Practice Exam 3

1) (a) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

(b) $(\det(A^4)) = (\det A)^4$
so $\det A = 0$.
$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = A$

(c) (1,1,1), (2,2,2), (3,3,3), (4,4,4),
(5,5,5), (6,6,6)

(d)
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(e) No such c exists. Consistent if $c=0$,
if $c \neq 0$, then

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & c & c \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \tfrac{1}{c}R_3$$

$$\downarrow$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \begin{matrix} R_1 - R_3 \\ R_2 - R_3 \\ \end{matrix}$$

which is consistent.

(f) No such A exists, since if A is invertible,
then $(A^2)^{-1} = A^{-1}A^{-1}$.

(g) $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

(h) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

2) (a) $tr(A^2 B (A^{-1})^2) = tr(A^2 (A^{-1})^2 B)$
$= tr(B) = 1+2+0+1 = 4$

(b) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$A^3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

so $A^{100} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ too.

(c) $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \end{bmatrix}$ det $A = 0$ since columns 1 and 3 are the same.

(d) $M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -10 & 2 & 0 & 0 \\ -10 & -7 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$

$= \begin{bmatrix} 1-\lambda & 0 & 0 & 0 \\ -10 & 2-\lambda & 0 & 0 \\ -10 & -7 & -2-\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{bmatrix}$ ← lower triangular

$\det M = (1-\lambda)(2-\lambda)(-2-\lambda)(-\lambda)$

invertible when $\det M \neq 0$. $\boxed{\lambda \neq 0, -2, 2, 1}$

(e)
$$\left[\begin{array}{ccc|c} 3 & 1 & -1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \rightarrow \begin{array}{c} \left[\begin{array}{ccc|c} 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \end{array} \begin{array}{l} R_1-3R_3 \\ R_2-R_3 \\ \end{array}$$

$$\swarrow$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] R_2-R_1 \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\uparrow$ free variable

$$(s,-2s,s) = \boxed{s\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}}$$

3) (a)
$$\left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & -1 & 1 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ \\ R_4-R_3 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{array}\right] \begin{array}{l} \\ \\ \\ \frac{1}{2}R_4 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -1 & 0 & -2 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{array}\right] \begin{array}{l} \\ R_2+R_4 \\ R_3-R_4 \\ \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1&0&0&0&-1&1&\frac{1}{2}&-\frac{1}{2} \\ 0&1&0&0&0&0&\frac{1}{2}&\frac{1}{2} \\ 0&0&1&0&2&-1&-\frac{1}{2}&\frac{1}{2} \\ 0&0&0&1&0&0&-\frac{1}{2}&\frac{1}{2}\end{array}\right] \begin{array}{l} R_1+R_3 \\ \\ -R_3 \\ \\ \end{array}$$

$$\boxed{A^{-1} = \begin{bmatrix} -1&1&\frac{1}{2}&-\frac{1}{2} \\ 0&0&\frac{1}{2}&\frac{1}{2} \\ 2&-1&-\frac{1}{2}&\frac{1}{2} \\ 0&0&-\frac{1}{2}&\frac{1}{2}\end{bmatrix}}$$

(b)
$$\begin{bmatrix} 1&0&3&0 \\ 2&0&1&1 \\ 0&1&0&-1 \\ 0&1&0&1\end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4\end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4\end{bmatrix}$$

$$A^{-1}Ax = A^{-1}b$$

$$x = \begin{bmatrix} -1&1&\frac{1}{2}&-\frac{1}{2} \\ 0&0&\frac{1}{2}&\frac{1}{2} \\ 2&-1&-\frac{1}{2}&\frac{1}{2} \\ 0&0&-\frac{1}{2}&\frac{1}{2}\end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4\end{bmatrix}$$

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4\end{bmatrix} = \begin{bmatrix} -b_1+b_2+\frac{1}{2}b_3-\frac{1}{2}b_4 \\ \frac{1}{2}b_3+\frac{1}{2}b_4 \\ 2b_1-b_2-\frac{1}{2}b_3+\frac{1}{2}b_4 \\ -\frac{1}{2}b_3+\frac{1}{2}b_4\end{bmatrix}}$$

4) (a)
$$\begin{bmatrix} 1&2&-1 \\ 1&3&1 \\ 1&2&1\end{bmatrix} \rightarrow \begin{bmatrix} 1&2&-1 \\ 0&1&2 \\ 0&0&2\end{bmatrix} \begin{array}{l} \\ R_2-R_1 \\ R_3-R_1\end{array} \rightarrow \begin{bmatrix} 1&2&0 \\ 0&1&0 \\ 0&0&2\end{bmatrix} \begin{array}{l} R_1+\frac{1}{2}R_3 \\ R_2-R_3 \\ \\ \end{array}$$

$$\rightarrow \begin{bmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1\end{bmatrix} \begin{array}{l} R_1-2R_2 \\ \\ \frac{1}{2}R_3\end{array}$$

Since A is square and row equivalent to the identity, it is invertible, so $Ax=0$ only has the trivial solution (so the columns of A are linearly independent) and $Ax=b$ has a unique solution for each b (so the columns of A span $\mathbb{R}^3$).

(b) If the columns of a square matrix are linearly independent, then $Ax=0$ has only the trivial solution. So A is invertible and hence $Ax=b$ has a unique solution for each b. So the columns of A span $\mathbb{R}^3$.

5) $f(x)=ax^2+bx+c$

$f(-1)=1,\ f(1)=1,\ f(2)=0$

$$1=a(-1)^2+b(-1)+c$$
$$1=a(1)^2+b(1)+c$$
$$0=a(2)^2+b(2)+c$$

$$\begin{cases} a-b+c=1 \\ a+b+c=1 \\ 4a+2b+c=0 \end{cases} \qquad \begin{array}{ccc|c} a & b & c & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 4 & 2 & 1 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & -2 & -3 & -4 \end{array}\right] \begin{array}{l} \\ R_2-R_1 \\ R_3-4R_2 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -3 & -4 \end{array}\right] R_2+R_3$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{4}{3} \end{array}\right] \begin{array}{l} \\ \frac{1}{2}R_2 \\ -\frac{1}{3}R_3 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{4}{3} \end{array}\right] R_1+R_2$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{4}{3} \end{array}\right] R_1-R_3$$

$a=-\frac{1}{3},\ b=0,\ c=\frac{4}{3}$

$$\boxed{f(x)=-\frac{1}{3}x^2+\frac{4}{3}}$$

This quadratic polynomial is unique since $A=\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}$ has the identity matrix as its reduced row echelon form. So the system has a unique solution by the Invertible Matrix Theorem.