Rightarrow\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{matrix}R_1+\frac{\sqrt{3}}{4}R_2\\\frac{1}{4}R_2\\ \end{matrix}$$

So A is invertible, hence $T(x)=Ax$ is an isomorphism.

Problem 4

$$\begin{bmatrix} 2 & 1 & 1 & -1 \\ 0 & 2 & 2 & 2 \\ 1 & 3 & 3 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 2 & 2 & 2 \\ 2 & 1 & 1 & -1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & -5 & -5 & -5 \end{bmatrix} \begin{matrix} \\ \frac{1}{2}R_2 \\ R_3 - 2R_1 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 - 3R_2 \\ \\ R_3 + 5R_2 \end{matrix}$$

free

$ker(T_A)$ = all x such that $\underbrace{T_A(x) = 0}_{Ax=0}$

$$= \boxed{x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}}$$

not surjective since $Ax = b$ is not always consistent (since RREF of A has a zero row).

So not an isomorphism.

To find vectors in range, we must find all b such that $Ax = b$ for some x (such that $Ax = b$ is consistent.)

$$\begin{bmatrix} 2 & 1 & 1 & -1 & | & a \\ 0 & 2 & 2 & 2 & | & b \\ 1 & 3 & 3 & 2 & | & c \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 3 & 2 & | & c \\ 0 & 1 & 1 & 1 & | & \frac{1}{2}b \\ 2 & 1 & 1 & -1 & | & a \end{bmatrix}$$

$$\boxed{\begin{matrix} \text{range} = \\ (a,b,c) \text{ s.t.} \\ a + \frac{5}{2}b - 2c = 0 \end{matrix}}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 & | & c - \frac{3}{2}b \\ 0 & 1 & 1 & 1 & | & \frac{1}{2}b \\ 0 & -5 & -5 & -5 & | & a - 2c \end{bmatrix} \begin{matrix} \\ \\ R_3 - 2R_1 \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 & | & c - \frac{3}{2}b \\ 0 & 1 & 1 & 1 & | & \frac{1}{2}b \\ 0 & 0 & 0 & 0 & | & a + \frac{5}{2}b - 2c \end{bmatrix} \begin{matrix} \\ \\ R_3 + 5R_2 \end{matrix}$$

5) $(1,0,0) = c_1(1,1,0) + c_2(0,2,1) + c_3(1,1,1)$

$$\begin{aligned} c_1 \quad\quad + c_3 &= 1 \\ c_1 + 2c_2 + c_3 &= 0 \\ c_2 + c_3 &= 0 \end{aligned}$$

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] R_2 - R_1$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 1 \end{array}\right] \begin{array}{l} \frac{R_2}{2} \\ R_3 - \frac{R_2}{2} \end{array} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -1 \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 1 \end{array}\right]$$

$$B^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -1 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} \quad \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = B^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$\begin{aligned} T(1,0,0) &= T\left(\tfrac{1}{2}(1,1,0) + \left(-\tfrac{1}{2}\right)(0,2,1) + \tfrac{1}{2}(1,1,1)\right) \\ &= \tfrac{1}{2}T(1,1,0) - \tfrac{1}{2}T(0,2,1) + \tfrac{1}{2}T(1,1,1) \\ &= \tfrac{1}{2}(0,1,-1) - \tfrac{1}{2}(1,0,-1) + \tfrac{1}{2}(0,-1,-3) \\ &= \left(-\tfrac{1}{2}, 0, -\tfrac{3}{2}\right) \end{aligned}$$

Similarly,

$$T(0,1,0) = T\left(\tfrac{1}{2}(1,1,0) + \tfrac{1}{2}(0,2,1) - \tfrac{1}{2}(1,1,1)\right)$$

$$= \tfrac{1}{2}T(1,1,0) + \tfrac{1}{2}T(0,2,1) + \left(-\tfrac{1}{2}\right)T(1,1,1)$$

$$= \tfrac{1}{2}(0,1,-1) + \tfrac{1}{2}(1,0,-1) - \tfrac{1}{2}(0,-1,-3)$$

$$= \left(\tfrac{1}{2}, 1, \tfrac{1}{2}\right)$$

$$T(0,0,1) = T\left(-1(1,1,0) + 0(0,2,1) + 1(1,1,1)\right)$$

$$= -T(1,1,0) + T(1,1,1)$$

$$= -(0,1,-1) + (0,-1,-3)$$

$$= (0,-2,-2)$$

So $\boxed{A = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -2 \\ -\frac{3}{2} & \frac{1}{2} & -2 \end{bmatrix}}$

$$\left[\begin{array}{ccc|ccc} -\frac{1}{2} & \frac{1}{2} & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ -\frac{3}{2} & \frac{1}{2} & -2 & 0 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & -1 & -2 & -3 & 0 & 1 \end{array}\right] \begin{array}{l} -2R_1 \\ \\ R_3-3R_1 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & -4 & -3 & 1 & 1 \end{array}\right] \begin{array}{l} \\ \\ R_2+R_3 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 1 \\ 0 & 0 & 1 & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right] \begin{array}{l} \\ \\ -\frac{1}{4}R_3 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right] \begin{array}{l} \\ R_2+2R_3 \\ \\ \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right] \begin{array}{l} R_1+R_2 \\ \\ \\ \end{array}$$

Since A is invertible, T is an isomorphism and its inverse is

$$\boxed{T^{-1}(y) = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}$$

6) The pivots of an upper triangular matrix with diagonal entries nonzero are all along the diagonal. For each row, we can scale the row so that the diagonal entries are all 1 and then we can use the diagonal entries to clear all entries above the diagonal so that the RREF is the identity matrix.

Hence, any upper triangular matrix with nonzero diagonal entries is invertible; hence, the linear transformation it defines is an isomorphism.

7)
$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 0 & 3 & 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] R_1 - R_3$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 0 & 0 & -2 & 1 & -3 & -1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] R_1 - 3R_2$$

$$\boxed{A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix}}$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} -\frac{1}{2} R_1 \\ R_2 + \frac{1}{2} R_1 \\ \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}\right] R_2 + R_3 \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} \end{array}\right]$$

$$Ax = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$$\boxed{x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}}$$