Quiz 2 Study Guide

Conceptual Questions:

- First, a homogeneous system is always consistent. The augmented matrix is of the form

  ← excluding the last one

  ← n cols →

  m rows

  where $m < n$, $m = \#$ of eqns
  $n = \#$ of variables

  There can only be at most m pivots since there are m rows. So there can only be at most m pivot columns. Since $n > m$ (see matrix above) there must be at least one free column, so since the system is consistent, it has infinitely many solutions.

- Yes. The trivial (zero solution) of all zeros is always a solution of every homogeneous system.

- $$\begin{aligned} x_1 + x_2 &= 2 \\ x_1 - 2x_2 &= -1 \\ 2x_1 - x_2 &= 1 \end{aligned}$$

  is consistent (solution is $(1,1)$)
  Reasoning to get this: The first two eqns give a consistent system and to get the third eqn, I just added the first two, which will not affect consistency)

  $$\begin{aligned} x_1 + x_2 &= 2 \\ x_1 - 2x_2 &= -1 \\ 2x_1 - x_2 &= 0 \end{aligned}$$

  is inconsistent
  Reasoning to get this: Any soln of the first two eqns satisfies $2x_1 - x_2 = 1$ (eqn 1 + eqn 2), which is incompatible with the given third eqn.

- False. Consider $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$.

  There are no free variables so the unique solution is $(1,-1)$.

Problem 1

① $\left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 2 & -5 & 3 & 1 \\ 4 & -7 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & -3 & 5 & -3 \\ 0 & -3 & 5 & -8 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3-4R_1 \end{array}$

$\to \left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & -3 & 5 & -3 \\ 0 & 0 & 0 & -5 \end{array}\right] \begin{array}{l} \\ \\ R_3-R_2 \end{array}$

$\boxed{\text{inconsistent}}$

② $\left[\begin{array}{cc|c} 1 & 4 & 0 \\ -2 & -1 & 0 \\ 3 & 2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 4 & 0 \\ 0 & 7 & 0 \\ 0 & -10 & 0 \end{array}\right] \begin{array}{l} \\ R_2+2R_1 \\ R_3-3R_1 \end{array}$

$\to \left[\begin{array}{cc|c} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & -10 & 0 \end{array}\right] \tfrac{1}{7}R_2 \to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_1-4R_2 \\ \\ R_3+10R_2 \end{array}$

no free vars.

$\boxed{\text{Soln: } (x_1, x_2) = (0,0)}$

③ $\left[\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ 1 & -1 & -1 & 0 \\ 0 & 2 & 1 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ 0 & -4 & -2 & -2 \\ 0 & 2 & 1 & 3 \end{array}\right] \begin{array}{l} \\ R_2-R_1 \\ \\ \end{array}$

$\to \left[\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ 0 & -4 & -2 & -2 \\ 0 & 0 & 0 & 4 \end{array}\right] \begin{array}{l} \\ \\ 2R_3+R_2 \end{array}$

$\boxed{\text{inconsistent}}$

④ $\begin{cases} x_1 + x_2 + 2x_3 + x_4 = 2 \\ x_1 - 2x_2 + 2x_3 - x_4 = 1 \end{cases}$

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 2 \\ 1 & -2 & 2 & -1 & 1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 2 \\ 0 & -3 & 0 & -2 & -1 \end{array}\right] R_2 - R_1$$

$$\to \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & \frac{2}{3} & \frac{1}{3} \end{array}\right] -\tfrac{1}{3}R_2$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & 2 & \frac{1}{3} & \frac{5}{3} \\ 0 & 1 & 0 & \frac{2}{3} & \frac{1}{3} \end{array}\right] R_1 - R_2$$

$\uparrow \ \uparrow$

x_3, x_4 free

$$x_1 + 2x_3 + \tfrac{1}{3}x_4 = \tfrac{5}{3}$$
$$x_2 + \tfrac{2}{3}x_4 = \tfrac{1}{3}$$

$$\Longrightarrow \quad x_1 = \tfrac{5}{3} - 2x_3 - \tfrac{1}{3}x_4$$
$$x_2 = \tfrac{1}{3} - \tfrac{2}{3}x_4$$

$$(x_1, x_2, x_3, x_4) = \left(\tfrac{5}{3} - 2x_3 - \tfrac{1}{3}x_4,\ \tfrac{1}{3} - \tfrac{2}{3}x_4,\ x_3,\ x_4\right)$$
$$= \left(\tfrac{5}{3}, \tfrac{1}{3}, 0, 0\right) + (-2x_3, 0, x_3, 0)$$
$$+ \left(-\tfrac{1}{3}x_4, -\tfrac{2}{3}x_4, 0, x_4\right)$$

$$= \boxed{\left(\tfrac{5}{3}, \tfrac{1}{3}, 0, 0\right) + x_3(-2, 0, 1, 0) + x_4\left(-\tfrac{1}{3}, -\tfrac{2}{3}, 0, 1\right)}$$

x_3, x_4 any real numbers

Problem 2

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & a \\ 2 & -1 & 3 & b \\ 4 & -3 & 5 & c \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 1 & a \\ 0 & 1 & 1 & -2a+b \\ 0 & 1 & 1 & -4a+c \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3-4R_1 \end{array}$$

$$\to \left[\begin{array}{ccc|c} 1 & -1 & 1 & a \\ 0 & 1 & 1 & -2a+b \\ 0 & 0 & 0 & -2a-b+c \end{array}\right] \begin{array}{l} \\ \\ R_3-R_2 \end{array} \quad \text{echelon}$$

← Note: We could have stopped here for this problem because you can tell whether a system is consistent or not in just the echelon form (don't need to go all the way to RREF if we only need to determine whether the system is consistent or not.)

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 2 & -a+b \\ 0 & 1 & 1 & -2a+b \\ 0 & 0 & 0 & -2a-b+c \end{array}\right] \quad \text{RREF}$$

need $\boxed{-2a-b+c=0}$

for system to be consistent

(no row of the form $[0 \cdots 0 \mid b]$ with $b \neq 0$)

Problem 3

$$\left[\begin{array}{ccc|c} 1 & 2 & h & 0 \\ 2 & 3 & -1 & 0 \\ 4 & 7 & 3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & h & 0 \\ 0 & -1 & -1-2h & 0 \\ 0 & -1 & 3-4h & 0 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3-4R_1 \end{array}$$

$$\to \left[\begin{array}{ccc|c} 1 & 2 & h & 0 \\ 0 & 1 & 1+2h & 0 \\ 0 & 0 & -2h+4 & 0 \end{array}\right] \begin{array}{l} \\ -R_2 \\ R_3-R_2 \end{array}$$

needs to be nonzero to be a pivot

For there to be a unique solution, we need a pivot in every column (no free columns).
We already have pivots in the first two columns.
For there to be a pivot in the third column,
we need $-2h+4 \neq 0$ so $\boxed{h \neq 2}$

4) $x_1 + 2x_2 - 3x_3 = 1$

pivot

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \end{array}\right]$$

(This is in RREF by default since the pivot is the only entry in its column!)

$\uparrow \quad \uparrow$

x_2, x_3 free

$$x_1 + 2x_2 - 3x_3 = 1 \Rightarrow x_1 = 1 - 2x_2 + 3x_3$$

$$(x_1, x_2, x_3) = (1 - 2x_2 + 3x_3, x_2, x_3)$$

$$= (1,0,0) + (-2x_2, x_2, 0) + (3x_3, 0, x_3)$$

$$= \boxed{(1,0,0) + x_2(-2,1,0) + x_3(3,0,1)}$$