Math 1A: Discussion 9/19/2018 Solutions

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Problem Set 1

Question 1

We have that the function x^2 is continuous (it is a polynomial) and $\arctan(x^2-2)$ is continuous since it is the composition of two continuous functions, $\arctan(x)$ and x^2-2. Then, the product of continuous functions is continuous, so $x^2\arctan(x^2-2)$ is continuous. So we can just plug in to find the limit.

$$\lim_{x\to-1} x^2\arctan(x^2-2) = (-1)^2\arctan(-1) = -\frac{\pi}{4}$$

Remember that $\arctan(x)$ only takes values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Problem Set 2

Question 2

We note that the function

$$f(x) = \frac{2x^2+3x+1}{x+1}$$

by factoring and canceling is the same as

$$f(x) = 2x+1 \text{ for } x \neq -1$$

where $f(x)$ is undefined at $x=-1$.

So consider $\epsilon > 0$. Then we want to find a $\delta > 0$ so that $0 < |x-(-1)| < \delta$ implies that $|f(x)-(-1)| < \epsilon$. Since the limit does not care about the point $x=-1$, we can just write this as follows. We want to find $\delta > 0$ so that $0 < |x+1| < \delta$ implies that $|2x+2| < \epsilon$, where we have just substituted in $f(x) = 2x+1$, because we are not considering $x \neq -1$.

From a diagram, we can see that a good choice of δ is $\frac{\epsilon}{2}$. Let us check that this choice works.

If $0 < |x+1| < \delta$, since $\delta = \frac{\epsilon}{2}$, then $|x+1| < \frac{\epsilon}{2}$. Multiplying by 2, this means that $|2x+2| < \epsilon$. So we have shown that for this choice of $\delta = \frac{\epsilon}{2}$, we have that if $0 < |x+1| < \delta$, then $|2x+2| < \epsilon$. This is what we wanted to show. So we are done.

Question 3

Use the Intermediate Value Theorem. We can see that the function f is continuous since it the sum of continuous functions. Since

$$f(1) = \sin\left(\frac{\pi}{2}\right) + 1 - 1 = 1$$

$$f(-1) = \sin\left(-\frac{\pi}{2}\right) + 1 - 1 = -1$$

we have by the Intermediate Value Theorem that there is a point $-1 < c < 1$ such that $f(c) = 0$, since $f(-1) < 0 < f(1)$ by the above computation.

Question 4

We want to show that $\lim_{x\to 0}|x| = 0$. Consider $\epsilon > 0$. We want to find a $\delta > 0$ so that if $0 < |x-0| < \delta$, then $||x|-0| < \epsilon$. Equivalently, we want to find a $\delta > 0$ so that if $0 < |x| < \delta$, then $|x| < \epsilon$. If we take $\delta = \epsilon$, then this works. Indeed, $0 < |x| < \delta$ implies $|x| < \epsilon$ if $\delta = \epsilon$. So we are done.

For the second part, consider $f(x)$ as given. It is the same as absolute value of x everywhere except at $x = 0$. So then,

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}|x| = 0$$

But $f(0) = 1$, so then

$$f(0) \neq \lim_{x\to 0} f(x)$$

This shows that f is not continuous at $x = 0$.

Question 5

Consider $\epsilon > 0$. We want to find a $\delta > 0$ so that if $0 < |x-0| < \delta$, then $|(x^4+2)-2| < \epsilon$. So we want to find $\delta > 0$ so that if $0 < |x| < \delta$, then $|x^4| < \epsilon$. If we take $\delta = \epsilon^{1/4}$, then this works. Because then if $0 < |x| < \delta$, then $|x| < \delta$, and since $\delta = \epsilon^{1/4}$, then $|x| < \epsilon^{1/4}$. Then taking fourth powers, we get that $|x|^4 < \epsilon$, or equivalently $|x^4| < \epsilon$ since $|x|^4 = |x^4|$. So we are done.

Problem Set 3

Question 6 (*)

We note that

$$-1 \leq \sin\left(\frac{1}{x^2}\right) \leq 1$$

Then, we have that

$$-|x| \leq x \sin\left(\frac{1}{x^2}\right) \leq |x|$$

Since $\lim_{x\to 0} - |x| = \lim_{x\to 0}|x| = 0$, we have by the Squeeze Theorem that

$$\lim_{x\to 0} x \sin\left(\frac{1}{x^2}\right) = 0$$

To prove the limit using ϵ-δ, consider any $\epsilon > 0$. We want to find $\delta > 0$ so that if $0 < |x - 0| < \delta$, then

$$\left|x \sin\left(\frac{1}{x^2}\right)\right| < \epsilon$$

We can choose $\delta = \epsilon$. Because then, if $0 < |x - 0| < \delta$, then $|x| < \delta$ so $|x| < \epsilon$ since $\delta = \epsilon$. Then,

$$\left|x \sin\left(\frac{1}{x^2}\right)\right| \leq |x| < \epsilon$$

where we are using the fact that at any $x \neq 0$,

$$\left|\sin\left(\frac{1}{x^2}\right)\right| \leq 1$$

So we are done.