Quiz 3 Study Guide

Conceptual Questions:

① $\mathbb{R}^n$ only has n linearly independent directions, so given $k>n$ directions, at least one must be redundant. (Rigorously, if you do the system out, you will need to solve $Ax=0$ where A has more columns than rows, so the RREF of A must have a free variable, so there is not just the trivial soln.)

② $\mathbb{R}^n$ has n linearly independent directions so k directions are not enough to fill $\mathbb{R}^n$ if $k<n$. (Rigorously, this will be a problem of showing $Ax=b$ is not always consistent for all b in $\mathbb{R}^n$, where A is $n\times k$, but since $k<n$, the RREF must have a zero row, so $Ax=b$ indeed is not always consistent.)

③ This is easiest to see by example. Consider $n=3$ and vectors

$v_1=(a,b,c)$
$v_2=(d,e,f)$
$v_3=(g,h,i)$

$\rightarrow$ linear independence means $x_1(a,b,c)+x_2(d,e,f)+x_3(g,h,i)=(0,0,0)$ only has the trivial soln.

$$\begin{aligned} ax_1+dx_2+gx_3&=0\\ bx_1+ex_2+hx_3&=0\\ cx_1+fx_2+ix_3&=0\end{aligned}$$

So RREF of $A=\begin{bmatrix} a & d & g\\ b & e & h\\ c & f & i\end{bmatrix}$ is $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

which also has no zero row. Hence, $A\vec{x}=\vec{b}$ is consistent for all $\vec{b}$ in $\mathbb{R}^3$, so in particular,

$$x_1(a,b,c)+x_2(d,e,f)+x_3(g,h,i)=\overbrace{(j,k,\ell)}^{\vec{b}}$$

always has a soln (x_1,x_2,x_3) for all (j,k,ℓ).

So v_1, v_2, v_3 also span $\mathbb{R}^3$.

(4) Similar argument to (3). Any n vectors that span $\mathbb{R}^n$ must be linearly independent, and hence form a basis.

(5) False. E.G. $(1,1,1), (1,0,0), (2,1,1)$ are linearly dependent since $(2,1,1) = 1(1,1,1) + 1(1,0,0)$ but no vector is a scalar multiple of the other.

(6) Let $v_1, v_2, \dots, v_k$ be a basis for $\mathbb{R}^n$. These vectors are linearly independent (def of basis) so from (1), $k \not> n$. These vectors also span $\mathbb{R}^n$ (def of basis) so from (2), $k \not< n$. Since $k \not> n$ and $k \not< n$, we must have $k = n$.

Problems

1) We need to show

$$c_1(1,2,1) + c_2(-1,0,-1) + c_3(0,-1,2) + c_4(-1,1,1) = (a,b,c)$$

has a soln (c_1, c_2, c_3, c_4) for every (a,b,c).

$$\Downarrow$$

$$(c_1 - c_2 - c_4,\ 2c_1 - c_3 + c_4,\ c_1 - c_2 + 2c_3 + c_4) = (a,b,c)$$

$$\begin{cases} c_1 - c_2 \qquad - c_4 = a \\ 2c_1 \qquad - c_3 + c_4 = b \\ c_1 - c_2 + 2c_3 + c_4 = c \end{cases}$$

need to show this is consistent for every (a,b,c).

$$A = \begin{bmatrix} 1 & -1 & 0 & -1 \\ 2 & 0 & -1 & 1 \\ 1 & -1 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 2 & 2 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - R_1 \end{matrix}$$

echelon form

$$\to \begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{matrix} \\ \frac{1}{2}R_2 \\ \frac{1}{2}R_3 \end{matrix} \to \begin{bmatrix} 1 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{matrix} R_1 + R_2 \\ R_2 + \frac{1}{2}R_3 \\ \\ \end{matrix}$$

true, since A has no zero row

For $(1,1,1)$ solve $c_1(1,2,1) + c_2(-1,0,-1) + c_3(0,-1,2) + c_4(-1,1,1) = (1,1,1)$

$$\begin{matrix} c_1 - c_2 \qquad - c_4 = 1 \\ 2c_1 \qquad - c_3 + c_4 = 1 \\ c_1 - c_2 + 2c_3 + c_4 = 1 \end{matrix} \xRightarrow{\text{row reduce}} c_1 = \tfrac{1}{2} \quad c_2 = -\tfrac{1}{2} \quad c_3 = 0 \quad c_4 = 0$$

$$\boxed{(1,1,1) = \tfrac{1}{2}(1,2,1) + \left(-\tfrac{1}{2}\right)(-1,0,-1) + 0(0,-1,2) + 0(-1,1,1)}$$

2) Solve $c_1(1,-1,2,1)+c_2(2,1,0,1)+c_3(0,1,1,1)=(0,0,0,0)$

$$(c_1+2c_2,\ -c_1+c_2+c_3,\ 2c_1+c_3,\ c_1+c_2+c_3) = (0,0,0,0)$$

$$\begin{cases} c_1+2c_2 = 0 \\ -c_1+c_2+c_3 = 0 \\ 2c_1 + c_3 = 0 \\ c_1+c_2+c_3 = 0 \end{cases}$$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & -2 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{matrix} \\ R_1+R_2 \\ R_3-2R_1 \\ R_4-R_1 \end{matrix}$$

$$\to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{matrix} \\ \frac{1}{2}R_2 \\ R_2+R_3 \\ R_4-\frac{1}{2}R_2 \end{matrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1-R_2 \\ R_2-R_3 \\ \\ R_4+2R_3 \end{matrix}$$

no free var, so only have the trivial soln $(c_1,c_2,c_3,c_4) = (0,0,0,0)$.

So vectors are linearly independent.

3) ① Show linear independence.

$$c_1(1,-1,1)+c_2(-2,0,1)+c_3(2,1,-1) = (0,0,0)$$

$$\begin{aligned} c_1 - 2c_2 + 2c_3 &= 0 \\ -c_1 \qquad + c_3 &= 0 \\ c_1 + c_2 \quad - c_3 &= 0 \end{aligned}$$

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 1 & -2 & 2 \\ 1 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & -2 & 3 \\ 0 & 1 & 0 \end{bmatrix} \begin{matrix} R_2-R_1 \\ R_3-R_1 \\ \end{matrix}$$

$$\to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{matrix} \\ R_3 \\ R_2+2R_3 \end{matrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1+\frac{1}{3}R_3 \\ \\ \frac{1}{3}R_3 \end{matrix}$$

no free vars, so only soln is $(c_1,c_2,c_3) = (0,0,0)$ so the vectors are linearly ind.

$\longrightarrow$

② Show the vectors span $\mathbb{R}^3$.

We need to show

$$c_1(1,-1,1)+c_2(-2,0,1)+c_3(2,1,-1)=(a,b,c)$$

has a soln (c_1,c_2,c_3) for every (a,b,c). This is equivalent to showing the system

$$\begin{aligned} c_1 - 2c_2 + 2c_3 &= a \\ -c_1 \qquad\quad + c_3 &= b \\ c_1 + c_2 - c_3 &= c \end{aligned}$$ is consistent for all (a,b,c).

But this is true since in part ①, we saw that

$$A=\begin{bmatrix} 1 & -2 & 2 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \longrightarrow \text{has RREF} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which has no zero row. So the vectors span $\mathbb{R}^3$.

Since the vectors are linearly independent and span $\mathbb{R}^3$, they are a basis for $\mathbb{R}^3$.

4) We need to find all (a,b,c,d) so that

$$c_1(1,2,1,-1)+c_2(-1,0,2,1)+c_3(-1,2,5,1)=(a,b,c,d)$$

has a soln (c_1,c_2,c_3). This is equivalent to the system

$$\begin{aligned} c_1 - c_2 - c_3 &= a \\ 2c_1 \qquad + 2c_3 &= b \\ c_1 + 2c_2 + 5c_3 &= c \\ -c_1 + c_2 + c_3 &= d \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & a \\ 2 & 0 & 2 & b \\ 1 & 2 & 5 & c \\ -1 & 1 & 1 & d \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & a \\ 0 & 2 & 4 & -2a+b \\ 0 & 3 & 6 & -a+c \\ 0 & 0 & 0 & a+d \end{array}\right] \begin{array}{l} R_2-2R_1 \\ R_3-R_1 \\ R_4+R_1 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & a \\ 0 & 1 & 2 & -a+\frac{1}{2}b \\ 0 & 1 & 2 & -\frac{1}{3}a+\frac{1}{3}c \\ 0 & 0 & 0 & a+d \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & a \\ 0 & 1 & 2 & -a+\frac{1}{2}b \\ 0 & 0 & 0 & \frac{2}{3}a-\frac{1}{2}b+\frac{1}{3}c \\ 0 & 0 & 0 & a+d \end{array}\right] R_3-R_2$$

$$\boxed{\frac{2}{3}a-\frac{1}{2}b+\frac{1}{3}c=0 \text{ and } a+d=0}$$

$\boxed{(1,0,0,1) \text{ is not in the span}}$

5) If $v_1, v_2, \ldots, v_k$ are linearly dependent,

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0 \text{ for } c_i \text{ not all zero.}$$

So $\underline{c_1} v_1 + \underline{c_2} v_2 + \cdots + \underline{c_k} v_k + \underline{0} w = 0$

where not all of the underlined scalars are all zero.

Thus, $v_1, v_2, \ldots, v_k, w$ are linearly dependent.

6) Since $v_1, v_2, \ldots, v_k$ span $\mathbb{R}^n$, for every b in $\mathbb{R}^n$, there exist $c_1, c_2, \ldots, c_k$ such that

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = b$$

and hence

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k + 0 w = b.$$

So $v_1, v_2, \ldots, v_k, w$ span $\mathbb{R}^n$.

7) $\{v_1, v_2, \ldots, v_n, w\}$ contains $(n+1)$ vectors in $\mathbb{R}^n$ so by the first conceptual question, these vectors cannot be linearly independent in $\mathbb{R}^n$ (since $n+1 > n$) and hence, they cannot be a basis for $\mathbb{R}^n$.