Practice Exam 2

1) (a) Not possible, row equivalent coefficient matrices have the same solution to the homogeneous system.

(b)
$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) $(1,0), (0,1), (1,1)$

(d)
$$A = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(e) Not possible, since then $(0,0,0)$ is a solution but this is only possible if b is the zero vector.

(f) $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2I$

(g) Not possible, see PS 3 (since $\det(A) = \det(A^t)$ for square matrices)

(h)
$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

2) (a) det is 0 since columns 1 and 4 are identical.

(b) $X \neq 0$ for system to be consistent. If $x \neq 0$,

$$\begin{bmatrix} 0 & 1 & 3 & 2 & -2 \\ 0 & 0 & 0 & y & 1 \\ 0 & 0 & 0 & x & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 3 & 2 & -2 \\ 0 & 0 & 0 & y & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tfrac{1}{x}R_3 \rightarrow \begin{bmatrix} 0 & 1 & 3 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{matrix} R_1 - 2R_3 \\ R_2 - yR_3 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 0 & 1 & 3 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Every row has a pivot, so $\boxed{\text{consistent if } x \neq 0}$

(c) $M = A^{-1}\begin{bmatrix}3&0\\0&2\end{bmatrix}A$ since $M^2 = A^{-1}\begin{bmatrix}3&0\\0&2\end{bmatrix}^2 A$

$= A^{-1}DA$

$$\left[\begin{array}{cc|cc}1&1&1&0\\0&1&0&1\end{array}\right] \downarrow \left[\begin{array}{cc|cc}1&0&1&-1\\0&1&0&1\end{array}\right] R_1-R_2$$

$A^{-1} = \begin{bmatrix}1&-1\\0&1\end{bmatrix}$

$$\begin{bmatrix}1&-1\\0&1\end{bmatrix}\begin{bmatrix}3&0\\0&2\end{bmatrix}\begin{bmatrix}1&1\\0&1\end{bmatrix}$$

$$= \begin{bmatrix}3&-2\\0&2\end{bmatrix}\begin{bmatrix}1&1\\0&1\end{bmatrix} = \boxed{\begin{bmatrix}3&1\\0&2\end{bmatrix}}$$

(d) No. Consider $A = \begin{bmatrix}1&0\\0&1\\1&1\end{bmatrix}$.

There is no solution for $b = (1,1,0)$.

$$\left[\begin{array}{cc|c}1&0&1\\0&1&1\\1&1&0\end{array}\right] \to \left[\begin{array}{cc|c}1&0&1\\0&1&1\\0&1&-1\end{array}\right] R_3-R_1 \to \left[\begin{array}{cc|c}1&0&1\\0&1&1\\0&0&-2\end{array}\right] R_3-R_2$$

(e) Det. of upper or lower triangular matrix is the product of the diagonal entries.

So $\det A = 120$, $\det B = 1$. Hence,

$$\det(B^4 A) = (\det B)^4 \det A = \boxed{120}$$

3)

$$M = \begin{bmatrix} 3 & 2 & 4 & 0 & 4 & 0 \\ 2 & 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 3 & 0 \\ 7 & 3 & 4 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 2 & 3 & 4 & 1 & 1 \end{bmatrix} \qquad \begin{matrix} + & - & + & - & + & - \\ - & + & - & + & - & + \\ + & - & + & - & + & - \\ - & + & - & + & - & + \\ + & - & + & - & + & - \\ - & + & - & + & - & + \end{matrix}$$

$$\det M = 1 \begin{vmatrix} 3 & 2 & 4 & 0 & 4 \\ 2 & 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 7 & 3 & 4 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} \qquad \begin{matrix} + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \end{matrix}$$

$$= 1 \cdot 1 \begin{vmatrix} 3 & 2 & 4 & 0 \\ 2 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 7 & 3 & 4 & 1 \end{vmatrix} \qquad \begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix}$$

$$= 1 \cdot 1 \cdot 1 \begin{vmatrix} 3 & 2 & 4 \\ 2 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} \qquad \begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

$$= 1 \cdot 1 \cdot 1 \cdot 1 \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} = -1$$

Since $\det M \neq 0$ and M is a square matrix, this is equivalent to the column vectors of M being linear independent and having $\mathbb{R}^6$ as their span.

4) $\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} x_1 + 2x_3 & x_2 + 2x_4 \\ 3x_1 - x_3 & 3x_2 - x_4 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 \quad + 2x_3 \quad &= 3 \\ x_2 \quad + 2x_4 &= -1 \\ 3x_1 \quad - x_3 \quad &= 0 \\ 3x_2 \quad - x_4 &= 0 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 0 & 2 & -1 \\ 3 & 0 & -1 & 0 & 0 \\ 0 & 3 & 0 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 0 & 2 & -1 \\ 0 & 0 & -7 & 0 & -9 \\ 0 & 0 & 0 & -7 & 3 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 3R_1 \\ R_4 - 3R_2 \end{array}$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 0 & 2 & -1 \\ 0 & 0 & 1 & 0 & \frac{9}{7} \\ 0 & 0 & 0 & 1 & -\frac{3}{7} \end{array}\right] \begin{array}{l} \\ \\ -\frac{1}{7}R_3 \\ -\frac{1}{7}R_4 \end{array} \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{3}{7} \\ 0 & 1 & 0 & 0 & -\frac{1}{7} \\ 0 & 0 & 1 & 0 & \frac{9}{7} \\ 0 & 0 & 0 & 1 & -\frac{3}{7} \end{array}\right] \begin{array}{l} R_1 - 2R_3 \\ R_2 - 2R_4 \\ \\ \\ \end{array}$$

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{3}{7} \\ -\frac{1}{7} \\ \frac{9}{7} \\ -\frac{3}{7} \end{bmatrix}}$$

5) (a) $$\left[\begin{array}{cccc|cccc} 1 & 2 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ \\ R_3 - R_4 \\ \\ \end{array}$$

$$\to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 2 & 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 + R_3 \\ R_2 + R_3 \\ \\ \\ \end{array}$$

$$\to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \frac{1}{2} R_2 \\ -R_3 \\ \\ \end{array}$$

$$\boxed{A^{-1} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

(b) If all diagonal entries are nonzero, you can use the diagonal entries to clear the entries above the diagonal entries using elementary row operations, so that the RREF is an identity matrix after multiplying by constants to make the diagonal entry on each row equal to 1.