Math 1A: Discussion 9/28/2018 Solutions

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Problem Set 1

Question 1

We calculate

$$f'(x) = 4x^3 + 1 + e^x$$

So then

$$f'(0) = 2$$

Because $f(0) = 1$, we have that the tangent line at $x = 0$ has slope 2 and passes through $(0, 1)$. So the equation of the tangent line is

$$y = 2x + 1$$

The slope of the normal line is $-1/2$ (the negative reciprocal of the slope of the tangent line). The normal line passes through $(0, 1)$. So the equation of the normal line is

$$y = -\frac{1}{2}x + 1$$

Problem Set 2

Question 2

First, consider $y = e^{-2x} + x$. We have that

$$\begin{aligned} y' = \lim_{h\to 0} \frac{e^{-2x-2h} + x + h - e^{-2x} - x}{h} &= \lim_{h\to 0} \frac{e^{-2x-2h} - e^{-2x} + h}{h} \\ = \lim_{h\to 0} \frac{e^{-2x-2h} - e^{-2x}}{h} + \frac{h}{h} &= 1 + e^{-2x} \cdot \lim_{h\to 0} \frac{e^{-2h} - 1}{h} \\ = 1 - 2e^{-2x} \cdot \lim_{h\to 0} \frac{e^{-2h} - 1}{-2h} &= 1 - 2e^{-2x} \cdot \lim_{u\to 0} \frac{e^{u} - 1}{u} \\ &= 1 - 2e^{-2x} \end{aligned}$$

where we set $u = -2h$ and noted that as $h \to 0$, we have that u also goes to 0. So the derivative here is

$$y' = 1 - 2e^{-2x}$$

Next, consider $y = 3^x - 3$. We have that

$$\begin{aligned} y' = \lim_{h\to 0} \frac{3^{x+h} - 3 - (3^x - 3)}{h} &= \lim_{h\to 0} \frac{3^{x+h} - 3^x}{h} = 3^x \cdot \lim_{h\to 0}\frac{3^h - 1}{h} \\ &= \lim_{h\to 0} 3^x \cdot \lim_{h\to 0}\frac{(e^{\ln(3)})^h - 1}{h} = \lim_{h\to 0} 3^x \cdot \lim_{h\to 0} \frac{e^{h\cdot\ln(3)} - 1}{h} \\ &= \ln(3) \cdot 3^x \cdot \lim_{h\to 0} \frac{e^{h\cdot\ln(3)} - 1}{h \cdot \ln(3)} = \ln(3) \cdot 3^x \cdot \lim_{u\to 0} \frac{e^u - 1}{u} \\ &= \ln(3) \cdot 3^x \end{aligned}$$

where we set $u = h \cdot \ln(3)$ and note that $u \to 0$ has $h \to 0$.

Question 3

For $f(x) = \frac{1}{2}e^{2x}$ and $g(x) = 5e^x - 4x$, we compute

$$f'(x) = e^{2x}$$

$$g'(x) = 5e^x - 4$$

So we want to solve the equation

$$e^{2x} = 5e^x - 4$$

This is the same as

$$e^{2x} - 5e^x + 4 = 0$$

Setting $y = e^x$, this is the same as

$$y^2 - 5y + 4 = 0$$

$$(y - 1)(y - 4) = 0$$

So $y = 1$ or $y = 4$. Since $y = e^x$, we have that $e^x = 1$ or $e^x = 4$. So then, $x = 0$ or $x = \ln(4)$. So for $a = 0$ and $a = \ln(4)$, the tangent lines to f and g at $x = a$ have the same slope.

Problem Set 3

Question 4 (*)

We first need the function to be continuous. So using continuity at $x = 1$, we have that

$$ae^{1-1} + 2b\sqrt{1} = b(1)^3 + 2$$

since the left and right hand limits must be the same. So we have that

$$a + 2b = b + 2$$

and hence $a + b = 2$.

Next, for differentiability, we need the left and right hand derivatives to be the same. The left hand derivative at $x = 1$ is

$$ae^{x-1} + \frac{b}{\sqrt{x}}$$

evaluated at $x = 1$, which is just $a + b$. The right hand derivative at $x = 1$ is

$$3bx^2$$

evaluated at $x = 1$, which is just $3b$. So then,

$$a + b = 3b$$

for the function to be differentiable at $x = 1$. So we have that $a = 2b$.

Solving $a + b = 2$ and $a = 2b$, we have that

$$a = \frac{4}{3}$$
$$b = \frac{2}{3}$$

Question 5

- A function $f(x)$ such that $f'(x) = f(x)$ is $f(x) = e^x$.
- Two functions such that $f''(x) = f(x)$ are $y = e^x$ and $y = e^{-x}$.
- Try a solution of the form $y = e^{\lambda x}$. Then, $y' = \lambda e^{\lambda x}$ and $y'' = \lambda^2 e^{\lambda x}$. So then, using $y = e^{\lambda x}$, we would have to have that

  $$(\lambda^2 - \lambda - 6)e^{\lambda x} = 0$$

  so that $\lambda^2 - \lambda - 6 = 0$, or $(\lambda - 3)(\lambda + 2) = 0$, so that $\lambda = 3$ or $\lambda = -2$. So two solutions are

  $$y = e^{3x}$$
  $$y = e^{-2x}$$

- A general process is to guess a solution of the form $y = e^{\lambda x}$. The solutions that will work are the solutions with λ satisfying

$$\lambda^2 + a\lambda + b = 0$$

But this process only works if we have real solutions to this equation. If we have complex solutions, then this will not work. For example, to solve $f''(x) + f(x) = 0$ using this way, we would need to solve

$$\lambda^2 + 1 = 0$$

which has no real solutions. The solutions to $f''(x) + f(x) = 0$ actually turn out to be $\sin(x)$ and $\cos(x)$ (and their linear combinations).