Practice Exam 4

1) (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$ $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$

(c) not possible since $2(1,2,1)+1(-2,-4,-2)+0v$
$= (0,0,0)$

(d) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ no solution for $b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

(e) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$tr(AB) = tr\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 2$

$\neq tr(A)\,tr(B) = 2 \cdot 2 = 4$

(f) $\det(A^{-1}) = \frac{1}{\det(A)}$ since $\det(AA^{-1}) = \det(I) = 1$
$= \det(A)\det(A^{-1})$
so $\det(A)\det(A^{-1}) = 1$

$\det(A) = 4$ $\boxed{A = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}}$

(g) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

(h) not possible, if the second row is four times the first row, the determinant is 0 so the matrix cannot be invertible.

2) (a) $(A^{-1}BA)^{101} = A^{-1}B^{101}A$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] R_2 - R_3$$

$$\boxed{A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}$$

$(A^{-1}BA)^{101} = A^{-1}B^{101}A$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$(-1)^{101} = -1, \ 1^{101} = 1$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}}$$

(b) A cannot have a row of zeros in its reduced row echelon form. So the reduced row echelon form is the identity matrix, so A is invertible.

(c) Find all λ so that det is nonzero.

$$\begin{vmatrix} 0 & \frac{6}{5} & 1 \\ \lambda & \lambda & 1 \\ 5 & \lambda & 0 \end{vmatrix} = -\lambda \begin{vmatrix} \frac{6}{5} & 1 \\ \lambda & 0 \end{vmatrix} + 5 \begin{vmatrix} \frac{6}{5} & 1 \\ \lambda & 1 \end{vmatrix}$$

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix} \qquad \begin{aligned} &= -\lambda(-\lambda) + 5\left(\tfrac{6}{5} - \lambda\right) \\ &= \lambda^2 - 5\lambda + 6 \\ &= (\lambda - 2)(\lambda - 3) \end{aligned}$$

$$\boxed{\lambda \neq 2, 3}$$

(d) $\left[\begin{array}{cc|c} 1 & 1 & c \\ 3 & d & 2d \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & c \\ 0 & d-3 & 2d-3c \end{array}\right] R_2 - 3R_1$

no solution: $d=3$, $c=0 \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 6 \end{array}\right]$

exactly one solution: $d=4$, $c=0 \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 8 \end{array}\right]$

infinitely many solutions: $d=3$, $c=2 \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$

(e) $\det B = 0$ (since B is upper triangular, so $\det B$ is the product of the diagonal entries)

$\det(A^{2019}B^{2019}) = (\det A)^{2019}(\det B)^{2019} = \boxed{0}$

3) (a) $AB = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & 1 \\ -1 & 2 & 0 & 1 \\ 0 & -1 & -1 & 3 \\ 4 & 2 & 1 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} -2 & 6 & 2 & 4 \\ -1 & 2 & 0 & 1 \\ 0 & -1 & -1 & 3 \\ 4 & 2 & 1 & 1 \end{bmatrix}}$

(b) ~~$R_2 + 3R_1$ in row 2~~ R1 + 3R2 in row 1

(c) $\left[\begin{array}{cccc|cccc} 1 & 3 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] R_1 - 3R_2$

$\boxed{A^{-1} = \begin{bmatrix} 1 & -3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$

4) Need to find a, b, c, d such that

$$x_1 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

only has the zero solution $(x_1, x_2, x_3) = (0,0,0)$.

$$-x_1 + x_2 + a x_3 = 0$$
$$x_1 + 2x_2 + b x_3 = 0$$
$$x_1 + x_2 + c x_3 = 0$$
$$d x_3 = 0$$

$$\left[\begin{array}{ccc|c} -1 & 1 & a & 0 \\ 1 & 2 & b & 0 \\ 1 & 1 & c & 0 \\ 0 & 0 & d & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 0 & 2 & a+c & 0 \\ 0 & 1 & b-c & 0 \\ 1 & 1 & c & 0 \\ 0 & 0 & d & 0 \end{array}\right] \begin{array}{l} R_1+R_3 \\ R_2-R_3 \\ \\ \\ \end{array}$$

$$\to \left[\begin{array}{ccc|c} 0 & 0 & a-2b+3c & 0 \\ 0 & 1 & b-c & 0 \\ 1 & 0 & -b+2c & 0 \\ 0 & 0 & d & 0 \end{array}\right] \begin{array}{l} R_1-2R_2 \\ \\ R_3-R_2 \\ \\ \end{array} \to \left[\begin{array}{ccc|c} 1 & 0 & -b+2c & 0 \\ 0 & 1 & b-c & 0 \\ 0 & 0 & a-2b+3c & 0 \\ 0 & 0 & d & 0 \end{array}\right] \begin{array}{l} R_3 \\ \\ R_1 \\ \\ \end{array}$$

pivot

need either $\boxed{a-2b+3c \neq 0 \text{ or } d \neq 0}$

for system to have three pivots

5)

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 + x_3 & x_1 + x_2 + x_4 & x_2 + x_3 + x_4 & -x_1 + x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 \quad + x_3 \quad &= 1 \\ x_1 + x_2 \quad + x_4 &= 0 \\ x_2 + x_3 + x_4 &= 0 \\ -x_1 \quad\quad + x_4 &= 0 \end{aligned} \qquad \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ \\ R_1 + R_4 \end{array} \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right] \begin{array}{l} \\ \\ R_3 - R_2 \\ \\ \end{array}$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{array}\right] \begin{array}{l} \\ \\ \frac{1}{2} R_3 \\ R_4 - \frac{1}{2} R_3 \end{array} \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{array}\right] \begin{array}{l} R_1 - R_3 \\ R_2 + R_3 \\ \\ \\ \end{array}$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{array}\right] \begin{array}{l} \\ R_2 - R_4 \\ \\ \\ \end{array} \qquad \boxed{\begin{aligned} x_1 &= \tfrac{1}{2} \\ x_2 &= -1 \\ x_3 &= \tfrac{1}{2} \\ x_4 &= \tfrac{1}{2} \end{aligned}}$$