Math 54 Quiz 4

September 19, 2019

Question 1 (2 points)

Directions: For each item, circle either True or False. (0.5 points each)

- (True/False) There is a surjective linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^4$.
- (True/False) Every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ is an isomorphism.
- (True/False) If A is an $n \times n$ matrix, then if $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbb{R}^n$, then $A\mathbf{x} = \mathbf{0}$ has a unique solution.
- (True/False) If A and B are both invertible square matrices of the same size, then $A + B$ is also invertible.

Question 2 (6 points)

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that

$$T(1,1) = (1,1,1) \qquad T(2,-1) = (0,1,-1)$$

Determine if T is injective, surjective, bijective.

$(1,0) = c_1(1,1) + c_2(2,-1)$

$c_1 + 2c_2 = 1$
$c_1 - c_2 = 0$
$c_1 = \frac{1}{3}, \ c_2 = \frac{1}{3}$

$(0,1) = c_1(1,1) + c_2(2,-1)$

$c_1 + 2c_2 = 0$
$c_1 - c_2 = 1$
$c_1 = \frac{2}{3} \quad c_2 = -\frac{1}{3}$

$T(1,0) = \frac{1}{3}T(1,1) + \frac{1}{3}T(2,-1) = (\frac{1}{3}, \frac{2}{3}, 0)$

$T(0,1) = \frac{2}{3}T(1,1) + (-\frac{1}{3})T(2,-1) = (\frac{2}{3}, \frac{1}{3}, 1)$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

T is injective, but it is not surjective or bijective.

$Ax = b$ not consistent for all b (zero row)
$Ax = 0$ only has soln $x = 0$ (no free var)

Question 3 (7 points)

Define the cyclic permutation linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ by

$$T(x_1, x_2, x_3, x_4) = (x_4, x_1, x_2, x_3)$$

- Find the matrix A for T.
- Is T injective, surjective, bijective? (Hint: Your row reduction can easily be done by just switching rows and using no other elementary row operations.)
- If T is bijective, find a formula for $T^{-1}(y)$, where $y = (y_1, y_2, y_3, y_4)$.

$T(1,0,0,0) = (0,1,0,0)$
$T(0,1,0,0) = (0,0,1,0)$
$T(0,0,1,0) = (0,0,0,1)$
$T(0,0,0,1) = (1,0,0,0)$

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$Ax = 0$ has only soln $x = 0$
$Ax = b$ always consistent

T is surjective, injective and bijective.

Find A^{-1}.

$$\left[\begin{array}{cccc|cccc} 0&0&0&1&1&0&0&0 \\ 1&0&0&0&0&1&0&0 \\ 0&1&0&0&0&0&1&0 \\ 0&0&1&0&0&0&0&1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1&0&0&0&0&1&0&0 \\ 0&0&0&1&1&0&0&0 \\ 0&1&0&0&0&0&1&0 \\ 0&0&1&0&0&0&0&1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1&0&0&0&0&1&0&0 \\ 0&1&0&0&0&0&1&0 \\ 0&0&0&1&1&0&0&0 \\ 0&0&1&0&0&0&0&1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1&0&0&0&0&1&0&0 \\ 0&1&0&0&0&0&1&0 \\ 0&0&1&0&0&0&0&1 \\ 0&0&0&1&1&0&0&0 \end{array}\right] \qquad A^{-1} = \begin{bmatrix} 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \\ 1&0&0&0 \end{bmatrix}$$

$$T^{-1}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \\ 1&0&0&0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} y_2 \\ y_3 \\ y_4 \\ y_1 \end{bmatrix}$$