Practice Exam 1

1) (a) Not possible, since $Ax=0$ always has the zero solution.

(b)

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{since RREF is} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

which has no free variables.

(c) Diagonal matrices commute so

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(d) I since $I^{2019} = I$

(e) Not possible since k vectors cannot be linearly independent in $\mathbb{R}^n$ if $k > n$.

(f)

$$\det A^{-1} = 1 \begin{vmatrix} 4 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \cdot 1 \begin{vmatrix} 4 & 2 \\ 2 & 1 \end{vmatrix} = 0.$$

Not possible, since if such an A exists, then $(A^{-1})^{-1} = A$, but A^{-1} is not invertible.

(g) Not possible, since the product of invertible matrices is invertible. In particular, $(AB^2)^{-1} = B^{-1}B^{-1}A^{-1}$.

(h)

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \leftarrow \det A = 0 \text{ since all columns are identical}$$

2) (a) $(BA)^{-1} = A^{-1}B^{-1}$

$$= \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 4 & 1 & 3 \\ 0 & -1 & 2 \\ 3 & 1 & 2 \end{bmatrix}}$$

(b)

$$M = \begin{bmatrix} 2 & 0 & 1 & 9 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 9 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & 0 & 1 & 9 \\ 0 & -\lambda & 1 & 9 \\ 0 & 0 & 1-\lambda & 9 \\ 0 & 0 & 0 & 9-\lambda \end{bmatrix}$$

Determinant of an upper triangular matrix is the product of the diagonal elements.

So $\det M = (2-\lambda)(-\lambda)(1-\lambda)(9-\lambda)$

$$\boxed{\lambda = 2, 0, 1, 9}$$

(c)

$$\left[\begin{array}{ccccc|c} 1 & 0 & 2 & -1 & 2 & 1 \\ 0 & 1 & 0 & 3 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{array}\right] \rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{array}\right] \begin{array}{l} R_1 + R_3 \\ R_2 - 3R_3 \end{array}$$

$\uparrow \quad \uparrow$

free variables

$x_3 = s$, $x_5 = t$

$x_1 + 2x_3 + 2x_5 = 0 \Rightarrow x_1 = -2s - 2t$

$x_2 + x_5 = 5 \rightarrow x_2 = -t + 5 \qquad x_4 = -1$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s-2t \\ -t+5 \\ s \\ -1 \\ t \end{bmatrix} = \boxed{\begin{bmatrix} 0 \\ 5 \\ 0 \\ -1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}}$$

(d) Since the solution to $Ax = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$

is unique, the RREF of A has no free columns.
So $Ax = 0$ also has a unique solution.

(e) The matrix is not invertible. Since the last row is the sum of the first three,

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 2 & 2 & 2 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_4 - R_1, R_4 - R_2, \\ R_4 - R_3 \end{matrix}$$

so there is a row of zeros in RREF.
So RREF is not the identity.

3) We need to find for which (a, b, c) is there a solution to

$$c_1 \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ -3 \\ -6 \end{bmatrix} + c_4 \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{cases} 2c_1 + c_2 + 3c_4 = a \\ -c_1 + c_2 - 3c_3 = b \\ -2c_1 + 2c_2 - 6c_3 = c \end{cases}$$

$$\left[\begin{array}{cccc|c} 2 & 1 & 0 & 3 & a \\ -1 & 1 & -3 & 0 & b \\ -2 & 2 & -6 & 0 & c \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 0 & 3 & -6 & 3 & a+2b \\ -1 & 1 & -3 & 0 & b \\ 0 & 0 & 0 & 0 & -2b+c \end{array}\right] \begin{matrix} R_1 + 2R_2 \\ \\ R_3 - 2R_2 \end{matrix}$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & -1 & 3 & 0 & -b \\ 0 & 1 & -2 & 1 & \frac{1}{3}a + \frac{2}{3}b \\ 0 & 0 & 0 & 0 & -2b+c \end{array}\right] \begin{matrix} -R_2 \\ \frac{1}{3}R_1 \end{matrix} \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & \frac{1}{3}a - \frac{1}{3}b \\ 0 & 1 & -2 & 1 & \frac{1}{3}a + \frac{2}{3}b \\ 0 & 0 & 0 & 0 & -2b+c \end{array}\right] R_1 + R_2$$

Need $-2b + c = 0$ to have a solution. $\rightarrow$

So $\boxed{c = 2b, \text{ a can be anything}}$

(a, b, c) is in the span if and only if $c = 2b$.
So $(1,1,1)$ is not in the span.

4) $$[x_1\ x_2\ x_3\ x_4]\begin{bmatrix}2\\1\\1\\1\end{bmatrix}=0$$

$$[x_1\ x_2\ x_3\ x_4]\begin{bmatrix}1\\1\\-1\\0\end{bmatrix}=0$$

$$[x_1\ x_2\ x_3\ x_4]\begin{bmatrix}2\\0\\-1\\-1\end{bmatrix}=0$$

$$\begin{cases} 2x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 - x_3 = 0 \\ 2x_1 - x_3 - x_4 = 0 \end{cases}$$

$$\left[\begin{array}{cccc|c} 2 & 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 2 & 0 & -1 & -1 & 0 \end{array}\right] \rightarrow \begin{array}{c}\left[\begin{array}{cccc|c} 0 & -1 & 3 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & -2 & 1 & -1 & 0 \end{array}\right]\end{array} \begin{array}{l} R_1-2R_2 \\ \\ R_3-2R_2 \end{array}$$

$$\rightarrow \left[\begin{array}{cccc|c} 0 & 1 & -3 & -1 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & -2 & 1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 0 & 1 & -3 & -1 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & -5 & -3 & 0 \end{array}\right] \begin{array}{l} \\ \\ R_3+2R_1 \end{array}$$

$$\rightarrow \left[\begin{array}{cccc|c} 0 & 1 & -3 & -1 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{5} & 0 \end{array}\right] \begin{array}{l} \\ \\ -\frac{1}{5}R_3 \end{array} \rightarrow \left[\begin{array}{cccc|c} 0 & 1 & 0 & \frac{4}{5} & 0 \\ 1 & 1 & 0 & \frac{3}{5} & 0 \\ 0 & 0 & 1 & \frac{3}{5} & 0 \end{array}\right] \begin{array}{l} R_1+3R_3 \\ R_2+R_3 \\ \\ \end{array}$$

$$\rightarrow \left[\begin{array}{cccc|c} 0 & 1 & 0 & \frac{4}{5} & 0 \\ 1 & 0 & 0 & -\frac{1}{5} & 0 \\ 0 & 0 & 1 & \frac{3}{5} & 0 \end{array}\right] \begin{array}{l} \\ R_2-R_1 \\ \\ \end{array} \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{1}{5} & 0 \\ 0 & 1 & 0 & \frac{4}{5} & 0 \\ 0 & 0 & 1 & \frac{3}{5} & 0 \end{array}\right] \begin{array}{l} R_2 \\ R_1 \\ \\ \end{array}$$

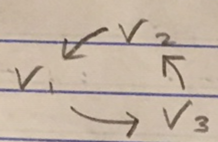
↑ free

Let $x_4 = s$. $x_1 = \frac{1}{5}s$, $x_2 = -\frac{4}{5}s$, $x_3 = -\frac{3}{5}s$.

$$\boxed{s\begin{bmatrix}\frac{1}{5}\\-\frac{4}{5}\\-\frac{3}{5}\\1\end{bmatrix} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}}$$

5) (a) $$Av = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \boxed{\begin{bmatrix} v_3 \\ v_1 \\ v_2 \\ v_4 \end{bmatrix}}$$

cyclically switches first three components of v.



(b)
$$\left[\begin{array}{cccc|cccc} 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \updownarrow$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \updownarrow$$

$$\boxed{A^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

$$\boxed{A^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ too so } A^2 = A^{-1}}$$

(c)

$$A^3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \boxed{I}$$

so $A^{2019} = (A^3)^{673} = I^{673} = \boxed{I}$