Math 1A: Discussion 8/29/2018 Solutions

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August 29, 2018

Some quick remarks:

- Remember that the questions in Problem Set 3 are always intended to be extra challenges. So they are completely **optional**. These questions often address concepts that are beyond the scope of the class.
- Focus on the problems in Problem Set 1 and Problem Set 2.

1 Problem Set 1

1.1 Question 1

To calculate $\sin\left(\frac{5\pi}{3}\right)$, we note that the reference angle (the acute angle made with the x-axis of a ray that makes an angle of $5\pi/3$) is $\pi/3$. We have that $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ by considering a $30-60-90$ right triangle. Since sin is only positive in the first and second quadrant, we have that

$$\sin\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

Similarly, the reference angle for $\frac{11\pi}{4}$ is $\frac{\pi}{4}$ and this angle is in the second quadrant. We have that $\tan\left(\frac{\pi}{4}\right) = 1$ and that tangent is only positive in the first and third quadrants. So we have that

$$\tan\left(\frac{11\pi}{4}\right) = -1$$

We note that $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$. So we just need to figure out what quadrant the angle is in. We note that cosine is negative only in the second and third quadrants. Since it is also periodic with period 2π, we have that

$$x = \frac{2\pi}{3} + 2\pi k, \frac{4\pi}{3} + 2\pi k$$

for integers k.

Finally, the domain of tangent is all real numbers except odd multiples of $\frac{\pi}{2}$. Equivalently, we can describe this as all real numbers except numbers of the form

$$(2k+1)\frac{\pi}{2}$$

for integers k.

1.2 Question 2

No, the domains are not the same. The domain of $f(s)$ is $s \leq 2$, since we require $2 - s \geq 0$. However, the domain of $g(s)$ is $s \leq 2$ and $s \neq 1$, since we have the extra condition that the denominator cannot be zero. In particular,

$$1 - \sqrt{2-s} \neq 0$$

so $s \neq 1$ also.

2 Problem Set 2

2.1 Question 3

Let us consider

$$f(t) = \frac{1}{\sqrt{t^2 - 3t + 2}}$$

We must have

$$t^2 - 3t + 2 \geq 0$$

and

$$t^2 - 3t + 2 \neq 0$$

We have that $t^2 - 3t + 2 \geq 0$ is equivalent to $(t-1)(t-2) \geq 0$, so $t \leq 1$ or $t \geq 2$.

But since $t^2 - 3t + 2 \neq 0$, we also have $t \neq 1, t \neq 2$. So our domain is

$$t < 1 \text{ or } t > 2$$

Next, consider the function

$$g(t) = \frac{1}{2 + \frac{1}{t-2}}$$

We have two denominators here, so we have two restrictions. We have that

$$t - 2 \neq 0$$

$$2 + \frac{1}{t-2} \neq 0$$

Solving these conditions, we get $t \neq 2$ and $t \neq 3/2$ respectively. So the domain is all real numbers except $3/2$ and 2.

Next, consider

$$h(t) = \frac{1}{\frac{4}{\sqrt{t^2-1}} - 1}$$

We have three conditions to consider.

$$t^2 - 1 \geq 0$$

$$t^2 - 1 \neq 0$$

$$\frac{4}{\sqrt{t^2-1}} - 1 \neq 0$$

We consider these conditions one by one:

- The first condition is equivalent to $t^2 \geq 1$, so $t \leq -1$ or $t \geq 1$.

- $t^2 - 1 \neq 0$ means that $t \neq -1$, $t \neq 1$.
- $\frac{4}{\sqrt{t^2-1}} - 1 \neq 0$ means that $t \neq -\sqrt{17}$ and $t \neq \sqrt{17}$.

Combining all of these restrictions together, we have that the domain is all real numbers t such that $t < -1$ or $t > 1$, except for $t = -\sqrt{17}$ and $t = \sqrt{17}$.

Next, consider

$$u(t) = \frac{\tan(t)}{1 + \sin(t)}$$

We have two restrictions here. First, tangent must be defined. Also, $1 + \sin(t) \neq 0$. Since tangent must be defined, we cannot have any odd multiples of $\frac{\pi}{2}$ (numbers of the form $(2k+1)\frac{\pi}{2}$ for integers k). Since $1+\sin(t) \neq 0$, we have that $\sin(t) \neq -1$, so t cannot be any number of the form $\frac{3\pi}{2} + 2\pi k$ for any integer k. So the domain is all real numbers except any numbers of the form

$$(2k+1)\frac{\pi}{2} \text{ or } \frac{3\pi}{2} + 2\pi k$$

for integers k.

Next, consider

$$v(t) = \frac{t}{\sqrt[3]{t^2 - 1}}$$

Cube roots do not give any domain restrictions (it is possible to take the cube root of negative numbers!). So we only have that $\sqrt[3]{t^2 - 1} \neq 0$. So $t \neq -1$ and $t \neq 1$. So the domain is all real numbers except $t = -1$ and $t = 1$.

Finally, consider

$$s(t) = \frac{t}{\sqrt{(\sqrt{t}) - 1}}$$

We have three restrictions here.

$$t \geq 0$$
$$\sqrt{t} - 1 \geq 0$$
$$\sqrt{(\sqrt{t}) - 1} \neq 0$$

The first restriction is just $t \geq 0$. The second restriction is $\sqrt{t} - 1 \geq 0$, which is equivalent to $\sqrt{t} \geq 1$, which implies that $t \geq 1$. The third restriction $\sqrt{(\sqrt{t}) - 1} \neq 0$ is equivalent to $\sqrt{t} \neq 1$, so $t \neq 1$. So combining all of these restrictions together, we have that the domain is all real numbers $t > 1$.

2.2 Question 4

For the first part, we have that the function must be of the form

$$f(x) = b(x - 1)^2$$

for some constant b. (This is because multiplying by any nonzero constant does not change the location of the zeros. For example, $(x-1)^2$, $2(x-1)^2$, and $-10(x-1)^2$ all have exactly one zero at $x = 1$). However, we can find what b is by using the fact that $(0, 3)$ is a point on the graph. In particular, $f(0) = 3$. So

$$3 = b(0 - 1)^2$$

So $b = 3$. So $f(x) = 3(x-1)^2 = 3x^2 - 6x + 3$.

For the second part, we note that the function $h(x)$ is of the form

$$h(x) = b(x+1)(x-2)$$

where b is some constant. Using the fact that $(3, -8)$ is on the graph, we have that

$$-8 = b(3+1)(3-2)$$

So $b = -2$, and hence

$$h(x) = -2(x+1)(x-2) = -2(x^2 - x - 2) = -2x^2 + 2x + 4$$

For the last part, consider the function $g(x) = x^2 + 1$. This has no zeros, because $x^2 \geq 0$, so $x^2 + 1 \geq 1$ for all real numbers x.

3 Problem Set 3

3.1 Question 5 (*)

The restrictions we have are that

$$2\sin^3(x) - 3\sin^2(x) - 2\sin(x) \geq 0$$

and also that

$$\sqrt{2\sin^3(x) - 3\sin^2(x) - 2\sin(x)} \neq 0$$

We can combine these two restrictions into one inequality:

$$2\sin^3(x) - 3\sin^2(x) - 2\sin(x) > 0$$

Factoring, we get that

$$\sin(x)(2\sin(x) + 1)(\sin(x) - 2) > 0$$

We note that the solution to the inequality

$$y(2y+1)(y-2) > 0$$

is $-1/2 < y < 0$ or $y > 2$. So using the substitution $y = \sin(x)$, we have that the domain is all x such that

$$-1/2 < \sin(x) < 0$$

or

$$\sin(x) > 2$$

But $\sin(x)$ is never greater than 2. So our domain is just all x such that $-1/2 < \sin(x) < 0$. Solving this inequality, we have that the domain is t in any of the intervals

$$\left(\pi, \frac{7\pi}{6}\right)$$

$$\left(\frac{11\pi}{6}, 2\pi\right)$$

and any translations of these intervals by $2\pi k$ for any integer k.

3.2 Question 6 (**)

For the first part, we note that since $a^2 + b^2 = 1$, there is a value δ such that

$$\cos(\delta) = a$$

$$\sin(\delta) = b$$

Then,

$$a\ \sin(t) + b\ \cos(t) = \cos(\delta)\sin(t) + \sin(\delta)\cos(t) = \sin(t + \delta)$$

For the next part, set $C = \sqrt{A^2 + B^2}$. (If $C = 0$, then $A = B = 0$, and the result is clear). Then, since

$$\left(\frac{A}{C}\right)^2 + \left(\frac{B}{C}\right)^2 = 1$$

we have that

$$\frac{A}{C}\sin(t) + \frac{B}{C}\cos(t) = \sin(t + \delta)$$

from the first part. Multiplying both sides by C, we get

$$A\ \sin(t) + B\ \cos(t) = C\ \sin(t + \delta)$$